Fibered Cones and translation length on sphere complex

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July 21, 2022

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arXiv:2011.08034

Sphere graphs

- Let M be d dimensional compact manifold, the **sphere graph** has vertices are isotopy classes of embedded S^{d-1} that do not bound a ball, there is an edge of length 1 between 2 vertices iff they are disjoint.
- An homeomorphism f on M induces isometry in the sphere graph S, the **stable translation length** is

$$l_f = \lim \inf_{n \to \infty} \frac{d(v, f^n(v))}{n}$$

Examples:

- $d = 2$: curve complex on compact surfaces. Masur-Minsky (hyperbolic, lox action if pseudo-Anosov), Bowditch (l_f rational), Gadre-Tsai (inf $l_f \sim g^{-2}$), Baik-Shin (inf $l_f \sim g^{-1}$ for Torelli f).
- $M = \#^n(S^1 \times S^2)$: Free splitting complex of free groups, or simplicial completion of the Culler-Vogtmann outer space. Handel-Mosher (hyperbolicity, lox)

Fried's Homological direction

- Let $f : M \to M$ be a homeomoerphism. Let N be the mapping torus of f, then N fibers over S^1 with fiber M, and the monodromy is f. Let $\tilde{N}$ be the maximal abelian cover, then its deck group Γ is the free part of $H_1(N)$. Consider points on $\tilde{N}$, looking at its trajectory under the map $(x, t) \mapsto (\tilde{f}(x), t + 1)$, then the possible directions in Γ that can be approximated by such (forward or backward) trajectories is called the **cone of homological direction**.
- The dual cone C of the cone of homological directions in $H^1(N)$ corresponds to classes that provides a fibering over S^1.
- For suitably chosen representatives of $MCG(S)$ this cone C is the Thurston's fibered cone.
- For $Out(F_n)$, this cone C contains the "positive cone" by Dowdall-Kapovich-Leininger.

Main theorem

Theorem: (Baik-Kim-W) Let C' be a proper subcone of C, L a n-dimensional slice of C' by a rational subspace of $H^1(N;\mathbb{R})$. $\|\|$ a norm on $H^1(N;\mathbb{R})$, then

$$I_{f_\beta} \lesssim \|\beta\|^{-1-1/(n-1)}$$

Proof idea for $d = 2$

M: manifold, N: mapping torus

- $\psi : M \to M$. Lift it to an invariant $\mathbb{Z}$ fold cover $\tilde{M} \to M$ as $\tilde{\psi}$. Let h be the deck transformation.
- Let $\tilde{N}$ be the $\mathbb{Z}^2$ cover of N, deck transformation group is generated by $\{\Psi, H\}$. Let $\{e_1, e_2\}$ be the dual basis. Then $\psi^p_{(p,q)} = \tilde{\psi}$, $M_{(p,q)} = \tilde{M}/\langle \tilde{\psi}^q h^{-p} \rangle$.
- When p is large, let D be a fundamental domain of $\tilde{M}$, then h^{-p} can be far from $\tilde{\psi}^q$, hence, if c is a curve in one fundamental domain in between, it would take many iterations of $\tilde{\psi}^q$ till it hits the whole $M_{(p,q)}$.

- Applications to curve complex, free splitting and free factor complexes.
- Case for curve complex was previously done by Gadre-Tsai, Kin-Shin and Baik-Shin-W
- Works for disc complexes as well.
  Corollary: (Baik-Kim-W) The minimal stable translation length on curve graphs for pseudo-anosovs on surface of genus g that come from handlebody mapping class decay as $O(g^{-2})$.

Further works

- Lower bound?
- Other complexes?
- Outer space for RAAGs?