Entropy and Kneading of Interval and Tree Maps

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Motivating question

Consider a continuous map from an interval or a tree to itself that satisfies certain properties and have a Markov decomposition. The leading eigenvalue of the incidence matrix is the exp of the entropy of this map. What other properties can we get about the Galois conjugates of these eigenvalues?

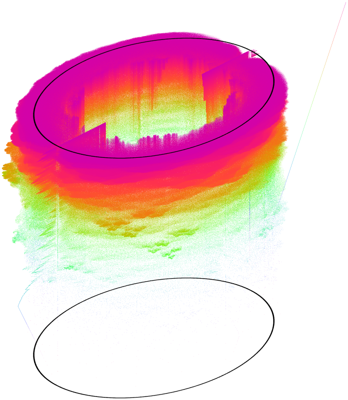
- Connection with β expandors, unimodal maps on intervals, Thurston's "Entropy in dimension 1" paper.
- Related to Hubbard Trees and Core entropy.
- Key tool: Milnor-Thurston kneading theory.

Hubbard Tree and Core Entropy

- Consider map $f_q : z \mapsto z^2 + c$. c is called **superattracting** if the orbit of critical point 0 is periodic, **post critically finite** if it is eventually periodic.
- Let c be a post-critically finite parameter of $z \mapsto z^2 + c$, then the **Hubbard tree** is the invariant tree in the filled Julia set that spans the critical orbit, and the map restricted to the Hubbard tree has a Markov decomposition, whose entropy is called **Core entropy**.
- A **principal vein** of the Mandelbrot set is a path connecting main cardoid to some c at the tip where Hubbard tree has q branches and $f_c^q(0)$ is a fixed point at the tip of one of the branches.
- When $q = 2$, $c = -2$, this is just unimodal maps, i.e. $c \in \mathbb{R}$.

Thurston's Master Teapot

- Let F be a family of interval maps or tree maps with Markov decomposition. The **Master Teapot** T_F is the closure of set of points $(z, h) \in \mathbb{C} \times \mathbb{R}$, where h is the leading root of the incidence matrix and z is a root.
- For example, if F is the map on Hubbard tree (i.e. interval) for superattracting parameters for the $q = 2$ principal vein, then this is the "Master Teapot" in Thurston's paper.



Statement of Main Results

Theorem: Let F be

- The family of Interval Maps with two critical values, one of which is a fixed point and the other has periodic orbit. (Bray-Davis-Lindsey-W, Farber-Lindsey-W)
- The map on Hubbard Trees of superattracting parameters of a given principal vein of the Mandelbrot Set. (Lindsey-Tiozzo-W)

Then:

- T_F has the following property: let H be the maximal height of points in T_F (can be infinite, e.g. in the first case), then $(z, y) \in T_F$, $|z| < 1$, implies $(z, y') \in T_F$ for all $y < y' < H$.
- If y is the leading eigenvalue of the incidence matrix of a map $f \in F$, then the height y-slice of T_F intersecting with the unit circle can be described as the parameters of some IFS where limit set contains 1.

For unimodal case, T_F can also be described as the closure of all (z, h) where h is the leading eigenvalue and z a Galois conjugate. (Lindsey-W)

Key Idea

- The critical points and vertices cut the underlying interval or trees into subintervals.
- To get characteristic polynomials of the incidence matrix one only need to understand the **itineraries** of critical points, i.e. which subintervals their forward orbits land.
- Topological constraints of interval or tree maps gives combinatorial constraints on these itineraries, called **“admissibility conditions”**.
- The proof is via careful study of these admissibility conditions.

Further Questions

- More General Tree Maps?
- Traintrack Maps?

References

- Harrison Bray, Diana Davis, Kathryn Lindsey, Chenxi Wu. The Shape of Thurston's Master Teapot, Adv. Math. 377, 2021
- Kathryn Lindsey, Chenxi Wu. A characterization of Thurston's Master Teapot, arXiv:1909.10675
- Kathryn Lindsey, Giulio Tiozzo and Chenxi Wu. Master Teapots and Entropy Algorithms for the Mandelbrot Set, arXiv:2112.14590