Types of connectedness.

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Definition 1 *A set A in a topological space (X, τ) is called* **connected** *if there are no sets $U, V \subset X$ such that*

1. *U and V are open;*

2. *$U \neq \emptyset \neq V$;*

3. *$U \cap V = \emptyset$, i.e., U and V are disjoint.*

4. *$A \subset U \cup V$.*

Otherwise A is **disconnected**.

The **connected component** *of $x \in A$ is the largest connected subset of A containing x. We call A* **totally disconnected** *if the connected component of every $x \in A$ is $\{x\}$ itself.*

The prime example of a totally disconnected is the Cantor set. The **middle third Cantor set** and a "two-dimensional version of it are both Cantor sets.

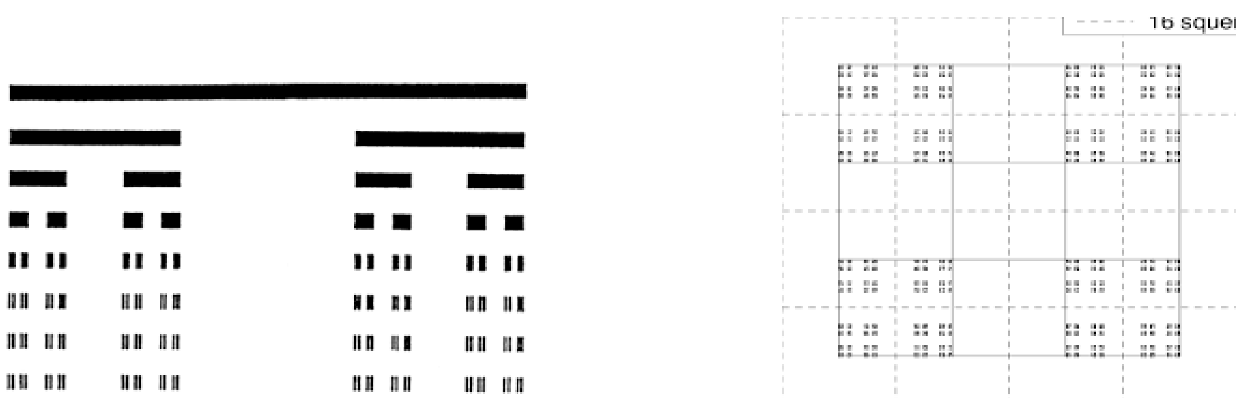

Figure 1: Two represnetations of the Cantor set.

Theorem 2 *Every two sets C and C' that are*

- *compact and non-empty;*

- *totally disconnected;*

- *perfect (i.e., without isolated points);*

are homeomorphic to each other. Such a set is called a **Cantor set**.

Definition 3 *An* **arc** *in a set A is the image of a continuous map $\gamma : [0,1] \to A$. We call A* **arc-connected** *or* **path-connacted** *if for every distinct $x, y \in A$ there is an arc in A wbetween x and y (i.e., $\gamma(0) = 0$ and $\gamma(1) = y$.*

Arc-connectedness implies connectedness, but not the other way around, as the $\sin \frac{1}{x}$-curve $\overline{\{(x, \sin \frac{1}{x}) : x \in (0, 1\}} \subset \mathbb{R}^2$ (connected but not arc-connected) shows. The Warsaw circle is arc-connected again.

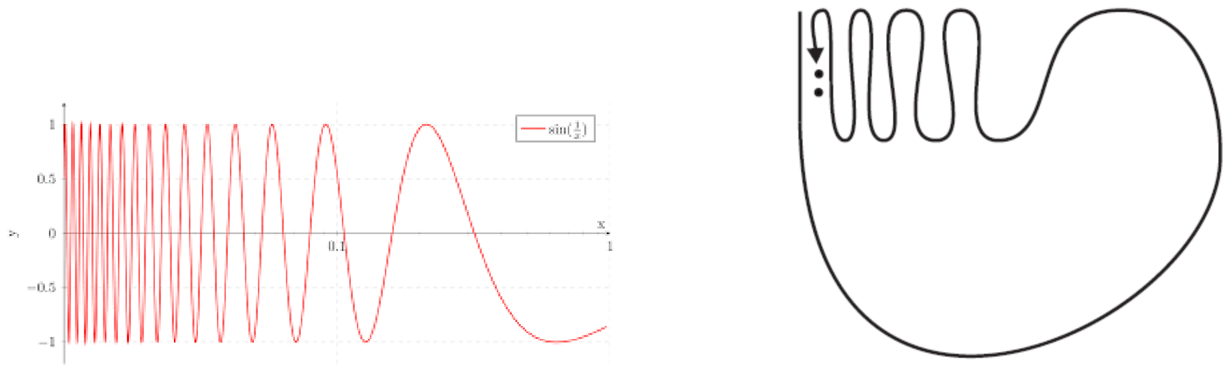

Figure 2: The $\sin \frac{1}{x}$-curve and Warsaw circle.

Definition 4 *An* **arc** *in a set A is the image of a continuous injective map $\gamma : [0,1] \to A$. We call A* **locally connected at** *$x \in A$ if for every neigborhhod $V \ni x$, there is an open neighborhood $x \in U \subset V$ that is connected. We call A* **locally connected** *if it is locally connected at every at $x \in A$.*

The Warsaw circle is connected, arc-connected but not locally connected. Local connectedness doesn't imply connectedness (sets with more than one connected component may be locally connected). Local connected plus connectedness together implies arc-connectedness.

Definition 5 *A* **loop** *in a set A is the image of a continuous injective map $\gamma : \mathbb{S}^1 \to A$. We call A* **simply connected** *if for every loop in A can be continuously contracted to a point.*

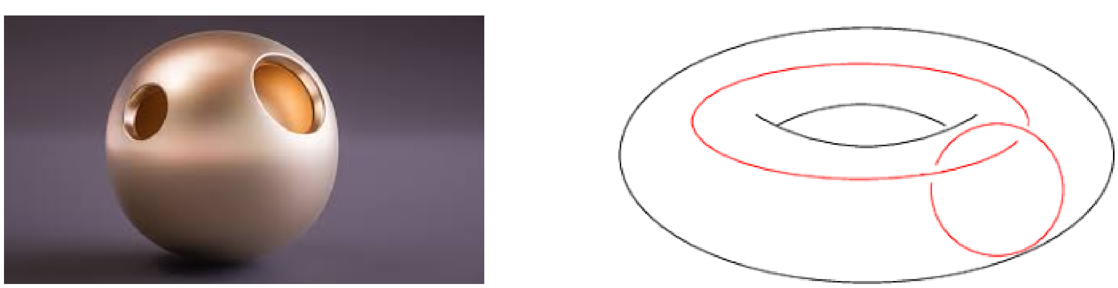

Figure 3: A sphere with two holes and a torus with two non-homotopic loops.

Simply connected means in a way: ”not too many holes“. The 2-sphere $\mathbb{S}^2$ with one hole is simply connected, but with two holes it is no longer simply connected. A torus is not simply connected: it has two ”non-homotopic“ non-contractible loops.

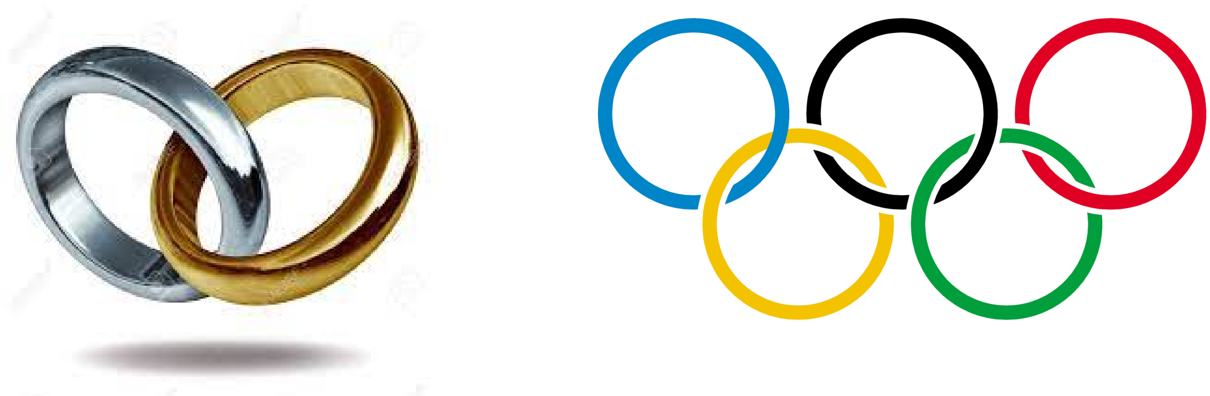

Figure 4: Linked rings and Olympic rings.

Illusionists have a trick to separate thwo linked rings R and R' (i.e., R, R' are tori $\mathbb{T}^2$ or solid tori $R = \mathbb{D}^2 \times \mathbb{S}^1$). Without trick, and inside $\mathbb{R}^3$, this is impossible.

> If $A, A' \subset \mathbb{R}^3$ are linked as links in a chain, do A, A' have to be non-simply connected?

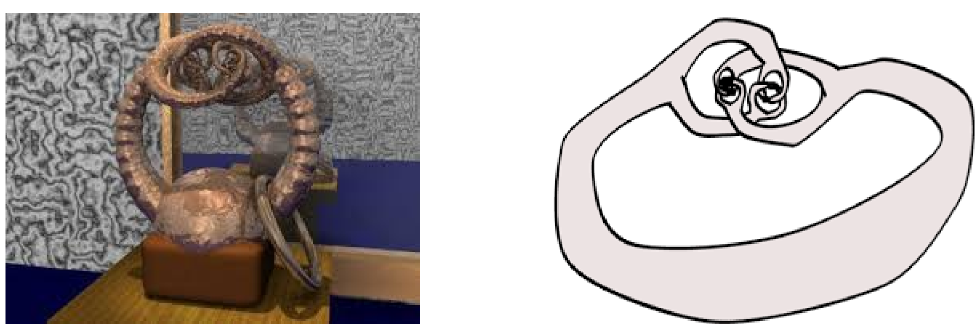

Figure 5: Alexander's horned sphere.

Theorem 6 *Let $f : X \to Y$ be a continuous map between topological spaces. If $A \subset X$ is connected/arc-wise connect/locally connected/simply connected then $f(X)$ has these same properties.*

Corollary 7 *If $f : SX \to \mathbb{R}$ is a continuous map and $A \subset X$ is connected, then for every $x, y \in f(A)$, also the interval $[x, y] \subset f(A)$.*

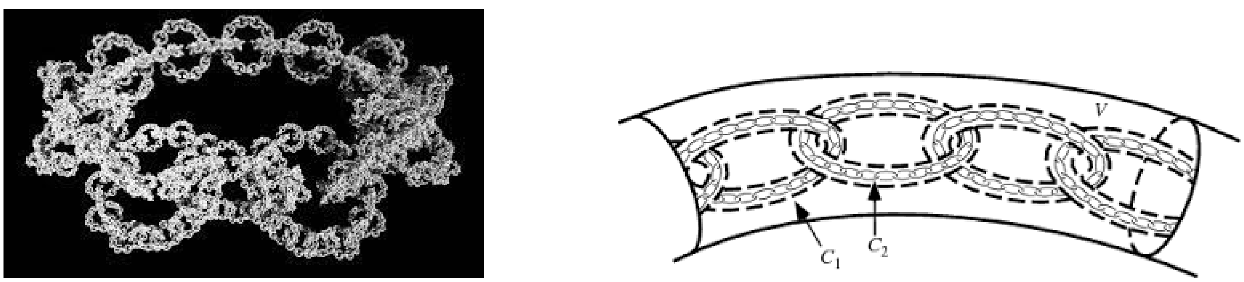

Figure 6: Antoine's necklace and a detail image.