Convergence (sequential and net-convergence) and compactness.

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1 Converging sequences

In metric spaces we know the notion of convergence, which we can translate directly to topological spaces.

Definition 1 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in topological space X. This sequence converges to x (notation: $x_n \to x$) if for*

$$\forall \text{ neighborhood } U \ni x \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 \ x_n \in U.$$

Instead of "$\forall$ neighborhood $U \ni x$" it suffices to take "$\forall U$ in a neighborhood basis of x". If X is a metric space, then the balls $\{B_{1/n}(x)\}_{1\leq n\in\mathbb{N}}$ serve as neighborhood basis, and the sequential convergence of Definition 1 reduces to the usual definition in metric spaces.

Example 2 *In the function space*

$$X = \mathbb{R}^{\mathbb{R}} = \prod_{\mathbb{R}} \mathbb{R} = \{f : \mathbb{R} \to \mathbb{R}\}$$

with product topology, a neighborhood basis of $f \in X$ is given by

$$U_{x,\varepsilon} = \{g \in X : |g(x) - f(x)| < \varepsilon\} \quad \text{for each } x \in \mathbb{R} \text{ and } \varepsilon > 0.$$

Hence $f_n \to f$ means that $\forall\ x \in \mathbb{R}\ \forall\ \varepsilon > 0\ \exists\ n_0\ \forall\ n \geq n_0$, $|f_n(x) - f(x)| < \varepsilon$. But this is pointwise convergence of $(f_n)_{n\in\mathbb{N}}$ to f.

Definition 3 *A topological space* (X, τ) *satisfies the* **first countability axiom** *if every* $x \in X$ *has a countable neighborhood basis. Such spaces are called* **AA1**, *for* 1**st Abzählbarkeitsaxiom**.

Since the balls $\{B_{1/n}(x)\}_{1 \le n \in \mathbb{N}}$ serve as **countable** neighborhood basis in a metric space, metric spaces are AA1 spaces.

For AA1 spaces two standard properties (that we know from metric spaces) hold:

Proposition 4 *Let* X *be an AA1 space and* $E \subset X$, $x \in X$. *Then* $x \in \overline{E}$ *if and only if there is a sequence* $(x_n)_{n \in \mathbb{N}}$ *in* E *such that* $x_n \to x$. *In other words, accumulation points*[1] *are limits of sequences.*

Proof. $\Rightarrow$ Let $\mathcal{B}(x) = \{U_n\}_{n \in \mathbb{N}}$ be a countable neighborhood basis of x. (We use AA1 here!).
We can assume that this basis is **nested**: $U_{n+1} \subset U_n$ for each $n \in \mathbb{N}$. If not, then take the neighborhood basis $\tilde{U}_n = \cap_{j \le n} U_j$ instead. Then $\tilde{U}_{n+1} \subseteq \tilde{U}_n$ automatically.
If $x \in \overline{E}$, then for every neighborhood of x, in particular for every U_n, $E \cap U_n \neq \varnothing$. Take $x_n \in U_n \cap E$. Then the sequence $(x_n)_{n \in \mathbb{N}}$ converges to x, because for every neighborhood U, we can find n_0 such that $U_{n_0} \subset U$, and then for every $n \ge n_0$, $x_n \in U_n \subset U_{n_0} \subset U$.

$\Leftarrow$ Assume $(x_n)_{n \in \mathbb{N}} \subset E$ converges to x. Let U be any neighborhood of x. Converges means that there is n_0 such that $x_n \in U$ for all $n \ge n_0$. In particular $x_{n_0} \in E \cap U \neq \varnothing$. But U was arbitrary, so $x \in \overline{E}$. $\square$

Theorem 5 *Let* X *be an AA1 space and* $f : X \to Y$ *a map from* X *to another topological space. Then*

> f *is continuous at* $x \in X$ *if and only if for every sequence* $x_n \to x$, *the images* $f(x_n) \to f(x)$.

The property on the right is called **sequential continuity**[2].

[1] Häufingspunkte
[2] Folgenstetigkeit

Proof. $\Rightarrow$ Let V be a neighborhood of $f(x)$. By continuity at x, there is a neighborhood $U \ni x$ such that $f(U) \subset V$. Suppose $x_n \to x$, so there is n_0 such that $x_n \in U$ for all $n \geq n_0$. But then also $f(x_n) \in V$ for all $n \geq n_0$, so $f(x_n) \to f(x)$.

$\Leftarrow$ Let $\{U_n\}_{n\in\mathbb{N}}$ be a neighborhood basis of x (we use AA1 here) and assume it is **nested** as in the previous proof. Assume by contradiction that f is **not** continuous at x, so there is a neighborhood $V \ni f(x)$ such that $f(U_n) \cap V^c \neq \varnothing$ for each $n \in \mathbb{N}$. So we can take $x_n \in U_n \setminus f^{-1}(V)$.
Let $U \ni x$ an arbitrary neighborhood, and find $n_0 \in \mathbb{N}$ such that $U_{n_0} \subset U$. Because the neighborhood U_n are nested, we have $x_n \in U_n \subset U_{n_0} \subset U$ for all $n \geq n_0$, so $x_n \to x$. By assumption, also $f(x_n) \to f(x)$. But this contradicts that $f(x_n) \notin V$ for all $n \in \mathbb{N}$. □

In both proofs, AA1 was used (and we indicated where). But we know that there are spaces that are simply too big for the AA1 property to hold. And then Proposition 4 and Theorem 5 can indeed fail.

Example 6 *Recall the orderings interval $\Omega = [0, \omega_1]$ of ordinal numbers, where ω_1 is the first uncountable ordinal.*

> *ω_1 is an accumulation point of $\Omega_0 := \Omega \setminus \{\omega_1\}$, but there is no sequence $(\alpha_n)_{n\in\mathbb{N}} \subset \Omega_0$ such that $\alpha_n \to \omega_1$.*

This shows that Proposition 4 fails. Also Theorem 5 fails.

> *Let $f : \Omega \to \{0,1\}$ be defined by*
>
> $$f(x) = \begin{cases} 0 & \text{if } x \in \Omega_0; \\ 1 & \text{if } x = \omega_1. \end{cases}$$
>
> *Then f is sequentially continuous, see above. But f is **not** continuous, because*
> $$f(\overline{\Omega_0}) = f(\Omega) = \{0,1\} \not\subset \{0\} = f(\Omega_0) = \overline{f(\Omega_0)}.$$
> *But one of the equivalent properties of continuity is $f(\overline{E}) \subset \overline{f(E)}$ (see the class notes Theorem 3.2, and that fails here for $E = \Omega_0$.*

Example 7 *We continue Example 2, i.e.,* $X = \mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$ *with product topology leading to pointwise convergence. Let*

$$E = \{f : \mathbb{R} \to \mathbb{R} : f(x) \neq 0 \text{ for at most finitely many points}\},$$

and $g(x) \equiv 1$*. Clearly* $g \notin E$*, but* $g \in \overline{E}$*. Indeed, take any finite subset* J *of* $\mathbb{R}$ *and* $\varepsilon > 0$*. Then*

$$U_{J,\varepsilon}(g) = \{f \in X : |f(x) - g(x)| < \varepsilon \ \forall x \in J\}$$

is a neighborhood of g*. The indicator function* $\mathbf{1}_J \in U_{J,\varepsilon}$ *and also* $\mathbf{1}_J \in E$*. Hence* g *is indeed an accumulation point of* E*.*

On the other hand, there is no sequence $(g_n)_{n\in\mathbb{N}} \subset E$ *such that* $g_n \to g$*. Indeed, of* $(g_n)_{n\in\mathbb{N}}$ *was such a sequence, let* $A_n = \{x \in \mathbb{R} : g_n(x) \neq 0\}$*. Each* A_n *is finite by the definition of* E*. Therefore* $A := \bigcup_{n\in\mathbb{N}} A_n$ *is countable. We can therefore find a finite set* $J \subset \mathbb{R}$ *that is disjoint from* A*. But then* $g_n \notin U_{J,\frac{1}{2}}$ *for all* $n \in \mathbb{N}$*, because* $g_n(x) = 0$ *for all* $x \in J$*.*

2 Nets

So if the problem is that neighborhood bases can be too large for sequences to deal with, we need a kind of sequence with an index set that is large enough. These are called **nets**.

We start with the properties imposed on the index set.

Definition 8 *Let* $(\mathcal{I}, \leq_{\mathcal{I}})$ *be a set with a (partial) order* $\leq_{\mathcal{I}}$*. It is called a* **directed set**[3] *if the following properties hold:*

(R1) $\forall i \in \mathcal{I} : i \leq_{\mathcal{I}} i$ *(*$\leq$ *is reflexive);*

(R2) $i \leq_{\mathcal{I}} j$ *and* $j \leq_{\mathcal{I}} k$ *implies* $i \leq_{\mathcal{I}} k$ *(*$\leq$ *is transitive);*

(R3) the order need not be total, but: $\forall\ i, j \in \mathcal{I}\ \exists k$ *such that* $i \leq_{\mathcal{I}} k$ *and* $j \leq_{\mathcal{I}} k$*.*

Definition 9 *A map* $x : \mathcal{I} \to X$ *from a directed set* $\mathcal{I}$ *into a topological space* X *is called a* **net**[4]*. As notation we use* $(x_i)_{i\in\mathcal{I}}$*.*

[3] gerichtete Menge
[4] Netz

Now we are ready to generalize limits (and accumulation points) from sequences to nets.

Definition 10 *A net $(x_i)_{i\in\mathcal{I}}$ in a topological space X*

- **converges to** x *if* $\forall$ *neighborhood* $U \ni x\ \exists\ i \in \mathcal{I}\ \forall\ j \geq_{\mathcal{I}} i\ x_j \in U$.
- *has* **accumulation point** x *if* $\forall$ *neighborhood* $U \ni x\ \forall\ i \in \mathcal{I}\ \exists\ j \geq_{\mathcal{I}} i\ x_j \in U$.

Example 11 *(a) If $\mathcal{I} = \mathbb{N}$ with the usual order, then nets reduce to sequences. Net-convergence reduces to convergence of sequences.*

(b) Let X be a topological space, and $x \in X$ arbitrary, having the neighborhood system $\mathcal{U}(x)$. Then $\mathcal{I} := \mathcal{U}(x)$ becomes directed if we take the order $U \leq_{\mathcal{I}} V$ if $V \subset U$. (Note that this is in general not a total order.) For each $U \in \mathcal{U}(x)$, choose a point $x_U \in U$. Then $(x_U)_{U\in\mathcal{U}(x)}$ is a net that converges to x. This works always, even if $\mathcal{U}(x)$ is uncountable or $\mathcal{U}(x)$ doesn't even have a countable basis.

(c) Given a real interval $[a, b]$, let

$$\mathcal{P} = \{a = p_0 < p_1 < \cdots < p_N = b\}$$

be the collection of all **finite partitions** *of $[a, b]$. The* **mesh** *of such a partition P is $\delta(P) := \max\{|p_i - p_{i-1}| : 1 \leq i \leq N\}$. The collection $\mathcal{P}$ becomes directed by setting $P \leq_{\mathcal{P}} P'$ if $\delta(P') \leq \delta(P)$.*

We extend this a bit by adding to $P \in \mathcal{P}$ an **intermediate vector**[5] $\xi = (\xi_1, \dots \xi_N)$ *with $p_{i-1} \leq \xi_i \leq p_i$ for all $1 \leq i \leq N$. Then*

$$\mathcal{I} = \{(P, \xi) : P \in \mathcal{P}, \xi \text{ is an intermediate vector to } P\}$$

becomes a directed set by setting $(P, \xi) \leq_{\mathcal{I}} (P', \xi')$ if $\delta(P') \leq \delta(P)$. So the intermediate vectors play no role in the definition of ξ.

Take a function $f : [a, b] \to \mathbb{R}$ and define **Riemann sums**

$$R^f_{(P,\xi)} \sum_{i=1}^{N} f(\xi_i)|p_i - p_{i-1}|$$

[5] Zwischenvektor

for $P = \{a = p_0 < p_1 \cdots < p_= b\}$, $\xi = (\xi_1 \ldots, \xi_N)$. *Then* $(R^f_{(P,\xi)})_{(P,\xi)\in\mathcal{I}}$ *is in fact a net on* $\mathbb{R}$. *One can show that* f *is Riemann integrable with integral* $x = \int_a^b f(t)\,dt$ *if and only if* $(R^f_{(P,\xi)})_{P,\xi)\in\mathcal{I}}$ *converges to* x.

We also need the analog of **subsequence** for nets:

Definition 12 *Let* $(\mathcal{H}, \leq_{\mathcal{H}})$ *and* $(\mathcal{I}, \leq_{\mathcal{I}})$ *be two directed sets, and* $(x_i)_{i\in\mathcal{I}}$ *a net in a topological space. A* **refinement** *or* **subnet**[6] *of* $(x_i)_{i\in\mathcal{I}}$ *is given by the composition*

$$x \circ \varphi : \mathcal{H} \to X$$

provided $\varphi : \mathcal{H} \to \mathcal{I}$ *has the properties:*

- φ *is* **monotone***:* $h_1 \leq_{\mathcal{H}} h_2$ *implies* $\varphi(h_1) \leq_{\mathcal{I}} \varphi(h_2)$.
- φ *is* **confinal***:* $\forall\ i \in \mathcal{I}\ \exists\ h \in \mathcal{H}$ *such that* $i \leq_{\mathcal{I}} \varphi(h)$.

The notation becomes $(x_{\varphi(h)})_{h\in\mathcal{H}}$

Remark 13 *If* $\mathcal{I} = \mathcal{H} = \mathbb{N}$ *(so* $(x_i)_{\mathcal{I}}$ *is a sequence) and* $\varphi : \mathcal{H} \to \mathcal{I}$ *is a strictly increasing map, say* $\varphi(h) = i_h$, *then the subnet* $(x_{\varphi(h)})_{h\in\mathcal{H}}$ *becomes the subsequence* $(x_{i_h})_{h\in\mathbb{N}}$.

However, not every subnet of a sequence is a subsequence. For example, if

$$\varphi : [0, 1) \to \mathbb{N}, \qquad \varphi(t) = \lfloor \frac{1}{1-t} \rfloor \in \mathbb{N},$$

then $(x_{\varphi(h)})_{h\in[0,1)}$ *is not a subsequence, but it is a subnet.*

Proposition 14 *Let* $(x_i)_{i\in\mathcal{I}}$ *be a net in a topological space* X.

(a) $(x_i)_{i\in\mathcal{I}}$ *converges to* x *if and only if every subnet* $(x_{\varphi(h)})_{h\in\mathcal{H}}$ *converges to* x *as well.*

(b) $(x_i)_{i\in\mathcal{I}}$ *has* x *as accumulation point if and only if there is a subnet* $(x_{\varphi(h)})_{h\in\mathcal{H}}$ *that converges to* x.

[6] Verfeinerung oder Teilnetz

Proof. (a) $\Rightarrow$ Let $\varphi : \mathcal{H} \to \mathcal{I}$ be a monotone confinal map. Let the net $(x_i)_{i\in\mathcal{I}}$ converge to x, so for every neighborhood $U \ni x$ there is $i \in \mathcal{I}$ such that $x_j \in U$ for all $j \geq_{\mathcal{I}} i$. Since φ is confinal, there is $h \in \mathcal{H}$ such that $\varphi(h) \geq_{\mathcal{I}} i$. Since φ is monotone and $\leq_{\mathcal{I}}$ is transitive, $\varphi(k) \geq_{\mathcal{I}} \varphi(h) \geq_{\mathcal{I}} i$ for all $k \geq_{\mathcal{H}} h$. But then $x_{\varphi(k)} \in U$ for all $k \geq_{\mathcal{H}} h$, and $(x_{\varphi(k)})_{h\in\mathcal{H}}$ converges to x.

$\Leftarrow$ For $\mathcal{H} = \mathcal{I}$, the identity map $\varphi(h) = h$ is monotone and confinal. By assumption $(x_{\varphi(h)})_{h\in\mathcal{H}}$ converges to x, so by indeed taking φ the identity, $(x_i)_{i\in\mathcal{I}}$ converges to x.

(b) $\Rightarrow$ Let $\mathcal{L} = \{(i, U) : i \in \mathcal{I}, U \in \mathcal{U}(x), x_i \in U\}$ with $(i, U) \leq_{\mathcal{L}} (i', U')$ if $i \leq_{\mathcal{I}} i'$ and $U' \subseteq U$. With this definition of order, $\mathcal{L}$ is directed. Also $\mathcal{L} \neq \varnothing$ because for $U = X$, $x_i \in U$ holds automatically.
Now define a monotone and confinal map $\varphi : \mathcal{L} \to \mathcal{I}$ by $\varphi(i, U) = i$. We need to show that $(x_{\varphi(i,U)})_{(i,U)\in\mathcal{L}}$ converges to x. Let $V \ni x$ be an arbitrary neighborhood and choose $i_0 \in \mathcal{I}$ with $x_{i_0} \in V$. Because x is an accumulation point, this is possible. In particular $(i_0, V) \in \mathcal{L}$. For any $(i, U) \geq_{\mathcal{L}} (i_0, V)$ we have $i \geq_{\mathcal{I}} i_0$ and $x_i \in U \subset V$. Therefore $x_{\varphi(i,U)} = x_i \in U \subset V$, so that indeed $(x_{\varphi(i,U)})_{(i,U)\in\mathcal{L}}$ converges to x.

$\Leftarrow$ Let $\varphi : \mathcal{H} \to \mathcal{I}$ a monotone confinal map such that the subnet $(x_{\varphi(i,U)})_{(i,U)\in\mathcal{L}}$ converges to x. Hence for any neighborhood $U \ni x$, there is $h_U \in \mathcal{H}$ such that $x_{\varphi(h)} \in U$ for all $h \geq_{\mathcal{H}} h_U$.
Choose a neighborhood $U \ni x$ and $i_0 \in \mathcal{I}$ arbitrary. Because φ is confinal, there is h_0 such that $\varphi(h_0) \geq_{\mathcal{I}} i_0$. Now choose $h \in \mathcal{H}$ so that both $h \geq_{\mathcal{H}} h_U$ and $h \geq_{\mathcal{H}} h_0$. Then $\varphi(h) \geq_{\mathcal{I}} \varphi(h_0) \geq_{\mathcal{I}} i_0$ and $x_{\varphi(h)} \in U$. That is, x is an accumulation point of $(x_i)_{i\in\mathcal{I}}$. □

Now we go back to Proposition 4 and Theorem 5, and show that they do hold for nets, also without the assumption that X is an AA1 space. We copy these results and proofs, indicating main changes in red.

Proposition 15 *Let X be a topological space and $E \subset X$, $x \in X$. Then $x \in \overline{E}$ if and only if there is a net $(x_i)_{i\in\mathcal{I}}$ in E that converges to x. In other words, accumulation* *points*[7] *are limits of nets.*

Proof. $\Rightarrow$ Let $\mathcal{U}(x)$ be a neighborhood system of x. Construct a net $(x_U)_{U\in\mathcal{U}(x)}$ with $x_U \in U$ as in Example 11(b). Since $x \in \overline{E}$, there is indeed such x_U that also belongs to E. Therefore $(x_U)_{U\in\mathcal{U}(x)}$ is inside E, and it indeed converges to x.

$\Leftarrow$ Take an arbitrary neighborhood $U \ni x$. Because $(x_i)_{i\in\mathcal{I}}$ converges to x, there is i_0 such that $x_i \in U$ for all $i \geq_{\mathcal{I}} i_0$. But $(x_i)_{i\in\mathcal{I}}$ belongs to E, so $x_{i_0} \in U \cap E \neq \varnothing$. Hence $x \in \overline{E}$. □

Theorem 16 *Let X be a topological space and $f : X \to Y$ a map between X and Y. Then*

> *f is continuous at $x \in X$ if and only if for every net $(x_i)_{i\in\mathcal{I}}$ converging to x, the image net $(f(x_i))_{i\in\mathcal{I}}$ converges $f(x)$.*

The property on the right is called **net continuity**[8].

Proof. $\Rightarrow$ Let V be a neighborhood of $f(x)$. By continuity at x, there is a neighborhood $U \ni x$ such that $f(U) \subset V$. Suppose $(x_i)_{i\in\mathcal{I}}$ converges to x, so there is i_0 such that $x_i \in U$ for all $i \geq_{\mathcal{I}} i_0$. But then also $f(x_i) \in V$ for all $i \geq_{\mathcal{I}} i_0$, so $f(x_i))_{i\in\mathcal{I}}$ converges to $f(x)$.

$\Leftarrow$ Assume by contradiction that f is **not** continuous at x, so there is a neighborhood $V \ni f(x)$ such that $f(U) \cap V^c \neq \varnothing$ for every $U \in \mathcal{U}(x)$, the neighborhood system of x. So we can take $x_U \in U \setminus f^{-1}(V)$.
The net $(x_U)_{U\in\mathcal{U}(x)}$ as in Example 11(b) converges to x. By assumption, also $(f(x_U))_{U\in\mathcal{U}(x)}$ converges to $f(x)$. But this contradicts that $f(x_U) \notin V$ for all $U \in \mathcal{U}(x)$. □

[7] Häufingspunkte
[8] Netzstetigkeit

3 Countability Axioms

Apart from AA1[9] from Definition 3 there is also AA2:

Definition 17 *A topological space satisfies the* **second countability axiom** *if it has a countable basis. Such spaces are called* **AA2**, *for* 2**nd Abzählbarkeitsaxiom**.

Clearly AA2 implies AA1 and from the exercises we know

- The euclidean line is AA1 and AA2;
- The Sorgenfrey line is AA1 but not AA2;
- $\mathbb{R}$ with discrete topology is AA1 (because $\mathcal{U}(x) = \{\{x\}\}$ suffices) but not AA2;
- $\mathbb{R}$ with cofinite topology is neither AA1 nor AA2. Indeed, if $\mathcal{B}(x) = \{B_i\}_{i \in N}$ were a countable neighborhood basis, then every B_i has an open U_i with $x \in U_i \subset B_i$. But $U_i = \mathbb{R} \setminus A_i$ for some finite A_i, and $A := \cup_{i \in \mathbb{N}} A_i$ is only countable, so there is $x \neq y \in \mathbb{R} \setminus A$. Hence there is no B_i contained in the open neighborhood $U := \mathbb{R} \setminus \{y\}$.

In general, AA2 is hard to check. But in metric spaces, it is easier, because it follows from (the usually easy to check) property of separability:

Definition 18 *A topological space is* **separable** *if it has a countable dense*[10] *subset.*

Every metric space is AA1 because $\{B_{1/n}(x)\}_{1 \leq n \in \mathbb{N}}$ is a neighborhood basis. Every separable (say with countable dense set A) metric space is AA2, because

$$\{B_{1/n}(a) : 1 \leq n \in \mathbb{N},\ a \in A\}$$

is a basis.

Without metric, a separable space need not be AA2; for example the Sorgenfrey line. However, every AA2 space is separable; just take one point in each element of the countable basis.

[9] Every point has a countable neighborhood basis

[10] dicht

4 Ultranets

By passing to a subsequence, you have a chance to make a non-convergent sequence convergent. For example, the real sequence $x_n = (-1)^n \frac{n}{n+1}$ doesn't converge; it has two accumulation points -1 and $+1$. By taking only the even terms, the sequence converges to 1. That is, by taking $n_k = 2k$, the subsequence $x_{n_k} = \frac{2k}{2k+1} \to 1$ as $k \to \infty$.

The same holds for nets $(x_i)_{i\in\mathcal{I}}$ in a topological space (X, τ). If E and E' are subsets of X with disjoint closures, and the intersections $(x_i)_{i\in\mathcal{I}} \cap E$ and $(x_i)_{i\in\mathcal{I}} \cap E'$ are both infinite, one can expect an accumulation point in E and another in E'. But we can take a subset of $(x_i)_{i\in\mathcal{I}}$ that only selects $x_i \in E$, and this subnet may even converge in $\overline{E}$.

An **ultimate** subnet would be one that distinguishes between **any** pair of disjoint sets E and E'. This is an **ultranet**:

Definition 19 *We call $(x_i)_{i\in\mathcal{I}}$ an* **ultranet**[11] *if for every $E \subset X$, there is $i_E \in \mathcal{I}$ such that either $x_i \in E$ for all $i \geq_{\mathcal{I}} i_E$ or $x_i \in E^c$ for all $i \geq_{\mathcal{I}} i_E$. (In this case, we say that either $x_i \in E$ eventually or $x_i \in E^c$ eventually.)*

Theorem 20 *Every net can be refined to an ultranet.*

Proof. We will skip the proof, see the last part of Chapter 5 in the class notes, with an argument via (ultra)filters, which we skip as well. Note, however, that the proof of this theorem relies on Zorn's Lemma, see the Appendix of the class notes. □

Lemma 21 *Let $f : X \to Y$ be a map between topological spaces. Then the image of an ultranet is again an ultranet.*

Proof. Let $F \subset Y$ be any set, and $E = f^{-1}(F)$. If $(x_i)_{i\in\mathcal{I}}$ is an ultranet in X, then either $x_i \in E$ eventually or $x_i \in E^c$ eventually. But then also either $f(x_i) \in F$ eventually or $f(x_i) \in F^c$ eventually. Since F was arbitrary, the image net $(f(x_i))_{i\in\mathcal{I}}$ is an ultranet. □

[11] Ultranetz oder ultrafeines Netz

Proposition 22 *If an ultranet $(x_i)_{i\in\mathcal{I}}$ has an accumulation point x, then it converges to x.*

Proof. Let U be an arbitrary neighborhood of x. Since $(x_i)_{i\in\mathcal{I}}$ is an ultranet, there is $i_0 \in \mathcal{I}$ such that either $x_i \in U$ for all $i \geq_{\mathcal{I}} i_0$ or $x_i \in U^c$ for all $i \geq_{\mathcal{I}} i_0$. But x is an accumulation point, there are is some $i \geq_{\mathcal{I}} i_0$ such that $x_i \in U$. Hence $x_i \in U$ for all $i \geq_{\mathcal{I}} i_0$. □

5 Compactness

The notion of compactness in euclidean space is simple: the Heine-Borel Theorem says that $A \in \mathbb{R}^n$ is compact if and only if it is closed and bounded.

In general, compactness is more involved. The basic definition is as follows:

Definition 23 *Let (X,τ) be a topological space and $E \subset X$. An* **open cover** *of E is a collection $\{U_i\}_{i\in\mathcal{I}}$ of open sets in X such that $E \subset \bigcup_{i\in\mathcal{I}} U_i$.*

The set $E \subset X$ of a topological space is **compact** *if every open cover has a finite subcover.*

In a metric space, every compact set is bounded and closed (in fact, compactness implies closedness in any Hausdorff space, but this implication can fail in non-Hausdorff spaces). However, there are metric spaces in which closed bounded sets are not compact. The standard example is the set $\{e_n\}_{n\in\mathbb{N}}$ of unit vectors $e_n = (0,0,\dots,0,\underbrace{1}_{\text{position } n},0\dots)$ in the sequence space ℓ_∞.

Another (non-equivalent) version of compactness is **sequentially compact**[12].

Definition 24 *A subset E in a topological space (X,τ) is* **sequentially compact** *if every sequence $(x_n)_{n\in\mathbb{N}}$ in E has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges in E (so also the limit $\lim_{k\to\infty} x_{n_k} \in E$).*

Proposition 25 *A subset E of a topological space X is sequentially compact if and only if every* **countable** *open cover of E has a finite subcover. (This latter property is called* **countably compact**[13]*.)*

[12] Folgenkompakt
[13] abzählbar kompakt

Proof. $\Rightarrow$ If $(x_n)_{n\in\mathbb{N}}$ contains the same value infinitely often, then there is a constant (and hence convergent) subsequence. So let us assume that as a set $A := \{x_n\}_{n\in\mathbb{N}}$ is countably infinite. Assume by contrapositive that A has no accumulation point in E, so for every $x \in E$, there is a neighborhood U_x such that $U_x \cap A = \varnothing$ or $= \{x\}$ itself, namely if $x \in A$. Let $U = \bigcup_{x\in E\setminus A} U_x$. Then $\{U\} \cup \{U_a : a \in A\}$ is a countable open cover of E without finite subcover.

$\Leftarrow$ Let $\{U_n\}_{n\in\mathbb{N}}$ be an open cover of E without finite subcover. That means, for every $n \in \mathbb{N}$, $E \setminus \bigcup_{j\le n} U_j \neq \varnothing$, so we can choose $x_n \in E \setminus \bigcup_{i\le n} U_j$, giving a sequence $(x_n)_{n\in\mathbb{N}}$. By sequential compactness, it has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$, say with limit $a \in E$. Since $\{U_n\}_{n\in\mathbb{N}}$ is an open cover, there is $m \in \mathbb{N}$ such that $a \in U_m$. By convergence, $x_{n_k} \in U_m$ for all k sufficiently large. But if this sufficiently large $n_k \ge m$, then $x_{n_k} \in U_m$ but also $x_{n_k} \notin \bigcup_{i\le n_k} U_i$. This is a contradiction. □

Remark 26 *If the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to x, then x is an accumulation point of $(x_n)_{n\in\mathbb{N}}$. But let us remember the definition of accumulation point x of a sequence: For every neighborhood $U \ni x$, there is $n \in \mathbb{N}$ such that $x_n \in U \setminus \{x\}$. Consider the space $X = \mathbb{N}$ with basis of the topology $\{0, n\}_{n\ge 1}$. Then the sequence $(x_n = n)_{n\in\mathbb{N}}$ has indeed 0 as accumulation point, but $(x_n)_{n\in\mathbb{N}}$ has no subsequence that converges, to 0 or anywhere. Hence in this space, every sequence has 0 as an accumulation point, but the only sequences with a convergent subsequence are those that repeat the same number infinitely often.*

As a consequence, sequential compactness can be a bit of a mess. In a metric space, fortunately, every sequence having an accumulation point, sequential compactness, countable compactness and compactness are all equivalent.

Example 27 *Apparently, sequential compactness implies, but is not the same as, compactness. To see the difference, we take again the order interval $\Omega = [0, \omega_1]$ where ω_1 is the first uncountable ordinal. Then $\Omega_0 = \Omega \setminus \{\omega_1\}$ is not compact, because $\{[0, \alpha)\}_{\alpha\in\Omega_0}$ is an open cover that has no finite subcover.*

It is good to note here, that Ω_0 is an uncountable set, even though every $\alpha \in \Omega_0$ is a (finite or) countable ordinal. This is of the same gist as: $\mathbb{N}$ is infinite, but every $n \in \mathbb{N}$ is finite.

Now to show that Ω_0 is sequentially compact, take a sequence $A := (\alpha_n)_{n\in\mathbb{N}}$ in Ω_0. By the well-ordering of Ω_0, we can create an increasing subsequence $(\beta_n)_{n\in\mathbb{N}}$ as follows:

$$\beta_0 = \min A, \qquad \beta_{n+1} = \min\{\alpha \in A : \alpha > \beta_n\}.$$

Let $\beta = \sup\{\beta_n : n \in \mathbb{N}\}$. As in Example 2.23 of the class notes, $\beta < \omega_1$, so $\beta \in \Omega_0$. We want to show that $\beta = \lim_{n\to\infty} \beta_n$. Suppose not, so there is a neighborhood $U \ni \beta$ (or in fact a basis set $(\gamma_0, \gamma_1) \ni \beta$), such that $\beta_n \notin U$ for infinitely many $n \in \mathbb{N}$. But $(beta_n)_{n\in\mathbb{N}}$ is increasing and bounded by β, so $\beta_n \notin U$ for all $n \in \mathbb{N}$. But β is the smallest upper bound of (β_n), so $(\gamma_0, \beta) = \varnothing$. That is, β is the first ordinal after γ_0, and $\gamma_0 = \beta_n$ for some $n \in \mathbb{N}$. But this leaves no room for β_{n+1}, which is a contradiction.

Hence, every infinite sequence in Ω_0 has an increasing subsequence which converges to its supremum $\beta \in \Omega_0$. This implies sequential compactness, and by Proposition 25 also countable compactness.

So we need to replace sequential compactness by net-compactness: there is no difference between compactness and net-compactness.

Theorem 28 *In a topological space (X, τ), the following statements are equivalent:*

(i) X is compact;

(ii) X has the **finite intersection property**[14]*: if $(F_i)_{i\in\mathcal{I}}$ is a collection of closed sets with $\bigcap_{i\in\mathcal{I}} F_i = \varnothing$, then there is a finite subset $J \subset \mathcal{I}$ such that $\bigcap_{i\in J} F_i = \varnothing$;*

(iii) Every net in X has an accumulation point;

(iv) Every ultranet in X converges.

[14]endliche Durchschnittseigenschaft

Proof. *(i)* $\Rightarrow$ *(ii)* Let X be compact and take the open sets $U_i = F_i^c$. Because $\bigcap_{i\in\mathcal{I}} F_i = \varnothing$, $\bigcup_{i\in\mathcal{I}} U_i = X$, so $\{U_i\}_{i\in\mathcal{I}}$ is an open cover. Let J be a finite subset of $\mathcal{I}$ such that $\bigcup_{i\in J} U_i = X$. Then $\bigcap_{i\in J} F_i = \varnothing$ as required.

(ii) $\Leftarrow$ *(i)* Reverse the proof of the previous step.

(ii) $\Rightarrow$ *(iii)* Let $(x_i)_{i\in\mathcal{I}}$ be a net in X and for each $i \in \mathcal{I}$ define the closed set

$$F_i = \overline{\{x_j : j \geq_{\mathcal{I}} i\}}.$$

If $J \subset \mathcal{I}$ is a finite set, then by property (R3) of directed sets, together with induction, we can find $i_J \in \mathcal{I}$ such that $i_J \geq_{\mathcal{I}} i$ for each $i \in J$. But then $x_J \in \bigcap_{i\in J} F_i$, that is: no finite intersection of F_i's is empty.
By property (ii), $F := \bigcap_{i\in\mathcal{I}} F_i$ is not empty, so we can choose $x \in F$. Let U be an arbitrary neighborhood of x. Then $x \in F_i$ for each $i \in \mathcal{I}$, and $U \cap \{x_j : j \geq_{\mathcal{I}} i\} \neq \varnothing$ because x belongs to the closure of $\{x_j : j \geq_{\mathcal{I}} i\}$. That means that, for every $i \in \mathcal{I}$, there is $j \geq_{\mathcal{I}} i$ so that $x_j \in U$. In other words: x is an accumulation point of $(x_i)_{i\in\mathcal{I}}$.

(ii) $\Leftarrow$ *(iii)* Assume that $(F_i)_{i\in\mathcal{I}}$ is a collection of closed sets such that $\bigcap_{i\in J} F_i \neq \varnothing$ for every finite subset $J \subset \mathcal{I}$. To show that also $\bigcap_{i\in\mathcal{I}} F_i \neq \varnothing$ we need to construct a net.
Set $\mathcal{J} = \{J \subset \mathcal{I} : J \text{ is finite}\}$ with directed order $J \leq_{\mathcal{J}} J'$ if $J \subset J'$. Since $\bigcap_{i\in J} F_i \neq \varnothing$, we can choose $x_J \in \bigcap_{i\in J} F_i$. This makes $(x_J)_{J\in\mathcal{J}}$ into a net.
Since we assume (iii), this net has an accumulation point, say x. We will show that $x \in \bigcap_{i\in\mathcal{I}} F_i$, and then we are done.
Let U be an arbitrary neighborhood of x. Let $i \in \mathcal{I}$ be arbitrary, so as a singleton $\{j\} \in \mathcal{J}$. Since x is an accumulation point of $(x_J)_{J\in\mathcal{J}}$, there is $J \geq_{\mathcal{J}} \{j\}$ (that is: $j \in J$) such that $x_J \in U$. But then $x_J \in \bigcap_{i\in J} F_i \subset F_j$, so $U \cap F_j \neq \varnothing$. But the neighborhood $U \ni x$ and $j \in \mathcal{I}$ were arbitrary, so $x \in F_i$ for all $i \in \mathcal{I}$. That is $x \in \bigcap_{i\in\mathcal{I}} F_i \neq \varnothing$.

(iii) $\Rightarrow$ *(iv)* Let $(x_i)_{i\in\mathcal{I}}$ be an ultranet in X. It is a net, with an accumulation point according to *(iii)*. By Proposition 22 it must converge.

(iii) $\Leftarrow$ *(iv)* Every net has a convergent subnet by Theorem 20. By (iv) this subnet converges, say to x. But then x is an accumulation point of the original net. □

A main compactness result is the Theorem of Tychonov concerning product spaces. Before stating and proving it, we need a lemma.

Lemma 29 *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ a product space with product topology, and $(x_i)_{i \in \mathcal{I}}$ a net in X. Then $(x_i)_{i \in \mathcal{I}}$ converges to $p = (p_\lambda)_{\lambda \in \Lambda} \in X$ if and only if $(\pi_\lambda(x_i))_{i \in \mathcal{I}}$ converges to $p_\lambda \in X_\lambda$ for every $\lambda \in \Lambda$.*

Proof. $\Rightarrow$ The projection π_λ is continuous for every $\lambda \in \Lambda$. By Theorem 5, the image net $(\pi_\lambda(x_i))_{i \in \mathcal{I}}$ converges to $\pi_\lambda(p) = p_\lambda$.

$\Leftarrow$ Choose any finite subset $(\lambda_1, \ldots, \lambda_N)$ of Λ and take $V = \pi_{\lambda_1}^{-1}(U_{\lambda_1}) \cap \pi_{\lambda_2}^{-1}(U_{\lambda_2}) \cap \cdots \cap \pi_{\lambda_N}^{-1}(U_{\lambda_N})$ for open sets U_{λ_j} with $p_{\lambda_j} \in U_{\lambda_j} \subset X_{\lambda_j}$. That is, V is a basis neighborhood of $p \in X$.
By assumption, for each $1 \leq k \leq N$ we have that $(\pi_{\lambda_k}(x_i))_{i \in \mathcal{I}}$ converges to p_{λ_k}, so there is $i_k \in \mathcal{I}$ such that $\pi_{\lambda_k}(x_i) \in U_{\lambda_k}$ for all $i \geq_{\mathcal{I}} i_k$. By rule (R3) of directed sets together with induction, we can find $i_V \in \mathcal{I}$ such that $i_V \geq_{\mathcal{I}} i_k$ for each $k \in \{1, \ldots, N\}$. Therefore also $x_i \in V$ for all $i \geq_{\mathcal{I}} i_V$. Since V is arbitrary, $(x_i)_{i \in \mathcal{I}}$ converges to p. □

Theorem 30 (Tychonov's Theorem) *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product space with product topology. Then X is compact if and only if X_λ is compact for every $\lambda \in \Lambda$.*

Proof. $\Rightarrow$ The projection π_λ is continuous and also surjective for every $\lambda \in \Lambda$. Therefore $\pi_\lambda(X) = X_\lambda$ is compact, due to Theorem 7.6 of the class notes.

$\Leftarrow$ Let $(x_i)_{i \in \mathcal{I}}$ be an ultranet in X. By Lemma 21, $(\pi_\lambda(x_i))_{i \in \mathcal{I}}$ is an ultranet in X_λ for each $\lambda \in \Lambda$. But X_λ is compact, so by Theorem 28 this ultranet converges, say to $p_\lambda \in X_\lambda$. But Lemma 29 then implies that $(x_i)_{i \in \mathcal{I}}$ converges to $p = (p_\lambda)_{\lambda \in \Lambda} \in X$. This shows that an arbitrary ultranet $(x_i)_{i \in \mathcal{I}}$ in X converges. Using Theorem 28 again, X is compact. □