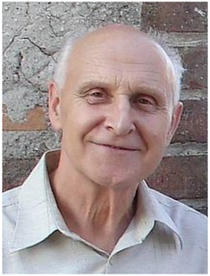
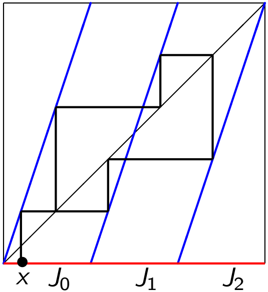
Symbolic dynamics, Markov partitions and Sharkovskiy's Theorem.



Andrei Markov (1856 - 1922) and Oleksander Sharkovsky (1936 -)

An example



$T : \mathbb{S}^1 \to \mathbb{S}^1$
$x \mapsto 3x \bmod 1$

Partition $\mathbb{S}^1$ as $\mathcal{J} = \{J_0, J_1, J_2\}$

Itinerary of $x \in \mathbb{S}^1$:

$$\boldsymbol{i}(x)_n = \begin{cases} 0 & T^n(x) \in J_0, \\ 1 & T^n(x) \in J_1, \\ 2 & T^n(x) \in J_2. \end{cases}$$

$$\boldsymbol{i}(x) = 001210\underbrace{1210\ 1210\ 1210\ 1210\ldots}_{\text{continues periodically}}$$

The (left-)shift $\sigma : \Sigma \to \Sigma$, $\quad x_0 x_1 x_2 x_3 \cdots \mapsto x_1 x_2 x_3 \ldots$ makes the diagram commute:

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{T} & \mathbb{S}^1 \\ \boldsymbol{i} \downarrow & & \downarrow \boldsymbol{i} \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array} \qquad \sigma \circ \boldsymbol{i} = \boldsymbol{i} \circ T$$

Shift spaces

Let $\mathcal{A} = \{0, 1, \ldots, N-1\}$ be a **finite** alphabet. Let

$$\Sigma := \mathcal{A}^{\mathbb{N}} \text{ or } \mathcal{A}^{\mathbb{Z}}$$

be the space of one-sided (or two-sided) sequences of letters of the alphabet.

Give Σ the product topology, i.e., the cylinder sets

$$Z_{a_n a_{n+1} \ldots a_{n+l-1}} := \{x \in \Sigma : x_n = a_n, \ldots, x_{n+l-1} = a_{n+l-1}\}$$

are all open, and form a basis for the topology.

Remark: Every cylinder set is at the same time closed, because:

$$Z_{a_n a_{n+1} \ldots a_{n+l-1}} := \Sigma \setminus \bigcup_{b_n \ldots b_{n+l-1} \neq a_n \ldots a_{n+l-1}} Z_{b_n \ldots b_{n+l-1}}$$

The space Σ also has a metric (that induces the same topology):

$$d_\Sigma(x, y) = \begin{cases} 2^{-\max\{k \,:\, x_i = y_i \ \forall |i| < k\}} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Is the itinerary map surjective?

Let $T : X \to X$ be a map, say on a compact metric space (X, d). If the partition of $\{J_0, \dots, J_{N-1}\}$ is such that $T(J_i) = X$ for each i, then $\boldsymbol{i} : X \to \Sigma$ is surjective (modulo a small set, see later).

This suggests way where symbolic dynamics can prove chaos in the sense of Devaney (provided $\boldsymbol{i}$ is also injective and continuous):

- The periodic sequences Per are dense in Σ. Therefore $\boldsymbol{i}^{-1}(Per)$ is a dense set of periodic points in X.
- Suppose for simplicity $N = \#\mathcal{A} = 3$. Then the sequence

$$s = 0\ 1\ 2\ 00\ 01\ 02\ 10\ 11\ 12\ 20\ 21\ 22\ 000\ 001\ \dots$$

  has a dense σ-orbit. Hence $\boldsymbol{i}^{-1}(s)$ has a dense T-orbit in X.
- σ has sensitive dependence on initial conditions:
  Take $\delta = \frac{1}{2}$, and $s = s_0 s_1 s_2 \dots \in \Sigma$, $\varepsilon > 0$ arbitrary.
  Let $n \in \mathbb{N}$ be so large that $2^{-n} < \varepsilon$.
  Take $t = (s_0 s_1 s_2 \dots s_{n-1} s'_n \dots)$ for $s'_n \neq s_n$. Then

$$d_\Sigma(\sigma^n(s), \sigma^n(t)) = d_\Sigma(s_n \dots, s'_n \dots) = 1 > \delta.$$

  Consequently, T has sensitive dependence on initial conditions

Is the itinerary map injective?

Definition: A map $T : X \to X$ on a metric space (X, d) is expansive with expansivity constant δ, if for every $x \neq y \in X$ there is $n \in \mathbb{Z}$ such that $d(T^n(x), T^n(y)) > \delta$.

Lemma: If T has expansivity constant δ and the partition $\{J_0, \dots, J_{N-1}\}$ is such that $\mathrm{diam}(J_i) \leq \delta$ for all i, then the itinerary map $\boldsymbol{i} : X \to \Sigma$ is injective.

Proof: Since $d(T^n(x), T^n(y)) > \delta$ for some n, $T^n(x)$ and $T^n(y)$ cannot belong to the same J_i, so $\boldsymbol{i}(x)_n \neq \boldsymbol{i}(y)_n$.

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For our example of the tripling map on the circle, T is expansive and every $\delta < \frac{1}{3}$ is an expansivity constant. But $\mathrm{diam}(J_i) = \frac{1}{3}$. However, if you take half-open intervals

$$J_0 = [0, \frac{1}{3}), \quad J_1 = [\frac{1}{3}, \frac{2}{3}), \quad J_2 = [\frac{2}{3}, 1),$$

then $\boldsymbol{i}$ is injective.

Is the itinerary map well-defined?

The problem with well-definedness is the boundary points of $\partial\mathcal{J} = \cup_i \partial J_i$. Which symbol to give if $x \in \overline{J}_i \cap \overline{J}_j$?

1. If $\mathcal{J}$ is a true partition, i.e., $J_i \cap J_j = \emptyset$ for $i \neq j$, then there is no ambiguity. But $\boldsymbol{i}$ is discontinuous at $\partial\mathcal{J}$ and $\boldsymbol{i}(X)$ is not closed (and $\boldsymbol{i}$ not surjective). E.g., for the tripling map
$$\lim_{x \nearrow \frac{1}{3}} \boldsymbol{i}(x) = 012222\cdots \neq 1000\cdots = \boldsymbol{i}(\frac{1}{3})$$
and there is no point $x \in \mathbb{S}^1$ with $\boldsymbol{i}(x) = 012222\dots$
2. Ignore the points $x \in X$ that ever hit $\partial\mathcal{J}$. This is usually a small set (countable if X is one-dimensional), but clearly $\boldsymbol{i}$ is not defined everywhere, and not surjective.
3. $x \in \overline{J_i} \cap \overline{J_j}$ gets both symbols i and j. Effectively you "double the point" x into x^- (with symbol i) and x^+ (with symbol j). This changes the topology of X, but can make $\boldsymbol{i} : X \to \Sigma$ into a homeomorphism. (Take care when $\mathrm{orb}(x)$ visits $\partial\mathcal{J}$ multiple times.)

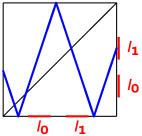
Is the itinerary map continuous?

Usually, $\boldsymbol{i} : X \to \Sigma$ is discontinuous at every point x such that $\mathrm{orb}(x) \cap \partial \mathcal{J} \neq \emptyset$. But this is, in general, a small set and we can ignore it in, for example, the verification of Devaney chaos.

One-dimensional horse-shoes

Definition: Let $T : I \to I$ a map on a one-dimensional space (e.g., the interval or the circle). If $I_0, \ldots, I_{N-1}$ are disjoint subintervals such that

$$T(I_i) \supset \bigcup_{j=0}^{N-1} I_j$$



then we say that T has an (N-fold) one-dimensional horse-shoe.

The restriction of T to

$$\Lambda = \{x \in I : T^n(x) \in \bigcup_{j=0}^{N-1} I_j \text{ for all } n \geq 0\}$$

can be described symbolically by (Σ, σ). If T is also expansive, then $T : \Lambda \to \Lambda$ and $\sigma : \Sigma \to \Sigma$ are conjugate.
In particular, $T : \Lambda \to \Lambda$ is chaotic in the sense of Devaney.

Remark: For most purposes we can relax the definition and allow I_i and I_j to intersect at their end-points. This brings ambiguity of $\boldsymbol{i}$ at these intersections, but this affects only a countable set of points.

Markov partitions

Definition Let $T : I \to I$ be a one-dimensional map. A partition $\mathcal{J} = \{J_0, \ldots, J_{N-1}\}$ f X is called a Markov partition if

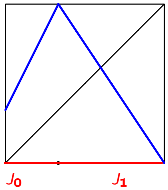
$$T(J_i) \supset J_j \text{ whenever } T(J_i) \cap J_j \neq \emptyset.$$

We can assign a transition matrix to this partition:

$$A = (a_{ij})_{i,j=0}^{N-1} \text{ is an } N \times N \text{ matrix s.t. } a_{i,j} = \begin{cases} 1 & \text{if } T(J_i) \supset J_j, \\ 0 & \text{otherwise.} \end{cases}$$

Example:

$T(J_0) = J_1$
$T(J_1) = J_0 \cup J_1$



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Markov partitions, an example

If a Markov partition $\mathcal{J} = \{J_0, \dots, J_{N-1}\}$ is used to define the itinerary map, then

$$\boldsymbol{i}(X) \supset \Sigma_A := \{s \in \{0, \dots, N_1\}^{\mathbb{N}} : a_{s_n s_{n+1}} = 1 \text{ for all } n \geq 0\}.$$

We call (Σ_A, σ) as subshift of finite type (SFT) because a finite number of words (of length 2) are forbidden, name $s_n s_{n+1}$ with $a_{s_n s_{n+1}} = 0$; for the rest, everything is allowed.

Lemma: If T has a Markov partition with transition matrix A, then the number of n-periodic orbits of T is $\geq \text{trace}(A^n)$.

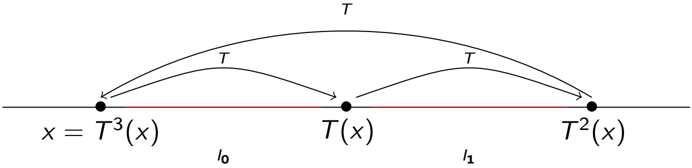
In the example,

$A^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$ for the Fibonacci numbers $F_0, F_1, F_2, F_3, F_4, F_5 \cdots = 0, 1, 1, 2, 3, 5 \dots$

Period 3 implies chaos

Theorem (Li & Yorke 1975): Let T be any continuous map on $\mathbb{R}$. If T has a periodic point of period 3, then T has a periodic point of period p for every $p \geq 1$. In addition, T is Li-Yorke chaotic (i.e., has an uncountable scrambled set.)

Proof of existence of periodic orbits: If $T^3(x) = x < T(x) < T^2(x)$, then there are closed interval I_0 and I_1 such that $T(I_0) \supset I_1$ and $T(I_1) \supset I_0 \cup I_1$:



Hence, symbolically, the restriction of T to

$$\Lambda = \{x \in I : T^n(x) \in I_0 \cup I_1 \text{ for all } n \geq 0\}$$

is the subshift of finite type Σ_A with $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Period 3 implies chaos: Proof continued

By the previous lemma, Σ_A has periodic sequences of every period. In fact, for $n \geq 2$, choose an n-periodic sequence as

$$s = 0\underbrace{11\ldots1}_{n-1 \text{ times}}\ 0\underbrace{11\ldots1}_{n-1 \text{ times}}\ 0\underbrace{11\ldots1}_{n-1 \text{ times}}\ \ldots$$

There is a subinterval $K = I_{0\underbrace{11\ldots1}_{n-1\text{times}}}$ such that $T^i(K) \subset I_{s_i}$ for $0 < i < n$, and $T^n(K) \supset K$. This follows from the Intermediate Value Theorem, which also gives the existence of an n-periodic point in $p \in K \subset I_0$. Because $T^i(p) \in I_1$ for $0 < i < n$, the smallest period is indeed n.

Finally, because $T(I_1) \supset I_1$, there must be a fixed point in I_1 by the Intermediate Value Theorem. This ends the proof.

Sharkovsky's Theorem

Unbeknownst to Li & Yorke (1975), the Ukrainian mathematician Sharkovsky had proved in 1963 a far more general result.

Theorem (Sharkovsky 1963): Consider the following order (called Sharkovskiy order) on the positive integers:

$3 \succ 5 \succ 7 \succ 9 \succ \dots$	odd numbers increasing
$\succ 6 \succ 10 \succ 14 \succ 18 \succ \dots$	$2 \times$ odd numbers increasing
$\succ 12 \succ 20 \succ 28 \succ 36 \succ \dots$	$4 \times$ odd numbers increasing
$\vdots \qquad \vdots$	$\vdots$
$\dots \succ 16 \succ 8 \succ 4 \succ 2 \succ 1$	powers of 2 decreasing.

If T be a continuous map on $\mathbb{R}$ has a periodic point of period p, then T has a periodic point of period q for all $q \prec p$.