INDUCING AND UNIQUE ERGODICITY OF DOUBLE ROTATIONS

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Abstract. In this paper we investigate "double rotations", *i.e.,* interval translation maps that when considered on the circle, have just two intervals of continuity. Using the induction procedure described by Suzuki et al., we show that Lebesgue a.e. double rotation is of finite type, *i.e.,* it reduces to an interval exchange transformation. However, the set of infinite type double rotations is shown to have Hausdorff dimension strictly between 2 and 3, and carries a natural induction-invariant measure. It is also shown that non-unique ergodicity of infinite type double rotations, although occurring, is a-typical with respect to every induction-invariant probability measure in parameter space.

1. Introduction

Interval translation maps (ITMs) are a generalisation of interval exchange transformations, first introduced by Boshernitzan and Kornfeld in 1995, [BK]. They are a class of piecewise isometry constructed by partitioning an interval, I (usually $[0,1)$) into d subintervals and applying a rotation to each partition element. Unlike IETs these rotations are not such that the map is surjective.

Thus in order to understand the dynamics upon an ITM, T it is useful to compute the attractor; $\Omega = I \cap TI \cap T^2I \cap \dots$. If there exists an $n \in \mathbb{N}$ such that $I \cap TI \cap T^2I \cap \dots \cap T^nI = I \cap TI \cap T^2I \cap \dots \cap T^nI \cap T^{n+1}I$ then we say, following [BK], that T is of *finite type*, and thus reduces to an IET. However if no such n exists then Ω is a Cantor set and T is said to be of *infinite type.* Infinite ITMs have properties such as minimality, provided Ω contains a dense orbit, [BT].

In the parameter space P of ITMs, of dimension $2d-1$, we would like to find the set of parameters for which the corresponding ITM is of infinite type. Boshernitzan and Kornfeld [BK], in analogy to the well-known Rauzy induction for interval exchange transformations (IETs), introduced an induction and renormalisation (*i.e.,* first return map rescaled to unit size) method to study their examples. In [BT] this was extended to a two-parameter slice of the three-dimensional parameter space for ITMs with two intervals. In the paper *Double Rotations*, [SIA], Suzuki et al. describe induction suitable for general ITMs of two intervals, which we shall term Suzuki induction. Using this method, we confirm the result proved [SIA] for the parameter slices where α and β are rationally independent.

Theorem 1. *For Lebesgue a.e. parameter in P, the corresponding ITM is of finite type.*

In fact, the subset of infinite type ITMs in P has a fractal structure, and our estimates indicate that its Hausdorff dimension lies in $[2, 2.88]$, see Section 4. Since Suzuki induction S is a non-conformal map, finding (a S-invariant measure equivalent to) Hausdorff measure is a non-trivial matter, as is a description of the measure of maximal dimension.

The question of unique ergodicity is a central theme for IETs. The speciality of early examples of non-unique ergodicity by Keynes & Newton [KN] and Keane [K] led to conjecture that typically, an IET (based on an irreducible permutation) is uniquely ergodic. This has been confirmed by the seminal work of Veech [V] and Masur [M], using the aforementioned Rauzy induction. In [BT] it

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was shown that there are infinite type ITMs that are not uniquely ergodic, although the number of ergodic measures is bounded by $\lceil (d+1)/2 \rceil$, see [BH]. Hence for the ITMs in this paper, two is the maximal number of ergodic measure. Although Rauzy induction, and little of the other advanced machinery of IETs are available for ITMs, non-unique ergodicity turns out to be extremely rare:

Theorem 2. *For every S-invariant measure μ such that $\mu(Abyss) = 0$, μ-a.e. parameter corresponds to a uniquely ergodic ITM.*

In Section 2 we introduce an ITM of two intervals, and consider its partitioning. Section 3 contains a definition of the accelerated version of the map, and the calculations giving us bounded distortion (*i.e.,* equations (2a) and (2b)). From this we show that with respect to Lebesgue measure, almost every point is an Abyss point. Section 4 describes results related to the Hausdorff dimension of the set of non-Abyss ITMs. The symbolic dynamics of the Suzuki map are in Section 5. Finally, the proof of Theorem 2 occupies Section 6.

2. Definitions

We consider a map, $T_{(\alpha,\beta,c)} \colon [0,1) \to [0,1)$, of the form:

$$T_{(\alpha,\beta,c)}x = \begin{cases} x+\alpha \pmod 1 & \text{if } x \in [0,c) \\ x+\beta \pmod 1 & \text{if } x \in [c,1) \end{cases}$$

with $\alpha, \beta, c \in [0,1)$. These three values give us our parameter space, $P = [0,1) \times [0,1) \times [0,1)$, a three-dimensional cube. A double rotation is a piecewise translation defined on two to four subintervals of continuity (if we identified 0 and 1 to form a circle, there are always two intervals of continuity, hence the name, but in this paper we will use the interval approach). Upon this parameter space we can apply Suzuki induction and renormalisation, to produce a new double rotation. This involves computing the first return map to a subinterval of $[0,1)$, determined by order relations between the parameters α, β, c. This corresponds to a partitioning of P, viz.:

(1) $P_{(1)} = \{(\alpha,\beta,c) \colon \alpha < \beta,\ c \le 1-\beta\}$
(2) $P_{(2)} = \{(\alpha,\beta,c) \colon \alpha < \beta,\ 1-\beta < c < 1-\alpha\}$
(3) $P_{(3)} = \{(\alpha,\beta,c) \colon \alpha < \beta,\ 1-\alpha \le c\}$
(4) $P_{(4)} = \{(\alpha,\beta,c) \colon \beta < \alpha,\ c \le \beta\}$
(5) $P_{(5)} = \{(\alpha,\beta,c) \colon \beta < \alpha,\ \beta < c < \alpha\}$
(6) $P_{(6)} = \{(\alpha,\beta,c) \colon \beta < \alpha,\ \alpha \le c\}$.

Any parameter that is in $P_{(2)}$ or $P_{(5)}$ will reduce to a rotation. The remaining cases give us the following map on P:

$$S(\alpha,\beta,c) = \begin{cases} \left(\left\{\frac{\alpha}{1-\beta}\right\}, \left\{\frac{\beta}{1-\beta}\right\}, \frac{c}{1-\beta}\right) & \text{if } (\alpha,\beta,c) \in P_{(1)} \\ \left(\left\{\frac{\alpha-1}{\alpha}\right\}, \left\{\frac{\beta-1}{\alpha}\right\}, \frac{c+\alpha-1}{\alpha}\right) & \text{if } (\alpha,\beta,c) \in P_{(3)} \\ \left(\left\{\frac{\alpha-1}{\beta}\right\}, \left\{\frac{\beta-1}{\beta}\right\}, \frac{c}{\beta}\right) & \text{if } (\alpha,\beta,c) \in P_{(4)} \\ \left(\left\{\frac{\alpha}{1-\alpha}\right\}, \left\{\frac{\beta}{1-\alpha}\right\}, \frac{c-\alpha}{1-\alpha}\right) & \text{if } (\alpha,\beta,c) \in P_{(6)} \end{cases} \tag{1}$$

with the induction taking place on $[0, 1-\beta)$, $[1-\alpha, 1)$, $[0, c) \cup [1-\beta+c, 1)$, and $[0, c-\alpha) \cup [c, 1)$ for $P_{(1)}$, $P_{(3)}$, $P_{(4)}$, and $P_{(6)}$ respectively. See [SIA] for more details.

If a point (α,β,c) maps to $P_{(2)}$ or $P_{(5)}$ under iteration of S, then the corresponding ITM is of finite type. This motivates our next definition: the *Abyss* is the set of $(\alpha,\beta,c) \in P$ such that there exists $n \in \mathbb{N}_0$ with $S^n(\alpha,\beta,c) \in P_{(2)} \cup P_{(5)}$.

The set $P_{(1)}$ is a tetrahedron with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(1,1,1)$. We can divide this space by the discontinuity lines of the Suzuki induction map S on the space, *i.e.,* when $S(\alpha,\beta,c)$

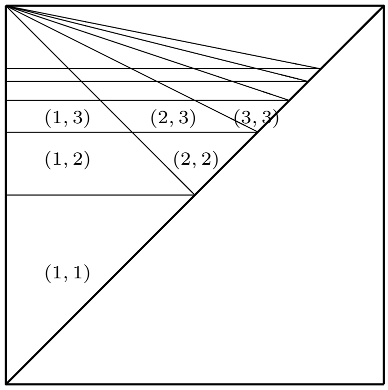

FIGURE 1. The subpartition of $P_{(1)}$ the brackets referring to the (m, n)

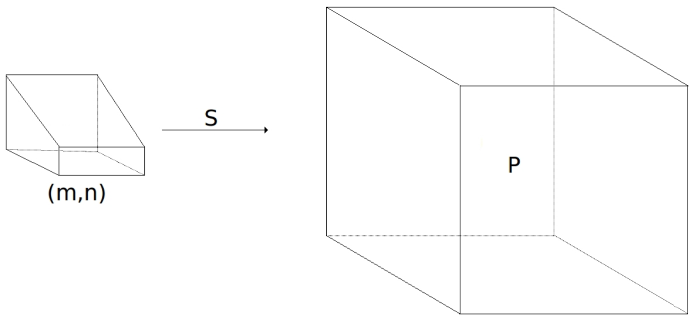

FIGURE 2. Each (m, n) box maps to the entire parameter space.

has an integer component. Let $S(\alpha_1, \beta_1, c_1) = (\alpha_2, \beta_2, c_2)$. We find $\beta_2 = 0$ when $\frac{\beta_1}{1-\beta_1} = n$ giving us the planes $\beta = \frac{n}{n+1}$ intersecting $P_{(1)}$ for all $n \in \mathbb{N}$. Similarly from $\frac{\alpha_1}{1-\beta_1} = m \in \mathbb{N}$ we get the planes $\beta_1 = 1 - \frac{\alpha}{m}$ intersecting $P_{(1)}$. This partitions the space $P_{(1)}$ into boxes made by the above mentioned planes and the plane $c = 1 - \beta$ (see Figure 1). We can give every point its address in the above in the form of (m, n).

For each box (m, n), the boundary $\partial(m, n) = \partial U_1 \cup \partial U_2 \cup \partial V_1 \cup \partial V_2 \cup \partial W_1 \cup \partial W_2$ corresponding to $c = 0$ (∂U_1), $c = 1 - \beta$ (∂U_2), $\beta = \frac{n-1}{n}$ (∂V_1), $\beta = \frac{n}{n+1}$ (∂V_2), $\alpha = (m-1)(1-\beta)$, or $\alpha = 0$ for $m = 1$, (∂W_1), and $\alpha = m(1-\beta)$ (∂W_2). These are mapped by S to $c = 0$, $c = 1$, $\beta = 0$, $\beta = 1$, $\alpha = 0$, and $\alpha = 1$ respectively. Thus each of these boxes maps to the entire parameter space, P. For the cases where $m = n$ we replace ∂V_2 and ∂W_2 with ∂D, corresponding to $\alpha = \beta$, this maps to $\alpha = \beta$ and thus (n, n) boxes map to $P_{(1)} \cup P_{(2)} \cup P_{(3)}$ for all $n \in \mathbb{N}$. Within each of the boxes the function is C^2.

There exist symmetries between the four non-Abyss cases acting as involutions

$$\begin{aligned} g(\alpha, \beta, c) &= (\beta, \alpha, 1-c) \\ h(\alpha, \beta, c) &= (1-\beta, 1-\alpha, 1-c) \end{aligned} \qquad \text{giving} \qquad \begin{array}{ccc} P_{(1)} & \overset{g}{\leftrightarrow} & P_{(6)} \\ \updownarrow h & & \updownarrow h \\ P_{(3)} & \overset{g}{\leftrightarrow} & P_{(4)} \end{array}$$

From this it follows that we shall see the same kinds of properties upon $P_{(3)}$, $P_{(4)}$, and $P_{(6)}$. Hence we can partition the space $P_{(1)} \cup P_{(3)} \cup P_{(4)} \cup P_{(6)}$ with each box having address (i, m, n) where m

and n are as above, and $i \in \{1,3,4,6\}$ signifying the $P_{(i)}$, we refer to such a partition as $P_{(i,m,n)}$. Further note that,

$$\bigcup_{i\in\{1,3,4,6\}} P_{(i)} \setminus \bigcup_{i\in\{1,3,4,6\}} \bigcup_{n\in\mathbb{N}} \bigcup_{m\leq n} P_{(i,m,n)}$$

has zero Lebesgue measure zero. Thus we have that the Suzuki induction map is a Markov map.

On each of the partition elements the expansion of S is determined by the denominators $1-\beta$, α, β, and $1-\alpha$. It turns out that each of the $P_{(i)}$s has a line of neutral fixed points, with $\beta = 0$ and $\alpha = 0$, for $P_{(1)}$ and $P_{(6)}$, and also $\alpha = 1$ and $\beta = 1$ for $P_{(4)}$ and $P_{(6)}$. So in order to obtain uniform expansion we shall accelerate the map.

Since $P_{(1,1,1)}$ is the part of $P_{(1)}$ where S has neutral fixed points, we shall speed up the subpartitioning of $P_{(1,1,1)}$. We divide the partition elements into slices: $\{(\alpha,\beta,c)\colon \beta \in (\frac{1}{t+1}, \frac{1}{t+2})\} \cap P_{(1,1,1)}$ for all $t \in \mathbb{N}$. If we examine the dynamics upon this box we see it maps to the box $\{(\alpha,\beta,c)\colon \beta \in (\frac{1}{t}, \frac{1}{t+1}), \alpha < \beta, c \in [0,1)\}$. So we can imagine that each of these boxes gets mapped into the area of the next box away from the neutral fixed point in the α and β directions, and gets its "roof" $c = 1-\beta$ stretched to $c = 1$.

We are interested in finding out when each point in each of these boxes gets mapped outside of $P_{(1,1,1)}$. There are three different cases. Firstly, the point might stay in $P_{(1,1,1)}$ until $S^k\beta \in (\frac{1}{2}, 1)$, where we write $S^k(\alpha,\beta,c) = (S^k\alpha, S^k\beta, S^kc)$. In this case $k = t$ and we define the *induce time*

$$\tau(\alpha,\beta,c) = t \qquad \text{where } t \text{ is as above.}$$

For the second type of point there exists an iteration where (α,β,c) is mapped outside of $P_{(1,1,1)}$ before $S^k\beta \in (\frac{1}{2}, 1)$. In this case the induce time τ is defined as,

$$\tau(\alpha,\beta,c) = \min\{k \in \mathbb{N}\colon S^k(\alpha,\beta,c) \in P_{(3)}\} + 1.$$

Finally we have the Abyss points, of which we need say no more. So using the above we get an accelerated version of Suzuki induction, $Z = S^\tau$, with the τ defined as above.

3. Almost Every Element of the Parameter Space is an Abyss Element

From the above we have a Markov map for S with the partition

$$\bigcup_{i\in\{1,3,4,6\}} \bigcup_{n\in\mathbb{N}} \bigcup_{m\leq n} P_{(i,m,n)} \cup P_{(2)} \cup P_{(5)}.$$

Thus we can describe an orbit of a point (α,β,c) with the addresses of its iterations: $\omega_j = i$ if $S^j(\alpha,\beta,c) \in P_{(i)}$. Obviously, the non-Abyss cases are those such that $\omega_j \in \{1,3,4,6\}$ for all $j \in \mathbb{N}_0$.

Now we consider the distortion properties of the map. For a map $f : X \to X$ with a partition, $\Phi = \{B_1, B_2, \dots\}$ of maximal regions of continuity, we say that f has bounded *distortion*, if there exists a $C > 0$ such that for every partition, B_i the $x, y \in B_i$ obey:

$$\left| \frac{\det(J_f(x))}{\det(J_f(y))} - 1 \right| \leq C\,|fx - fy|\,, \tag{2a}$$

and it has uniform expansion of the Jacobian if there are $K > 0$ and $\lambda > 1$ such that

$$|\det(J_{f^n}(x))| \geq K\lambda^n \tag{2b}$$

for all $n \in \mathbb{N}$, where J_f is the Jacobian matrix of f. Combining (2a) and (2b), we obtain bounded distortion for all iterates: there is C' such that

$$\left| \frac{\det(J_{f^k}(x))}{\det(J_{f^k}(y))} - 1 \right| \leq C' \left| f^kx - f^ky \right|,$$

for all $k \geq 1$ and all x, y in the same element of the k-fold refinement $\Phi \vee f^{-1}\Phi \vee \cdots \vee f^{1-k}\Phi$ of the partition Φ.

For Suzuki induction we compute the Jacobian matrix for $P_{(1)}$:

$$J_S(\alpha,\beta,c) = \begin{pmatrix} \frac{1}{1-\beta} & \frac{\alpha}{(1-\beta)^2} & 0 \\ 0 & \frac{1}{(1-\beta)^2} & 0 \\ 0 & \frac{c}{(1-\beta)^2} & \frac{1}{1-\beta} \end{pmatrix} \quad \text{with} \quad \det(J_S(\alpha,\beta,c)) = \frac{1}{(1-\beta)^4}.$$

Similarly, we have determinants $\frac{1}{\alpha^4}$, $\frac{1}{\beta^4}$, and $\frac{1}{(1-\alpha)^4}$ for points in $P_{(3)}$, $P_{(4)}$, and $P_{(6)}$ respectively. So now we consider if we can find a C for (2a), using the notation $S(\alpha,\beta,c) = (S\alpha, S\beta, Sc)$:

$$\begin{aligned}
\left|\frac{\det(J_S(\alpha_2,\beta_2,c_2))}{\det(J_S(\alpha_1,\beta_1,c_1))} - 1\right| &= \left|\left(\frac{1-\beta_1}{1-\beta_2}\right)^4 - 1\right| \\
&= \left|\left(\frac{1+S\beta_2}{1+S\beta_1}\right)^4 - 1\right| \\
&= \left|\left(1 + \frac{S\beta_2 - S\beta_2}{1+S\beta_1}\right)^4 - 1\right| \\
&\leq (1 + |S\beta_2 - S\beta_2|)^4 - 1 \leq 15|S\beta_2 - S\beta_2|.
\end{aligned}$$

Thus taking $C \geq 15$ suffices to satisfy (2a).

However we also have to consider the accelerated version of Suzuki induction, Z, for the points in the set $P_{(1,1,1)}$, of the first type mentioned above, those that stay in $P_{(1)}$ until it reaches $\{(\alpha,\beta,c)\colon \beta \in (\frac{1}{2},1)\}$. Hence we have:

$$Z(\alpha,\beta,c) = S^\tau(\alpha,\beta,c) = \left(\frac{\alpha}{1-\tau\beta}, \frac{\beta}{1-\tau\beta}, \frac{c}{1-\tau\beta}\right).$$

This has a Jacobian with determinant:

$$\det(J_Z(\alpha,\beta,c)) = \frac{1}{(1-\tau\beta)^4}.$$

So in this case $1 + \tau Z\beta = \frac{1}{1-\tau\beta}$ and because $Z\beta_2 \geq 1/2$ we have for (2a):

$$\begin{aligned}
\left|\frac{\det(J_Z(\alpha_2,\beta_2,c_2))}{\det(J_Z(\alpha_1,\beta_1,c_1))} - 1\right| &= \left|\left(\frac{1-\tau\beta_1}{1-\tau\beta_2}\right)^4 - 1\right| \\
&= \left|\left(\frac{1+\tau Z\beta_2}{1+\tau Z\beta_1}\right)^4 - 1\right| \\
&= \left|\left(1 + \tau\frac{Z\beta_2 - Z\beta_1}{1+\tau Z\beta_1}\right)^4 - 1\right| \\
&\leq (1 + 2|Z\beta_2 - Z\beta_2|)^4 - 1 \leq 80|Z\beta_2 - Z\beta_1|.
\end{aligned}$$

So any $C \geq 80$ satisfies (2a).

Finally we have the points of $P_{(1,1,1)}$ that get mapped to $P_{(3)}$ before they hit $\{(\alpha,\beta,c)\colon \beta \in (\frac{1}{2},1)\}$, which can happen only if $\beta < 1/(\tau+1)$. This will give us the accelerated Suzuki map of the form:

$$Z(\alpha,\beta,c) = \left(\left\{\frac{\alpha + (\tau-1)\beta - 1}{\alpha}\right\}, \left\{\frac{\tau\beta - 1}{\alpha}\right\}, \frac{\alpha + (\tau-1)\beta + c - 1}{\alpha}\right).$$

This has a determinant of $\frac{1}{\alpha^4}$. Notice that this is the same value as we get for a point in $P_{(3)}$.

To estimate the distortion in this case, we decompose $Z = S \circ \hat{Z}$ where $(\hat{\alpha},\hat{\beta},\hat{c}) = \hat{Z}(\alpha,\beta,c) = S^{\tau-1}(\alpha,\beta,c)$ is the $\tau-1$-fold application of S on $P_{(1)}$, and

$$Z(\alpha,\beta,c) = S(\hat{\alpha},\hat{\beta},\hat{c}) = \left(\left\{\frac{\hat{\alpha}-1}{\hat{\alpha}}\right\}, \left\{\frac{\hat{\beta}-1}{\hat{\alpha}}\right\}, \frac{\hat{c}+\hat{\alpha}-1}{\hat{\alpha}}\right)$$

is the final application of S on $P_{(3)}$.

Let $n \in \mathbb{N}$ be such that $\left\{\frac{\hat{\alpha}-1}{\hat{\alpha}}\right\} = \frac{n\hat{\alpha}-1}{\hat{\alpha}}$. Then for $\hat{\alpha}_1, \hat{\alpha}_2$ in the same region of continuity of S we have,

$$\begin{aligned}
\hat{\alpha}_2 - \hat{\alpha}_1 &= \hat{\alpha}_1 \left(\frac{n\hat{\alpha}_2 - 1}{\hat{\alpha}_1} - \frac{n\hat{\alpha}_1 - 1}{\hat{\alpha}_1} - (n-1)\frac{\hat{\alpha}_2 - \hat{\alpha}_1}{\hat{\alpha}_1} \right) \\
&= \hat{\alpha}_1 \left((1 + \frac{\hat{\alpha}_2 - \hat{\alpha}_1}{\hat{\alpha}_1}) S\hat{\alpha}_2 - S\hat{\alpha}_1 - (n-1)\frac{\hat{\alpha}_2 - \hat{\alpha}_1}{\hat{\alpha}_1} \right) \\
&= \hat{\alpha}_1 \left(S\hat{\alpha}_2 - S\hat{\alpha}_1 - (n-1-S\hat{\alpha}_2)\frac{\hat{\alpha}_2 - \hat{\alpha}_1}{\hat{\alpha}_1} \right),
\end{aligned}$$

so $|\hat{\alpha}_2 - \hat{\alpha}_1| = \frac{\hat{\alpha}_1}{n - S\hat{\alpha}_2} |S\hat{\alpha}_2 - S\hat{\alpha}_1|$. By a similar computation,

$$\begin{aligned}
|\hat{\beta}_2 - \hat{\beta}_1| &\leq \hat{\alpha}_1 \left(|S\hat{\beta}_2 - S\hat{\beta}_1| + (1-\beta_1)|S\hat{\alpha}_2 - S\hat{\alpha}_1| \right) \\
&\leq \hat{\alpha}_1 \left(|Z\beta_2 - Z\beta_1| + |Z\alpha_2 - Z\alpha_1| \right).
\end{aligned}$$

Since $\det J_Z = (\det J_S \circ \hat{Z}) \cdot \det J_{\hat{Z}} = (\frac{1}{\hat{\alpha}} \frac{1}{1-(\tau-1)\beta})^4$, we obtain for the distortion

$$\begin{aligned}
\frac{\det J_Z(\alpha_1, \beta_1, c_1)}{\det J_Z(\alpha_2, \beta_2, c_2)} &= \frac{\det J_S(\hat{\alpha}_1, \hat{\beta}_1, \hat{c}_1)}{\det J_S(\hat{\alpha}_2, \hat{\beta}_2, \hat{c}_2)} \frac{J_{\hat{Z}}(\alpha_1, \beta_1, c_1)}{J_{\hat{Z}}(\alpha_2, \beta_2, c_2)} \\
&\leq \left(1 + \frac{|Z\alpha_1 - Z\alpha_2|}{n - S\hat{\alpha}_2} \right)^4 \left(1 + \frac{(\tau-1)|\hat{\beta}_1 - \hat{\beta}_2|}{1 + (\tau-1)\hat{\beta}_1} \right)^4 \\
&\leq \left(1 + \frac{|Z\alpha_1 - Z\alpha_2|}{n - S\hat{\alpha}_2} \right. \\
&\qquad \left. + (1 + \frac{\tau - 1}{1 + (\tau-1)\hat{\beta}_1}) \hat{\alpha}_1 \left(|Z\beta_2 - Z\beta_1| + |Z\alpha_2 - Z\alpha_1| \right) \right)^4 \\
&\leq (1 + 3\|Z(\alpha_2, \beta_2, c_2) - Z(\alpha_1, \beta_1, c_1)\|_\infty)^4 \\
&\leq 1 + 269\|Z(\alpha_2, \beta_2, c_2) - Z(\alpha_1, \beta_1, c_1)\|_\infty,
\end{aligned}$$

where we used $\hat{\alpha}_1 \leq \hat{\beta}_1$ for the third inequality. Therefore

$$\left| \frac{\det J_Z(\alpha_1, \beta_1, c_1)}{\det J_Z(\alpha_2, \beta_2, c_2)} - 1 \right| \leq 269\|Z(\alpha_2, \beta_2, c_2) - Z(\alpha_1, \beta_1, c_1)\|_\infty.$$

It follows from the symmetries that the results for Suzuki induction and accelerated Suzuki induction will work on $P_{(3)}$, $P_{(4)}$, and $P_{(6)}$ also. Thus (2a) is satisfied. Now we consider (2b) for $P_{(1)}$ (as before this will be sufficient to demonstrate it holds for the parameter space). For Z we have that $|\det(J_Z(x))| = |\det(J_S(x))| \geq \left(\frac{1}{\frac{2}{3}}\right)^4 = \left(\frac{3}{2}\right)^4$. So for (2b) we have that:

$$|\det(J_{Z^n}(x))| \geq \lambda^n$$

where $\lambda = \left(\frac{3}{2}\right)^4$. Hence accelerated Suzuki induction Z satisfies the bounded distortion conditions above.

From [Mañ] we have the Folklore Theorem:

Theorem 3 (Folklore Theorem). *Let $f\colon M \longrightarrow M$ be a C^1 piecewise expanding map of a compact (n-dimensional) manifold M, with bounded distortion. Assume also that f is topologically mixing and preserves a Markov partition with finite image partition. Then f has an absolutely continuous invariant probability measure μ. Furthermore, μ is ergodic, its density is bounded and bounded away from zero and $\mu(A) = \lim_{n\longrightarrow\infty} Leb(f^{-n}(A))$ for each measurable set A.*

Hence accelerated Suzuki induction Z has a finite, invariant probability measure. It follows quite readily that the set of non-abyss cases has measure zero.

This result also follows from work by Suzuki et al. [SIA]. They show that for any (α, β) such that α and β are rationally independent, there exists subset of the $c \in [0,1]$, Γ which correspond to values of c where (α, β, c) is not an Abyss element. They then prove that Γ is a Cantor set with measure zero.

Corollary 4. *Lebesgue-a.e. point in the phase space belongs to the Abyss.*

Proof. Define the following map of the parameter space:

$$Z(\alpha,\beta,c) = \begin{cases} Z(\alpha,\beta,c) & \text{if } (\alpha,\beta,c) \in P_{(i)} \text{ for } i \in \{1,3,4,6\} \\ Z(\alpha,\beta,\{c\}_{1-\beta}) & \text{if } (\alpha,\beta,c) \in P_{(2)} \\ Z(\alpha,\beta,\{c\}_{\beta}) & \text{if } (\alpha,\beta,c) \in P_{(5)} \end{cases}$$

This map will have an ergodic, invariant measure by the Folklore theorem. Hence by ergodicity, μ-a.e. $(\alpha,\beta,c) \in P_{(2)} \cup P_{(5)}$ will visit every point in the parameter space. Thus μ-a.e. $(\alpha,\beta,c) \in P$ is an Abyss element, and as μ is absolutely continuous with respect to Lebesgue, we get the desired result. $\square$

4. Hausdorff Dimension of Non-Abyss Cases

We can consider the dimension of the set of non-Abyss points, NA. Noting that the set $P_{(1)} \cap \{(\alpha,\beta,c) \colon c = 0\}$ is invariant we can deduce that the dimension of the non-Abyss set must be at least 2. For an upper bound we compute an estimate for the box-dimension. This gives us the estimate 2.88. Thus we conclude that the Hausdorff dimension of NA satisfies $dim_H NA \in [2, 2.88]$.

Let $\epsilon = 2^{-n}$ and

$$\begin{cases} G_{p,q,r}(\epsilon) = [\epsilon p, (p+1)\epsilon) \times [\epsilon q, (q+1)\epsilon) \times [\epsilon r, (r+1)\epsilon), \\ N_{G_{p,q,r}(\epsilon)} = \sum_{p,q,r \in \{0,\dots,2^n-1\}} \Upsilon(G_{p,q,r}(\epsilon)), \end{cases}$$

where

$$\Upsilon(G_{p,q,r}(\epsilon)) = \begin{cases} 1 & \text{if } NA \cap G_{p,q,r}(\epsilon) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The approximation of the box-dimension comes from approximating

$$\lim_{\epsilon \to 0} -\frac{\log(N_{G_{p,q,r}(\epsilon)})}{\log(\epsilon)}$$

using grids, $G_{p,q,r}(2^{-n})$ for a finite range of $n \in \mathbb{N}$. Figure 3 shows a graph of $\log(N_{G_{p,q,r}(\epsilon)})$ versus $\log(\epsilon)$. Our approximation was based upon computing the values of $N_{G_{p,q,r}(\epsilon)}$ for the 420th iteration of Suzuki induction on the parameter space. With the parameter space divided up into $30^3 = 27000$, $31^3 = 29791, \dots, 300^3 = 27000000$ cubes. The line on the plot was given by the polyfit function in Matlab, which uses the least squares method. See *e.g.* Barnsley [Ba] for more details on the algorithm.

It would seem natural to use the measure of maximal dimension on NA as a reference measure to express the typicality of dynamic behaviours of $T_{(\alpha,\beta,c)}$. The distortion properties of the accelerated Suzuki induction, Z, are such that much of the standard theory of thermodynamic formalism should apply for the potential $-\log|\det(J_Z)|$. Hence, it can be expected that the pressure:

$$P(t) = \sup\left\{ h_S(\mu) t \int \log|\det(J_Z)| d\mu \colon \mu \text{ is } Z\text{-invariant and } \operatorname{supp}(\mu) = NA \right\} \tag{3}$$

is an analytic, convexly decreasing function for $t > 0$, and that for each t there is a unique equilibrium measure μ_t that maximises the right hand side in (3). However, since S is a non-conformal map, it does not follow, even for the minimal value $t = t_0$ where $P(t_0) = 0$, that μ_t is equivalent to

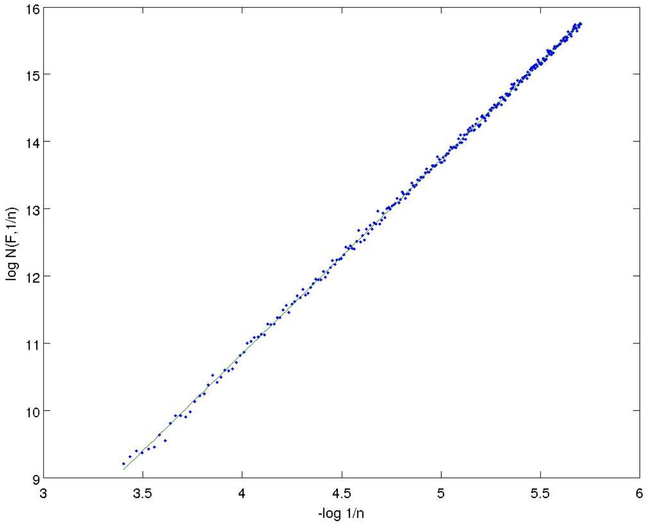

FIGURE 3. A plot of $\log(N_{G_{p,q,r}(\epsilon)})$ versus $\log(\epsilon)$.

the measure of maximal dimension. Hence there is no good motivation to single out such equilibrium measure among all other probability measures. Fortunately, in the case of unique ergodicity, there is a stronger notion of typicality that holds, and we will discuss this in the next sections.

5. Symbolic Dynamics

The usual way of applying symbolic dynamics is by assigning a label b to each continuity interval I_b of T, and then forming *itineraries* (strings of labels) for each $x \in I$ by

$$i(x) = i_0(x)i_1(x)i_2(x)\dots, \qquad \text{where } i_k(x) = b \quad \text{if} \quad T^k(x) \in I_b.$$

Renormalisation then takes the shape of a substitution on the space of itineraries, as follows. If $\hat{T} : J \to J$ is the first return map of T to a subinterval J, then $\hat{T}(x) = T^{\tau(x)}(x)$ for the first return time $\tau(x)$ of $x \in J$, and if b is the label of the continuity interval of $\hat{T}$ containing x, then we write

$$t(b) = b_0 b_1 \dots b_{\tau(x)-1} \qquad \text{where } T^i(x) \in I_{b_i} \text{ for } 0 \le i < \tau(x).$$

This substitution acts on strings of labels by concatenation:

$$t(b_0 b_1 \dots b_n) = t(b_0)t(b_1)\dots t(b_n).$$

In order to repeat this procedure for all subsequent renormalisation steps, *i.e.,* for all iterates S^k of the Suzuki induction, we require an alphabet $\mathcal{A}_k$ of labels at stage k, generating the space $\mathcal{A}_k^*$ of finite words of these labels, and substitutions

$$t_k : \mathcal{A}_k \to \mathcal{A}_{k-1}^*.$$

Let $J_k \subset [0,1)$ be the interval of the first return map which is represented by the k-th renormalisation. If $x \in \cap_k J_k$, then the itinerary of x can be found as

$$i(x) = \lim_{m\to\infty} t_1 \circ t_2 \dots t_m(b_m),$$

where b_m is the label of the continuity interval of J_m that contains x. By minimality of $T : \Omega \to \Omega$, each $y \in \Omega$ will have its itinerary in the orbit closure $\Sigma = \overline{\{\sigma^n(i(x)) : n \in \mathbb{N}\}}$, where σ denotes the left-shift.

The sequence of substitutions $(t_k)_{k\in\mathbb{N}_0}$ is determined by the addresses of $S^k(\alpha, \beta, c)$, as shown in equation (5), below. The shape of the renormalisation map (especially where the symmetries g

and h are concerned) makes it convenient to sometimes use more labels than there are intervals of continuity, but this doesn't affect the method.

It is obvious that all of the non-Abyss maps have either three or four intervals, so it might seem natural to use an alphabet $\{1,2,3\}$ (with 4 in some cases). However we need to divide the intervals into pieces such that the involutions g and h cannot split any of them into two pieces. As we will have problems for computing the substitution shift of, say P_1 or $P_{(3)}$ into $P_{(4)}$ or $P_{(6)}$. So for each map we need the alphabet $\{1,2,3,4\}$ (and in some cases 5).

Firstly we require a new partitioning of P:

$$\begin{array}{ll} Q_{(1)} = \{(\alpha,\beta,c)\colon \beta \geq \alpha \geq c\} & Q_{(4)} = \{(\alpha,\beta,c)\colon 1-\beta \geq 1-\alpha \geq c\} \\ Q_{(2)} = \{(\alpha,\beta,c)\colon \beta > c > \alpha\} & Q_{(5)} = \{(\alpha,\beta,c)\colon 1-\beta > c > 1-\alpha\} \\ Q_{(3)} = \{(\alpha,\beta,c)\colon c \geq \beta \geq \alpha\} & Q_{(6)} = \{(\alpha,\beta,c)\colon c \geq 1-\beta \geq 1-\alpha\} \end{array}$$

and from this and the P-partitioning we have the χ-partitions:

$$\chi_{(i,j)} = P_{(i)} \cap Q_{(j)}. \tag{4}$$

$$\begin{array}{ccccccc} & g & & \qquad & & h & \\ \chi_{(1,1)} & \leftrightarrow & \chi_{(6,6)} & & \chi_{(1,1)} & \leftrightarrow & \chi_{(3,3)} \\ \chi_{(1,2)} & \leftrightarrow & \chi_{(6,5)} & & \chi_{(1,2)} & \leftrightarrow & \chi_{(3,2)} \\ \chi_{(1,3)} & \leftrightarrow & \chi_{(6,4)} & & \chi_{(1,3)} & \leftrightarrow & \chi_{(3,1)} \\ \chi_{(3,1)} & \leftrightarrow & \chi_{(4,6)} & & \chi_{(4,4)} & \leftrightarrow & \chi_{(6,6)} \\ \chi_{(3,2)} & \leftrightarrow & \chi_{(4,5)} & & \chi_{(4,5)} & \leftrightarrow & \chi_{(6,5)} \\ \chi_{(3,3)} & \leftrightarrow & \chi_{(4,4)} & & \chi_{(4,6)} & \leftrightarrow & \chi_{(6,4)} \end{array}$$

We can use the above to make connections between properties of different partition elements. So now we have to consider the symbolic dynamics the Suzuki induction of each partition to every other partition. Firstly we ask the question of whether every element maps to every other element. This is not the case. For the cases $\chi_{(1,1)}$, $\chi_{(3,3)}$, $\chi_{(4,4)}$, and $\chi_{(6,6)}$ all map to all the other partitions. However, consider the partition element $\chi_{(1,2)}$ (where $c < 1-\alpha < 1-\beta$ and $\alpha < c < \beta$) in this α is less than c, hence $\alpha < 1-\beta$. From the definition of Suzuki induction we know that $\frac{\alpha}{1-\beta} < \frac{c}{1-\beta}$. Hence $\chi_{(1,2)}$ cannot map to points where $c < \alpha$ thus it cannot map to $\chi_{(1,1)}$, $\chi_{(3,1)}$, $\chi_{(4,4)}$, $\chi_{(4,5)}$, and $\chi_{(4,6)}$. In the case of the partitions that don't map to all the other partitions we have:

$$\begin{array}{lll} \chi_{(1,2)} & \mapsto & \chi_{(1,2)},\ \chi_{(1,3)},\ \chi_{(3,2)},\ \chi_{(3,3)},\ \chi_{(6,4)},\ \chi_{(6,5)},\ \text{and } \chi_{(6,6)} \\ \chi_{(3,2)} & \mapsto & \chi_{(1,1)},\ \chi_{(1,2)},\ \chi_{(3,1)},\ \chi_{(3,2)},\ \chi_{(4,4)},\ \chi_{(4,5)},\ \text{and } \chi_{(4,6)} \\ \chi_{(4,5)} & \mapsto & \chi_{(3,1)},\ \chi_{(3,2)},\ \chi_{(3,3)},\ \chi_{(4,5)},\ \chi_{(4,6)},\ \chi_{(6,5)},\ \text{and } \chi_{(6,6)} \\ \chi_{(6,5)} & \mapsto & \chi_{(1,1)},\ \chi_{(1,2)},\ \chi_{(1,3)},\ \chi_{(4,4)},\ \chi_{(4,5)},\ \chi_{(6,4)},\ \text{and } \chi_{(6,5)} \\ \chi_{(1,3)} & \mapsto & \chi_{(1,3)},\ \text{and } \chi_{(3,3)} \\ \chi_{(3,1)} & \mapsto & \chi_{(1,1)},\ \text{and } \chi_{(3,1)} \\ \chi_{(4,6)} & \mapsto & \chi_{(4,6)},\ \text{and } \chi_{(6,6)} \\ \chi_{(6,4)} & \mapsto & \chi_{(4,4)},\ \text{and } \chi_{(6,4)} \end{array}$$

For $\chi_{(1,1)}$ we have the subintervals; $[0,c)$, $[c,1-\beta)$, $[1-\beta,1-\beta+c)$, and $[1-\beta+c,1)$, labelled 1, 2, 3, and 4 respectively. In the case of $\chi_{(1,2)}$ the intervals are $[0,c-\alpha)$, $[c-\alpha,c)$, $[c,1-\beta)$, $[1-\beta,1-\beta+c)$, and $[1-\beta+c,1)$. With $\chi_{(1,3)}$ we have the intervals $[0,c-\alpha)$, $[c-\alpha,c)$, $[c,1-\beta)$, and $[1-\beta,1)$ see figure 4.

So we consider Suzuki induction on $\chi_{(1,1)}$. For $\chi_{(1,1)} \to \chi_{(1,1)}$ the first interval will be mapped to $[0,1-\beta)$ after $a-1 = \left\lceil \frac{\alpha}{1-\beta} \right\rceil - 1$ iterations in the fourth interval. The second interval splits into three, the left part maps to $[0,1-\beta)$ after $b-1 = \left\lceil \frac{\beta}{1-\beta} \right\rceil - 1$ iterations in the fourth interval. The centre part spends $b-1$ iterations in the fourth interval and then in the third interval before returning. The right part spends b iterations in the fourth interval.

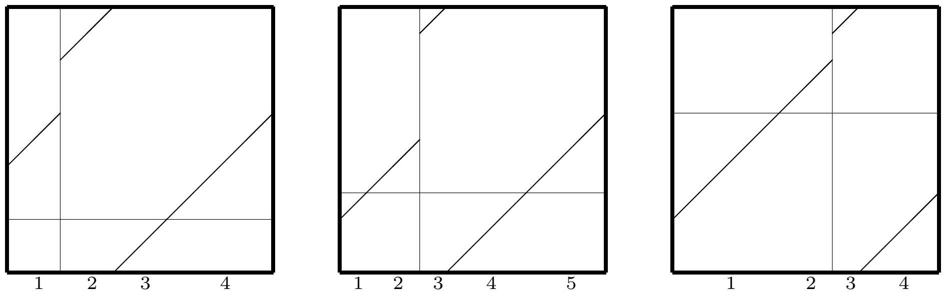

FIGURE 4. The division of $\chi_{(1,1)}$, $\chi_{(1,2)}$, and $\chi_{(1,3)}$ (from left to right).

For the case $\chi_{(1,1)} \to \chi_{(1,2)}$ the first interval in this case has a right and a left part. The left part spends $a-2$ in fourth interval before going to the third. The right part behaves the same as the first interval in the case above. These two cases correspond to the substitution shifts below:

$$\begin{array}{lll} 1 & \longrightarrow & 14^{a-1} \\ 2 & \longrightarrow & 24^{b-1} \\ 3 & \longrightarrow & 24^{b-1}3 \\ 4 & \longrightarrow & 24^{b} \end{array} \qquad \begin{array}{lll} 1 & \longrightarrow & 14^{a-2}3 \\ 2 & \longrightarrow & 14^{a-1} \\ 3 & \longrightarrow & 24^{b-1} \\ 4 & \longrightarrow & 24^{b-1}3 \\ 5 & \longrightarrow & 24^{b} \end{array}$$

In total there are 84 different cases. The other cases are listed in the appendix.

6. Unique Ergodicity

In [BK, Theorem 11] it was shown that the slice $c = 1-\alpha$ of the parameter space contains parameters (α, β) for which the ITM is of infinite type, but not uniquely ergodic. Also some weak conditions were given [BK, Theorem 12] that imply that $T_{\alpha,\beta} : \Omega \to \Omega$ preserves only one probability measure, showing that unique ergodicity is the typical situation in that parameter slice. It would be natural to express typical behaviour on the space of infinite type ITMs, in terms of S-invariant measures μ on P. One might take μ to be equivalent to measure of maximal dimension, if available, but in case of unique ergodicity, we can express its typical occurrence in a stronger way.

Theorem 5. *For any measure μ, invariant with respect to Suzuki induction S,*

$$\mu\{(\alpha, \beta, c) \in P \setminus \textit{Abyss}\colon T_{\alpha,\beta,c} \textit{ is not uniquely ergodic}\} = 0.$$

The matrices A_k associated to the substitutions t_k can be used to keep track of the number (and frequency) of symbols appearing in strings. Indeed, if $N_k(w)$ is the vector in $\mathbb{N}_0^{\#\mathcal{A}_k}$ indicating the number of appearances of the i-th symbol of $\mathcal{A}_k$ in w, then $N_{k-1}(t_k(w)) = A_{k-1}N_k(w)$ is the vector in $\mathbb{N}_0^{\#\mathcal{A}_{k-1}}$ indicating the number of appearances of the i-th symbol of $\mathcal{A}_{k-1}$ in $t_k(w)$. By normalising these vectors, we obtain the frequency of these symbols in w, etc. Alternatively, we can use the positive cone $C_k = \mathbb{R}_+^{\#\mathcal{A}_k}$ and projective (Hilbert) metric H to investigate whether the symbol frequencies in itineraries $i(x)$ are independent of the point $x \in \Omega$. This is important to determine if $T : \Omega \to \Omega$ is uniquely ergodic, and we can easily extend Lemma 17 of [BT] to the following:

Proposition 6. *Let (α, β, c) correspond to an infinite type ITM $T_{(\alpha,\beta,c}$. Then (T, Ω) is uniquely ergodic if and only if*

$$F_k = \cap_{m \geq k} A_k \cdot A_{k+1} \cdots A_m(C_{m+1}) \tag{5}$$

*is a single line (*i.e.*, single point in projective space) for all $k \in \mathbb{N}$.*

We will use *Hilbert Metric* on cones C, defined as

$$H(x, y) = \log \frac{\inf\{a \colon ax - y \in C\}}{\sup\{b \colon y - bx \in C\}}.$$

In fact, H is only a semi-metric, since $H(x, y) = 0$ if and only if x and y are on the same line through the origin, but H is a proper metric on projective space. In addition [Bi], it is a contraction under linear transformations: namely, if $A : C_2 \to C_1$ is a linear transformation between cones (equipped with Hilbert metric H_2 and H_1 respectively), then

$$H_1(Ax, Ay) \leq \tanh(\frac{D}{4}) H_2(x, y) \qquad \text{where } D = \mathrm{diam}_{H_1} A(C_2).$$

The diameter $\mathrm{diam}_{H_1}(AC_2)$ is finite if only if $A(C_2)$ is compactly contained in C_1 (*i.e.,* $\partial C_1 \cap \partial A(C_2)$ is only the origin of C_1). In our case, the C_m are cones within $\mathbb{R}_+^{\#\mathcal{A}_m}$ and the linear transformations are those represented by the matrices A_m. Only when A_m is positive (at every entry) we find $\mathrm{diam}_{H_m}(A_m \mathbb{R}_+^{\#\mathcal{A}_{m+1}}) < \infty$, and for this reason, we consider compositions $A_k \cdots A_m$ of blocks of matrices rather than single matrices. Let

$$P_m = \{(\alpha, \beta, c) \in P \setminus \text{Abyss} \colon A_1 \cdots A_m \text{ is a positive matrix}\},$$

where the matrices A_k, $k = 1, \dots, m$ are uniquely determines by the itinerary of (α, β, c) under Suzuki induction S. We have

Lemma 7. *$P \setminus Abyss = \bigcup_{m \in \mathbb{N}} P_m$.*

Proof. Since $T = T_{\alpha,\beta,c}$ is of infinite type, $T : \Omega \to \Omega$ is minimal, see [ST] and [BK, Theorem 1]. Due to compactness of Ω, there is $m \geq 1$ such that the first return map T^τ to the interval J_m that is associated to the m-th Suzuki induction has the following properties:

- The first return time τ is bounded;
- For each $x \in J_m \cap \Omega$ and each interval I_j with label $j \in \mathcal{A}_1$, there is $0 \leq k < \tau(x)$ such that $T^k(x) \in I_j$.

This means that the matrix composition $A_1 \cdots A_m$ is positive. □

Proof of Theorem 5. Let μ be any S-invariant measure such that $\mu(\text{Abyss}) = 0$; without loss of generality we can assume that μ is ergodic. By Lemma 7, there is m such that $\mu(P_m) > 0$. By taking a smaller $U \subset P_m$ if necessary, but still with $\mu(U) > 0$, we can assume that the sub-labels (a, b) in the addresses of cells $\chi_{i,j}$ used for the substitutions t_k (and hence matrices A_k) are all bounded. Thus, for each $(\alpha, \beta, c) \in U$, a the corresponding m matrices are chosen out of a finite collection, and hence $A_1 \cdots A_m$ is not only positive, but also bounded. Therefore

$$D = \sup\{\mathrm{diam}_{H_1}(A_1 \cdots A_m \mathbb{R}^{\#\mathcal{A}_{m+1}}) \colon (\alpha, \beta, c) \in U\} < \infty$$

uniformly on U. By the Ergodic Theorem, μ-a.e. $(\alpha, \beta, c) \in P$ has a sequence $(n_j)_{j \geq 1}$ with $n_{j+1} \geq n_j + m$, such that $S^{n_j}(\alpha, \beta, c) \in U$ for all $j \in \mathbb{N}$. At each visit to U, Hilbert distance will decrease by a factor $\lambda := \tanh(D/4)$ within the next m iterates, so we find

$$\mathrm{diam}_{H_1}(A_1 \cdots A_{n_j} \mathbb{R}^{\#\mathcal{A}_{n_j+1}}) \leq \lambda^{j-1} D.$$

It follows that $F_1 := \bigcap_j A_1 \cdots A_{n_j} \mathbb{R}^{\#\mathcal{A}_{n_j+1}}$ is a single line, and the same argument gives that also $F_k := \bigcap_j A_k \cdots A_{n_j} \mathbb{R}^{\#\mathcal{A}_{m+1}}$ is a single line for each $k \in \mathbb{N}$. Therefore Proposition 6 implies that $T_{\alpha,\beta,c}$ is uniquely ergodic. □

Appendix

The table in this appendix lists the possible transitions between regions in parameter space P that are allowed by the Suzuki induction S. In each of the boxes (there are in total 84 cases), we give the symbolic substitution that Suzuki induction represents in these regions, so the whole table indicates which substitution can be followed by which other substitution. Note that the exponents a and b stand for the ceiling function of $S\alpha$ and $S\beta$ respectively.

	$\chi_{(1,1)}$	$\chi_{(3,3)}$	$\chi_{(4,4)}$	$\chi_{(6,6)}$	$\chi_{(1,2)}$	$\chi_{(3,2)}$
$\chi_{(1,1)}$	$1 \to 14^{a-1}$ $2 \to 24^{b-1}$ $3 \to 24^{b-1}3$ $4 \to 24^{b}$	$1 \to 31^{b-1}2$ $2 \to 41^{a}2$ $3 \to 41^{a}$ $4 \to 41^{a-2}2$	$1 \to 12^{a-1}$ $2 \to 42^{b}$ $3 \to 42^{b-1}3$ $4 \to 42^{b-1}$	$1 \to 13^{b-1}2$ $2 \to 43^{a-2}2$ $3 \to 43^{a-1}$ $4 \to 43^{a-1}2$		$1 \to 31^{b-1}2$ $2 \to 42$ $3 \to 4$ $4 \to 5$
$\chi_{(1,2)}$	$1 \to 14^{a-2}3$ $2 \to 14^{a-1}$ $3 \to 24^{b-1}$ $4 \to 24^{b-1}3$ $5 \to 24^{b}$	$1 \to 31^{b}$ $2 \to 31^{b-1}2$ $3 \to 41^{a-1}2$ $4 \to 41^{a-1}$ $5 \to 41^{a-1}2$	$1 \to 12^{a-1}3$ $2 \to 12^{a-1}$ $3 \to 42^{b}$ $4 \to 42^{b-1}3$ $5 \to 42^{b-1}$	$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 43^{a-2}2$ $4 \to 43^{a-1}$ $5 \to 43^{a-1}2$	$1 \to 1$ $2 \to 2$ $3 \to 35^{b-1}$ $4 \to 25^{b-1}4$ $5 \to 25^{b}$	$1 \to 31^{b}$ $2 \to 31^{b-1}2$ $3 \to 42$ $4 \to 4$ $5 \to 5$
$\chi_{(1,3)}$	$1 \to 14^{a-2}3$ $2 \to 14^{a-1}$ $3 \to 24^{b-1}$ $4 \to 24^{b-1}3$	$1 \to 31^{b-1}$ $2 \to 31^{b-1}2$ $3 \to 41^{a-1}2$ $4 \to 41^{a-1}$	$1 \to 12^{a-1}3$ $2 \to 12^{a-1}$ $3 \to 42^{b}$ $4 \to 42^{b-1}3$	$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 43^{a-2}2$ $4 \to 43^{a-1}$	$1 \to 1$ $2 \to 2$ $3 \to 35^{b-1}$ $4 \to 35^{b-1}4$	
$\chi_{(3,1)}$	$1 \to 14^{a-1}$ $2 \to 14^{a-1}3$ $3 \to 24^{b-1}3$ $4 \to 24^{b-1}$	$1 \to 31^{b-1}2$ $2 \to 31^{b-1}$ $3 \to 41^{a-1}$ $4 \to 41^{a-2}2$	$1 \to 12^{a-1}$ $2 \to 12^{a-2}3$ $3 \to 42^{b-1}3$ $4 \to 42^{b-1}$	$1 \to 13^{b-1}2$ $2 \to 13^{b}$ $3 \to 43^{a-1}$ $4 \to 43^{a-1}2$		$1 \to 31^{b-1}2$ $2 \to 31^{b-1}$ $3 \to 4$ $4 \to 5$
$\chi_{(3,2)}$	$1 \to 14^{a-2}3$ $2 \to 14^{a-1}$ $3 \to 14^{a-1}3$ $4 \to 24^{b-1}3$ $5 \to 24^{b}$	$1 \to 31^{b}$ $2 \to 31^{b-1}2$ $3 \to 31^{b-1}$ $4 \to 41^{a-1}$ $5 \to 41^{a-2}2$	$1 \to 12^{a-1}3$ $2 \to 12^{a-1}$ $3 \to 12^{a-2}3$ $4 \to 42^{b-1}3$ $5 \to 42^{b-1}$	$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 13^{b}$ $4 \to 43^{a-1}$ $5 \to 43^{a-1}2$	$1 \to 1$ $2 \to 2$ $3 \to 24$ $4 \to 35^{b-1}4$ $5 \to 35^{b}$	$1 \to 41^{b}$ $2 \to 41^{b-1}2$ $3 \to 31^{b-1}$ $4 \to 4$ $5 \to 5$
$\chi_{(3,3)}$	$1 \to 14^{a-2}3$ $2 \to 14^{a-1}$ $3 \to 14^{a-1}3$ $4 \to 24^{b-1}3$	$1 \to 31^{b}$ $2 \to 31^{b-1}2$ $3 \to 31^{b-1}$ $4 \to 41^{a-1}$	$1 \to 12^{a-1}3$ $2 \to 12^{a-1}$ $3 \to 12^{a-2}3$ $4 \to 42^{b-1}3$	$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 13^{b}$ $4 \to 43^{a-1}$	$1 \to 1$ $2 \to 2$ $3 \to 24$ $4 \to 35^{b-1}4$	
$\chi_{(4,4)}$	$1 \to 14^{a-1}$ $2 \to 24^{b-1}$ $3 \to 24^{b-1}3$ $4 \to 24^{b}$	$1 \to 31^{b-1}2$ $2 \to 41^{a-1}2$ $3 \to 41^{a-1}$ $4 \to 41^{a-2}2$	$1 \to 12^{a-1}$ $2 \to 42^{b}$ $3 \to 42^{b-1}3$ $4 \to 42^{b-1}$	$1 \to 13^{b-1}2$ $2 \to 43^{a-2}2$ $3 \to 43^{a-1}$ $4 \to 43^{a-1}2$		$1 \to 31^{b-1}2$ $2 \to 42$ $3 \to 4$ $4 \to 5$
$\chi_{(4,5)}$	$1 \to 14^{a-1}$ $2 \to 14^{a-1}3$ $3 \to 24^{b-1}$ $4 \to 24^{b-1}3$ $5 \to 24^{b}$	$1 \to 31^{b-1}2$ $2 \to 31^{b-1}$ $3 \to 41^{a-1}2$ $4 \to 41^{a-1}$ $5 \to 41^{a-2}2$	$1 \to 12^{a-1}$ $2 \to 12^{a-1}3$ $3 \to 42^{b}$ $4 \to 42^{b-1}3$ $5 \to 42^{b-1}$	$1 \to 13^{b-1}2$ $2 \to 13^{b}$ $3 \to 43^{a-2}2$ $4 \to 43^{a-1}$ $5 \to 43^{a-1}2$		$1 \to 31^{b-1}2$ $2 \to 31^{b-1}$ $3 \to 42$ $4 \to 4$ $5 \to 5$
$\chi_{(4,6)}$	$1 \to 14^{a-1}$ $2 \to 14^{a-1}3$ $3 \to 24^{b-1}3$ $4 \to 24^{b}$	$1 \to 31^{b-1}2$ $2 \to 31^{b-1}$ $3 \to 41^{a-1}$ $4 \to 41^{a-2}2$	$1 \to 12^{a-1}$ $2 \to 12^{a-2}3$ $3 \to 42^{b-1}3$ $4 \to 42^{b-1}$	$1 \to 13^{b-1}2$ $2 \to 13^{b-1}$ $3 \to 43^{a-1}$ $4 \to 43^{a-1}2$		$1 \to 31^{b-1}2$ $2 \to 31^{b-1}$ $3 \to 4$ $4 \to 5$
$\chi_{(6,4)}$	$1 \to 14^{a-2}3$ $2 \to 14^{a-1}$ $3 \to 24^{b-1}$ $4 \to 24^{b-1}3$	$1 \to 31^{b}$ $2 \to 31^{b-1}2$ $3 \to 41^{a-1}2$ $4 \to 41^{a-1}$	$1 \to 12^{a-1}3$ $2 \to 12^{a-1}$ $3 \to 42^{b-1}$ $4 \to 42^{b-1}3$	$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 43^{a-2}2$ $4 \to 43^{a-1}$	$1 \to 1$ $2 \to 2$ $3 \to 35^{b-1}$ $4 \to 35^{b-1}4$	
$\chi_{(6,5)}$	$1 \to 14^{a-2}3$ $2 \to 14^{a-1}$ $3 \to 14^{a-1}3$ $4 \to 24^{b-1}$ $5 \to 24^{b-1}3$	$1 \to 31^{b}$ $2 \to 31^{b-1}2$ $3 \to 31^{b-1}$ $4 \to 41^{a-1}2$ $5 \to 41^{a-1}$	$1 \to 12^{a-1}3$ $2 \to 12^{a-1}$ $3 \to 12^{a-2}3$ $4 \to 42^{b}$ $5 \to 42^{b-1}3$	$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 13^{b}$ $4 \to 43^{a-2}2$ $5 \to 43^{a-1}$	$1 \to 1$ $2 \to 2$ $3 \to 24$ $4 \to 35^{b-1}$ $5 \to 35^{b-1}4$	
$\chi_{(6,6)}$	$1 \to 14^{a-2}3$ $2 \to 14^{a-1}$ $3 \to 14^{a-1}3$ $4 \to 24^{b-1}3$	$1 \to 31^{b}$ $2 \to 31^{b-1}2$ $3 \to 31^{b-1}$ $4 \to 41^{a-1}$	$1 \to 12^{a-1}3$ $2 \to 12^{a-1}$ $3 \to 12^{a-2}3$ $4 \to 42^{b-1}3$	$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 13^{b}$ $4 \to 43^{a-1}$	$1 \to 1$ $2 \to 2$ $3 \to 24$ $4 \to 35^{b-1}4$	

	$\chi_{(4,5)}$	$\chi_{(6,5)}$	$\chi_{(1,3)}$	$\chi_{(3,1)}$	$\chi_{(4,6)}$	$\chi_{(6,4)}$
$\chi_{(1,1)}$		$1 \to 13^{b-1}2$ $2 \to 4$ $3 \to 5$ $4 \to 52$		$1 \to 21$ $2 \to 31$ $3 \to 3$ $4 \to 4$		
$\chi_{(1,2)}$		$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 4$ $4 \to 5$ $5 \to 52$				
$\chi_{(1,3)}$		$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 4$ $4 \to 5$	$1 \to 1$ $2 \to 2$ $3 \to 3$ $4 \to 34$			
$\chi_{(3,1)}$	$1 \to 1$ $2 \to 2$ $3 \to 53^{b-1}4$ $4 \to 53^{b-1}$			$1 \to 21$ $2 \to 2$ $3 \to 3$ $4 \to 4$		
$\chi_{(3,2)}$	$1 \to 14$ $2 \to 1$ $3 \to 2$ $4 \to 53^{b-1}4$ $5 \to 53^{b-1}$					
$\chi_{(3,3)}$	$1 \to 14$ $2 \to 1$ $3 \to 2$ $4 \to 53^{b-1}4$		$1 \to 1$ $2 \to 2$ $3 \to 24$ $4 \to 34$			
$\chi_{(4,4)}$		$1 \to 13^{b-1}2$ $2 \to 4$ $3 \to 5$ $4 \to 52$				$1 \to 12$ $2 \to 3$ $3 \to 4$ $4 \to 42$
$\chi_{(4,5)}$	$1 \to 1$ $2 \to 2$ $3 \to 13^{b}$ $4 \to 13^{b-1}4$ $5 \to 53^{b-1}$	$1 \to 13^{b-1}2$ $2 \to 13^{b}$ $3 \to 4$ $4 \to 5$ $5 \to 52$				
$\chi_{(4,6)}$	$1 \to 1$ $2 \to 2$ $3 \to 53^{b-1}4$ $4 \to 53^{b-1}$				$1 \to 1$ $2 \to 2$ $3 \to 43$ $4 \to 4$	
$\chi_{(6,4)}$		$1 \to 13^{b-1}$ $2 \to 13^{b-1}2$ $3 \to 4$ $4 \to 5$				$1 \to 1$ $2 \to 12$ $3 \to 3$ $4 \to 4$
$\chi_{(6,5)}$	$1 \to 14$ $2 \to 1$ $3 \to 2$ $4 \to 53^{b}$ $5 \to 53^{b-1}4$	$1 \to 13^{b-1}$ $2 \to 53^{b-1}2$ $3 \to 53^{b}$ $4 \to 4$ $5 \to 5$				
$\chi_{(6,6)}$	$1 \to 14$ $2 \to 1$ $3 \to 2$ $4 \to 13^{b-1}4$				$1 \to 13$ $2 \to 1$ $3 \to 2$ $4 \to 43$	

References

[Ba] (MR0977274) M. Barnsley, "Fractals Everywhere", Academic Press Inc. (1988).

[Bi] G. Birkhoff, *Extensions of Jentzsch's theorem,* Trans. Amer. Math. Soc. **85** (1957) 219–227.
[BK] (MR1356616) M. Boshernitzan, I. Kornfeld, *Interval translation mappings,* Ergod. Th. Dyn. Sys. **15** (1995) 821–831.
[BT] (MR2013352) H. Bruin, S. Troubetzkoy, *The Gauss Map on a class of Interval Translation Mappings,* Israel J. Math. **137** (2003) 125-148.
[BH] (MR2054049) J. Buzzi, P. Hubert, *Piecewise monotone maps without periodic points: Rigidity, measures and complexity,* Ergodic Theory Dynam. Systems **24** (2004) 383–405.
[K] (MR0435353) M. Keane, *Non-ergodic interval exchange transformations,* Israel J. Math. **26** (1977) 188–196.
[KN] (MR0409766) H. B. Keynes, D. Newton, *A "minimal", non-uniquely ergodic interval exchange transformation,* Math. Z. **148** (1976), 101–105.
[Mañ] (MR0889254) R. Mañé "Ergodic Theory and Differentiable Dynamics", Springer-Verlag, 1987.
[M] (MR0644018) H. Masur, *Interval exchange transformations and measured foliations,* Ann. of Math. **115** (1982) 169–200.
[MS] (MR1239171) W. de Melo, S. van Strien, "One-Dimensional Dynamics", Springer-Verlag, 1996.
[SIA] (MR2152403) H. Suzuki, S. Ito, K. Aihara, *Double rotations,* Discrete Contin. Dyn. Sys. **13** (2005) 515–532.
[ST] (MR1796167) J. Schmeling, S. Troubetzkoy, *Interval translation mappings,* in "Dynamical systems from crystals to chaos" J.-M. Gambaudo et al. eds. World Scientific, Singapore (2000) 291–302.
[V] (MR0633764) W. Veech, *Gauss measures for transformations on the space of interval exchange maps,* Ann. of Math. **115** (1982) 201–242.

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