Verifying abstract hypotheses used in obtaining mixing rates for some non-uniformly hyperbolic semiflows

Henk Bruin Dalia Terhesiu

February 3, 2015

Abstract

We provide an example of a non-uniformly hyperbolic semiflow for which we obtain sharp decay rates of the correlation function for observables supported on a flow-box of unbounded length. We do so by verifying the hypotheses of our recent work which provides such results in an abstract set-up.

1 Introduction

In the work [1], we develop an abstract framework for obtaining sharp mixing rates for finite and infinite measure preserving suspension semiflows over non-uniformly hyperbolic maps. This framework is based on a renewal scheme that closely resembles the renewal scheme for the discrete time scenario, see [8, 3, 6]. A previous framework for Gibbs Markov semiflows has been developed in [7].

In our treatment of the semiflow f_t, we induce to a map Φ defined on a flow-box $\tilde{Y}$ of length $\tilde{h}$, that is: $\tilde{Y} = \bigcup_{y \in Y} \{y\} \times [0, \tilde{h}(y))$ where Y is a Poincaré section of the semiflow. One novelty of [1] is that we can treat unbounded length flow-boxes, *i.e.,* $\tilde{h}$ can be unbounded.

The first return map to the Poincaré section Y is denoted $F : Y \to Y$, so $F = f_{\varphi_0}$, where $\varphi_0 : Y \to \mathbb{R}^+$ is the first return time to Y, and F is assumed to be uniformly expanding and preserve a measure μ.

The map $\Phi = f_\varphi$ where the flow-time $\varphi : \tilde{Y} \to \mathbb{R}^+$ is constructed such that Φ becomes uniformly expanding also in the flow direction. For this purpose, we need to remetrize $\tilde{Y}$ (as explained in [1, Section 2.2]) so that the flow within $\tilde{Y}$ is no longer of unit speed. More precisely,

---

Mathematics Subject Classification (2010): Primary 37A25; Secondary 37A40, 37A50, 37D25
keywords: semiflow, rates of mixing

Faculty of Mathematics, University of Vienna, Oskar Morgensternplatz 1, 1090 Vienna, Austria; e-mail: henk.bruin@univie.ac.at

Faculty of Mathematics, University of Vienna, Oskar Morgensternplatz 1, 1090 Vienna, Austria; e-mail: daliaterhesiu@gmail.com

after the change of coordinates, we have $\tilde{Y} = Y \times [0, 1)$ with Φ and f_t acting on $\tilde{Y}$ as

$$\Phi(y, u) = f_{\varphi(y,u)}(y, u) = (Fy, 2u \bmod 1), \tag{1.1}$$

where

$$\varphi(y, u) = \varphi_0(y) + \begin{cases} (2\tilde{h}(Fy) - \tilde{h}(y))u & \text{if } 0 \le u < \frac{1}{2}, \\ (2\tilde{h}(Fy) - \tilde{h}(y))u - \tilde{h}(Fy) & \text{if } \frac{1}{2} \le u < 1, \end{cases} \tag{1.2}$$

and

$$f_t(y, u) = (y, u + t/\tilde{h}(y)) \qquad \text{for } 0 \le u + t/\tilde{h}(y) < 1.$$

The reason for this particular choice of Φ is that the transfer operator R associated with Φ will have good spectral properties (ensured by F being uniformly expanding). We notice that twisted transfer operators can be related to proper Laplace transforms of non delta functions. More precisely, the twisted version $R(e^{-s\varphi}v)$ of the transfer operator associated with Φ, can be related to $\int_0^\infty R_t v e^{-st}\, dt$, where $R_t v = R(1_{\{t<\varphi<t+1\}}v)$. For details we refer to [1, Section 3]. This makes it possible to show that many techniques/calculations from the discrete time scenario [8, 3, 6] carry over to the continuous case. Below we recall the abstract hypotheses and the main results of [1].

The aim of this work is to give an example to which the sharp rates of mixing obtained in [1] apply for both the finite and infinite measure preserving setting. It is the first such example with unbounded length flow-box. For an example with bounded length flow-box, see [1, Section 9]. In Proposition 2.2 (referring back to Propositions 1.1 and 1.2) we give the precise statement. While rather restrictive, this example addresses the difficulties posed by the required tail estimates (in hypotheses (H4) and (H5) below) of the norm $\|R_t\|$, which don't hold for standard norms. The norm we construct here is a combination of the Hölder norm and the L^1-norm where the integration is over "multivalued" curves that transversally intersects the sets $S_t = \{(y, u) \in \tilde{Y} : t < \varphi < t + 1\}$ appearing in the definition of R_t. The problem is not so much the discontinuity of the indicator function $1_{\{t<\varphi<t+1\}}$; this can in fact be dealt with in different ways. It is rather that $\|R_t\|$ should be roughly *proportional* with the measure of the sets S_t, which for unbounded $\tilde{h}$ is the real obstacle. Nonetheless, since our Banach spaces of observables are required to be embedded in L^∞ (which L^1 is not), we have to put serious restrictions of analyticity on our Banach space in Section 2. It would therefore be interesting to see if the generalized BV norms of Keller [4] and Saussol [9] can be adjusted to fulfill these requirements.

1.1 Recalling the abstract set-up of [1]

In this section we list the hypotheses in the abstract set-up of [1].

(H0) **i)** Finite case: $\mu_\Phi((y, u) \in \tilde{Y} : \varphi(y, u) > t) = O(t^{-\beta})$, $\beta > 1$.

ii) Infinite case: $\mu_\Phi((y, u) \in \tilde{Y} : \varphi(y, u) > t) = \ell(t)t^{-\beta}$ where ℓ is slowly varying and $\beta \in (1/2, 1)$.

We require that the height $\tilde{h}$ of the flow-box satisfies

(H1) $\inf_{y\in Y}\tilde{h}(y)\geq 1$ and that $\tilde{h}=\varphi_0^{\gamma}$, where

- **i)** Finite case. Under (H0) i), we assume that $\gamma\in(0,1)$.
- **ii)** Infinite case. Under (H0) ii), we assume that $\gamma\in(0,\min\{\frac{2\beta-1}{1-\beta},\frac{1-\beta}{2\beta-1},\beta\})$.

We require that Φ satisfies the functional analytic assumptions listed below. We assume that there exists a Banach space $\mathcal{B}$, with norm $\|.\|_{\mathcal{B}}$ such that

(H2)
- **i)** The space $\mathcal{B}$ contains constant functions and $\mathcal{B}\subset L^\infty(\mu_\Phi)$.
- **ii)** 1 is a simple eigenvalue for R, isolated in the spectrum of R.

Define the twisted transfer operator $\hat{R}(s)v=R(e^{-s\varphi}v)$ associated with the map Φ. By (H2) ii), 1 is an isolated eigenvalue in the spectrum of $\hat{R}(0)$. In addition to (H2) ii), we require

(H3) The spectral radius of $\hat{R}(s)$ is strictly less than 1 for $s\in\overline{\mathbb{H}}-\{0\}$ and is equal to 1 for $s=0$.

Set $R_{t,a}v=R(1_{\{t<\varphi<t+a\}}v)$ and define $\hat{L}_a(s)=\int_0^\infty R_{t,a}e^{-st}dt$. Given $a>0$ such that $e^{sa}\neq 1$, we make certain assumptions on $\|R_{t,a}\|$, which in the sequel will be used to obtain appropriate continuity properties for $\hat{R}$.

(H4) *Finite case*. Under (H0) i), we require that for any $\tau<\beta$, the following upper bound holds uniformly in $a\in[1,2]$:
$$\int_0^\infty \sigma^\tau\|R_{\sigma,a}\|_{\mathcal{B}}\,d\sigma<\infty,$$

(H5) *Infinite case*. Under (H0) ii), we require that there exists a Banach space $\mathcal{B}_0$ such that $\mathcal{B}\subset\mathcal{B}_0\subset L^\infty(\mu_\Phi)$ such that

- **i)** There exists constants $C_1>0$, $C_2<1$ and some $\theta\in(0,1)$ such that
$$\|\hat{R}^n(s)v\|_{\mathcal{B}}\leq C_1\theta^n\|v\|_{\mathcal{B}}+C_2\|v\|_{\mathcal{B}_0},\qquad \|\hat{R}(s)v\|_{\mathcal{B}_0}\leq\|v\|_{\mathcal{B}_0}.$$
- **ii)** The following upper bound holds uniformly in $a\in[1,2]$,
$$\int_0^\infty \sigma^\tau\|R_{\sigma,a}\|_{\mathcal{B}\to\mathcal{B}_0}\,d\sigma<\infty,$$
for $\max\{1-\beta,2\beta-1\}<\tau<\frac{\beta}{1+\gamma}$.

Hypothesis (H4) gives a good control of $(I-\hat{R}(a+ib))^{-1}$ for $a\geq 0$ and $|b|<1$. To be able to estimate the inverse Laplace transform $\rho_t(v,w)$ of $\hat{\rho}(s)$, we need a good understanding of the asymptotics of $(I-\hat{R}(a+ib))^{-1}$, for $a\geq 0$ and large values of b. For this purpose we assume

(H6) *Dolgopyat type inequality*. There exist $C>0$ and $\alpha>0$ such that for all $|b|\geq 1$
$$\|(I-\hat{R}(ib))^{-1}\|_{\mathcal{B}}\leq C|b|^\alpha.$$

1.2 Recalling the main results in [1]

In contrast to the discrete time operator renewal theory which is concerned with estimating the operators T_t in the norm of some appropriate Banach space, here we follow the strategy in [7]. Namely, we adapt renewal theory techniques to estimate the correlation function

$$\rho_t(v,w) = \int_{\tilde{Y}} vw \circ f_t \, d\hat{\mu},$$

where $d\hat{\mu} = \frac{\tilde{h}}{\bar{\varphi}_0} d\mu_\Phi$ for $\bar{\varphi}_0 = \int_Y \varphi_0 \, d\mu_\Phi$ in the finite case (under (H0) i)) and $d\hat{\mu} = \tilde{h} d\mu_\Phi$ in the infinite case (under (H0) ii)).

For the statement of the main results, define the class of observables

$$C^m(\tilde{Y}) = \{w : \tilde{Y} \to C, \, w = w^*/\tilde{h} \text{ with } w^* \in C^m(\tilde{Y}, \mu_\Phi)\}. \tag{1.3}$$

Recall that $\mathcal{B}$ is the Banach space defined by (H2) and (H3) and that the corresponding norm is denoted by $\|.\|_{\mathcal{B}}$.

Under (H0) i), we let $\epsilon > 0$ and define

$$\eta(t) = \frac{1}{\bar{\varphi}_0} \int_t^\infty \mu_\Phi(\varphi > \tau) \, d\tau, \quad \xi_{\beta,\epsilon}(t) = \begin{cases} t^{-(\beta-\epsilon)}, & \beta \geq 2, \\ t^{-(2\beta-2)}, & 1 < \beta < 2. \end{cases} \tag{1.4}$$

With these specified we recall:

Proposition 1.1 (Theorem 5.1. of [1]: Finite measure)**.** *Assume (H0) i), (H1) i), (H2), (H3), (H4) and (H6). Set α such that (H6) holds. Let $v = v^*/\tilde{h}$ with $v^* \in \mathcal{B}$, and $w \in C^m(\tilde{Y})$. The following hold for all $m \in \mathbb{N}$ such that $m \geq 3 + \alpha(\beta+1)$ and for any $\epsilon > 0$.*

(a) Let η and $\xi_{\beta-\epsilon}$ be as defined in (1.4). Then,

$$\rho_t(v,w) - \int_{\tilde{Y}} v \, d\hat{\mu} \int_{\tilde{Y}} w \, d\hat{\mu} = \eta(t) \int_{\tilde{Y}} v \, d\hat{\mu} \int_{\tilde{Y}} w \, d\hat{\mu} + O(\|v^*\|_{\mathcal{B}} \|w\|_{C^m(\tilde{Y})} \, \xi_{\beta,\epsilon}(t)).$$

(b) Suppose further that $\int v d\hat{\mu} = 0$. Then,

$$\rho_t(v,w) = O\big(\|v^*\|_{\mathcal{B}} \|w\|_{C^m(\tilde{Y})} t^{-(\beta-\epsilon)}\big).$$

Proposition 1.2 (Theorem 5.2. of [1]: Infinite measure)**.** *Assume (H0) ii), (H1) ii), (H2), (H3), (H5) and (H6). Set α such that (H6) holds. The following hold for all $m \in \mathbb{N}$ such that $m \geq 2(\alpha+1)$. Let $v = v^*/\tilde{h}$, with $v^* \in \mathcal{B}$, and $w \in C^m(\tilde{Y})$. Then*

$$\ell(t) t^{1-\beta} \rho_t(v,w) \to \frac{1}{\pi} \sin \pi\beta \int_{\tilde{Y}} v \, d\hat{\mu} \int_{\tilde{Y}} w \, d\hat{\mu}.$$

2 Semiflows over analytic maps: unbounded $\tilde{h}$

Markov maps represent a class of examples where the conditions of the abstract setting are likely to hold, except that condition (H4) is problematic for standard norms. The difficulty in checking condition (H4) for unbounded $\tilde{h}$ is that on the one hand $\|R_{t,a}\|$ needs to be proportional to $\mu_\Phi(S_{t,a})$ for

$$S_{t,a} = \{(y,u) \in \tilde{Y} : t < \varphi(y,u) < t+a\}. \tag{2.1}$$

We address this by letting $\| \ \|_{\mathcal{B}}$ involve integrals over diagonal multivalued curves. On the other hand $\mathcal{B}$ needs to be embedded in $L^\infty(\mu_\Phi)$ (and hence $\|v\|_\infty \ll \|v\|_{\mathcal{B}}$). Therefore we resort to piecewise analytic Markov maps with a class of observables v that are complex analytic in both directions, *i.e.,* there are $\rho > 0$ and complex ρ-neighborhoods Y_ρ in $\mathbb{C}$ of the real interval Y, and $[0,1)_\rho$ in $\mathbb{C}$ of $[0,1)$, such that for each $u \in [0,1)$, $v(\cdot, u)$ is complex analytic on Y_ρ and for each $y \in [0,1)$, $v(y,\cdot)$ is complex analytic on $[0,1)_\rho$. We call this class $\mathcal{B} = \mathcal{B}(\tilde{Y}_\rho)$, defining its norm in Section 2.2 below.

2.1 The set-up

Let $\mathcal{P}$ be the partition of Y into domains of continuity of F, and for $n \geq 1$, let $\mathcal{P}_n = \mathcal{P} \vee F^{-1}\mathcal{P} \vee \cdots \vee F^{-(n-1)}\mathcal{P}$ be the n-th joint of this partition. On $\tilde{Y}$, in the vertical direction, let $\mathcal{Q}$ be defined as the partition of $\tilde{Y}$ into the complementary domains of the line $\{(y,1/2) : y \in Y\}$, and the n-th joint $\mathcal{Q}_n$ as the partition of $\tilde{Y}$ into the complementary domains of the lines $\{(y, j2^{-n}) : y \in Y\}$ for the integers $0 < j < 2^n$. Then Φ is continuous on each element of the product partition $\tilde{\mathcal{P}}_n := \mathcal{P}_n \times \mathcal{Q}_n$. For $y_1, y_2 \in Y$, define the *separation time* $s(y_1,y_2)$ as the smallest integer $n \geq 0$ such that $F^n y_1$ and $F^n y_2$ lie in different elements of $\mathcal{P}$. Similarly for $\tilde{y}_1, \tilde{y}_2 \in \tilde{Y}$, let $\tilde{s}(\tilde{y}_1, \tilde{y}_2)$ be the smallest integer $n \geq 0$ such that $\Phi^n \tilde{y}_1$ and $\Phi^n \tilde{y}_2$ lie in different elements of $\tilde{\mathcal{P}}$.

For given $\theta \in (0,1)$, let $\mathcal{B}(\tilde{Y})$ be the Banach space of function v supported on $\tilde{Y}$, with norm $\|v\|_\theta = |v|_\theta + \|v\|_\infty$, where $\|v\|_\infty = \|v\|_{L^\infty(\mu_\Phi)}$ and the seminorm $|v|_\theta$ is defined as

$$|v|_\theta = \sup_{\tilde{y}_1 \neq \tilde{y}_2 \in \tilde{Y}} \theta^{-\tilde{s}(\tilde{y}_1,\tilde{y}_2)} |v(\tilde{y}_1) - v(\tilde{y}_2)|.$$

Let $f : X \to X$ be a non-uniformly expanding map with a single indifferent fixed point, say at $0 \in X$. Consider a suspension flow over f with continuous roof function h and assume that h is bounded and bounded away from zero. Assume that $F : Y \to Y$ is an induced map over f, with the following properties:

(1) F is full-branched, *i.e.,* $F(Z) = Y$ for every Z in the Markov partition $\mathcal{P}$, and the induced time $\tau_F : Y \to \mathbb{N}$ such that $F = f^{\tau_F}$ is constant on each $Z \in \mathcal{P}$.

(2) F is expanding and there is a distortion constant C_{dis} such that

$$\frac{|DF^k(y_1)|}{|DF^k(y_2)|} \leq C_{dis}, \tag{2.2}$$

for all $k \geq 0$, $Z \in \mathcal{P}_k$ and $y_1, y_2 \in Z$. This condition implies that F preserves a measure μ, absolutely continuous w.r.t. Lebesgue, such that $\frac{1}{C_\mu} \leq \frac{d\mu}{dx} \leq C_\mu$ for some $C_\mu > 0$.

Define the potential $p : Y \to \mathbb{R}$, $p = \log \frac{d\mu}{d\mu\circ F}$ and $p_n = \sum_{j=0}^{n-1} p \circ F^j$; we assume that there is a constant C_p such that

$$e^{p_n(y)} \leq C_p \mu(Z) \text{ for every } y \in Z, Z \in \mathcal{P}. \tag{2.3}$$

(3) The roof function of the induced system $F : Y \to Y$ is $\varphi_0 = \sum_{i=0}^{\tau_F - 1} h \circ f^i \leq \tau_F \sup h$. We assume that there exists $C_{\varphi_0} > 2$ such that

$$|\varphi_0(y_1) - \varphi_0(y_2)| \leq C_{\varphi_0} \theta^{s(y_1, y_2)}, \tag{2.4}$$

for all $y_1, y_2 \in Z$, $Z \in \mathcal{P}$.

If there is only one $Z \in \mathcal{P}$ with $\tau_F(Z) = n$, the following is immediate: There is $h_n = h(0)n + o(n)$ such that

$$|\varphi_0(y) - h_n| \leq C_{\varphi_0} \tag{2.5}$$

for all $y \in Z, Z \in \mathcal{P}$ with $\tau_F(Z) = n$. Let us write $\eta : \mathbb{R} \to \mathbb{N}$ for the asymptotic inverse of h_n in the sense that $\eta(t)$ is minimal such that $h_{\eta(t)} \geq t$. For example, for the case $\beta \in (0,1)$, if the roof function h is differentiable near 0, and the branch of f with the indifferent fixed point is $x \mapsto x + x^{1+1/\beta}$, then $h_n = h(0)n + \frac{h'(0)}{1-\beta} n^{1-\beta} + o(n^{1-\beta})$, and $\eta(t) = t/h(0) + O(t^{1-\beta})$.

(4) τ_F satisfies the tail condition

$$\mu(y \in Y : \tau_F(y) = n) = \begin{cases} O(n^{-(\beta+1)}), & \text{if } 1 < \beta, \qquad \text{(finite case)} \\ \ell(n) n^{-(\beta+1)}, & \text{if } 0 < \beta \leq 1, \qquad \text{(infinite case)} \end{cases} \tag{2.6}$$

for some slowly varying function ℓ. By the argument of [7, Proposition 2.6], the same tail condition holds for $\mu(y \in Y : \varphi_0(y) \in [h_n, h_n + 1])$.

(5) To make $\mathcal{B}(\tilde{Y}_\rho)$ invariant under the transfer operator associated with Φ, the inverse branches of $F : Y \to Y$ are assumed to be complex analytic as well. That is, for each $Z \in \mathcal{P}$ we can extend the inverse branches $F_Z^{-1} : Y \to Z$ to complex analytic maps $F_Z^{-1} : Y_\rho \to Z_\rho$, where the Z_ρ are appropriate neighborhoods of Z in $\mathbb{C}$. (In the u-direction, Φ^{-1} is clearly analytic, so we don't need extra assumptions there.)

(6) The parameter $\theta \in (0,1)$ is such that

$$\theta^{-1/\epsilon'} \leq \inf_{Z \in \mathcal{P}} \inf_{y_1 \neq y_2 \in Z} \frac{|Fy_1 - Fy_2|}{|y_1 - y_2|}$$

for some $\epsilon' \in (0, 1 - \frac{1}{p})$, where $p > 1$ is such that $\tilde{h} = \varphi_0^\gamma \in L^p(\mu_\Phi)$ as in (H1).

(7) We restrict $\mathcal{B}(\tilde{Y}_\rho)$ to those v which satisfy

$$v(y, 0) = v(y, 1) \qquad \text{for all } y \in Y. \tag{2.7}$$

2.2 The space $\mathcal{B}(\tilde{Y}_\rho)$ with norm $\| \|_{\mathcal{B}} = | |_\theta^* + \| \|_\infty^*$

The standard θ-Hölder norm on $\mathcal{B}(\tilde{Y})$ does not work well with the sets $S_{t,a}$ (defined in (2.1)): since $S_{t,a}$ is not aligned with $\tilde{\mathcal{P}}_n$, we get $\|R_{t,a}v\|_\theta = \infty$ for most $v \in \mathcal{B}(\tilde{Y})$. In this section we define a version of the θ-Hölder seminorm and the ∞-norm, where we first integrate over one-dimensional curves.

Let $\mathcal{G}_0$ be the collection of piecewise linear curves $G_0 = \{y, u(y)\}_{y\in Y}$ such that

(a) $\left|\frac{\partial u}{\partial y}\right| = 1/|Y|$ wherever the derivative is defined.

(b) For Lebesgue a.e. $u \in [0,1)$, there is exactly one $y \in Y$ such that $(y,u) \in G_0$.

Next let $\mathcal{G} = \cup_{r\geq 1}\mathcal{G}_r := \cup_{r\geq 1}\Phi^{-r}(\mathcal{G}_0)$, see Figure 1. Hence $\Phi^{-1}(\mathcal{G}) \subset \mathcal{G}$ and for every multival-

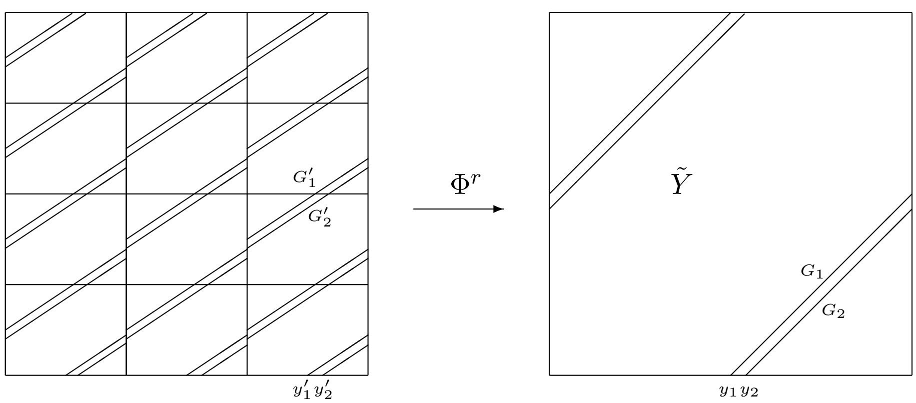


Figure 1: Schematic picture of two curves G_1 and G_2, and their Φ^r-preimages G_1' and G_2'.

ued curve $G = \{(y, \underline{u}(y))\}_{y\in Y} \in \mathcal{G}$ we can take $r = r(G) \geq 0$ such that $G \in \Phi^{-r}(\mathcal{G}_0)$. Then we have $r(\Phi^{-1}G) = r(G) + 1$ and

(c) For all $Z \in \mathcal{P}_r$ and Lebesgue a.e. $u \in [0,1)$, there is exactly one $y \in Z$ such that $(y,u) \in G$.

(d) For all $Z \in \mathcal{P}_r$ and $y \in Z$, there are exactly 2^r values $u_j \in [0,1)$ such that $(y,u_j) \in G$. (The notation $\underline{u}(y) = \{u_j(y),\, 0 \leq j < 2^r\}$ is our shorthand for this.)

For $r = r(G)$, let

$$\int_Y v(y, \underline{u}(y))\, d\mu(y) = \int_Y \frac{1}{2^r} \sum_{j=0}^{2^r-1} v(y, u_j(y))\, d\mu(y) \tag{2.8}$$

be our notation for the weighted integral of v over the multivalued curve G. In the sequel, we will usually estimate a single integral in this sum.

Given $G \in \mathcal{G}$ with $r(G) = r$, let $G(y_1)$ denote the multivalued curve translated in the u-direction $\bmod 1$ so that $(y_1, 0) \in G(y_1)$, and similar for $G(y_2)$. Let $(y, \underline{u}_1(y))$ and $(y, \underline{u}_2(y))$,

$y \in Y$ parametrize the multivalued curves $G(y_1)$ and $G(y_2)$, respectively. Due to property (a), $G(y_1)$ and $G(y_2)$ are vertical translation of each other by

$$u_{1,2} = \frac{|y_1 - y_2|}{2^r |Y|}. \tag{2.9}$$

Define the seminorm for $v \in \mathcal{B}(\tilde{Y}_\rho)$:

$$|v|_\theta^* = \sup_{G_1, G_2 \in \mathcal{G}} \theta^{-s(y_1,y_2)} \int_Y |v(y, \underline{u}_1(y)) - v(y, \underline{u}_2(y))| \, d\mu(y), \tag{2.10}$$

and weak norm

$$\|v\|_\infty^* = \sup_{G \in \mathcal{G}} \int_Y |v(y, \underline{u}(y)| \, d\mu(y). \tag{2.11}$$

The norm $\|v\|_{\mathcal{B}} = |v|_\theta^* + \|v\|_\infty^*$ will then make $\mathcal{B}(\tilde{Y}_\rho)$ into a Banach space. The choice of bi-analytic functions ensures that $\|v\|_\infty^*$ is actually equivalent to $\|v\|_\infty$:

Lemma 2.1. *There is C_ρ such that for all $v \in \mathcal{B}(\tilde{Y}_\rho)$*

$$\frac{1}{C_\rho} \|v\|_\infty^* \leq \|v\|_\infty \leq C_\rho \|v\|_\infty^* \tag{2.12}$$

and

$$\|\frac{\partial v}{\partial u}\|_\infty \leq \frac{1}{\rho} \|v\|_\infty \tag{2.13}$$

for all $v \in \mathcal{B}$.

Proof. Formula (2.13) follows directly from the Cauchy formula $\frac{\partial v(y,u)}{\partial u} = \frac{1}{2\pi i} \int_\Gamma \frac{v(y,\zeta)}{(\zeta - u)^2} \, d\zeta$ by taking Γ a circle of radius ρ around u.

The first inequality of (2.12) follows by taking $C_\rho = |Y|$.

The other inequality means roughly that v is not disproportionally large on small sets. To prove the inequality, let (y_0, u_0) be such that $|v(y_0, u_0)| = \|v\|_\infty$. If $|y - y_0| \leq A := \frac{1}{3} \frac{\|v\|_\infty}{\|\frac{\partial v}{\partial y}\|_\infty}$, then $|v(y_0, u_0) - v(y, u_0)| \leq |y_0 - y| \|\frac{\partial v}{\partial y}\|_\infty \leq \frac{1}{3} \|v\|_\infty$. Similarly, if $|u - u_0| \leq B := \frac{1}{3} \frac{\|v\|_\infty}{\|\frac{\partial v}{\partial u}\|_\infty}$, then $|v(y, u_0) - v(y, u)| \leq |u_0 - u| \|\frac{\partial v}{\partial u}\|_\infty \leq \frac{1}{3} \|v\|_\infty$.

This implies that $|v(y, u)| \geq \frac{1}{3} \|v\|_\infty$ for all (y, u) in the rectangle $([y_0 - A, y_0 + A] \cap Y) \times ([u_0 - B, u_0 + B] \cap [0, 1))$. If $G = \{(y, u(y))\}_{y \in Y} \in \mathcal{G}$ is a curve through (y_0, u_0), then (using Lemma 2.3 below)

$$\|v\|_\infty^* \geq \int_Y |v(y, u(y))| \, d\mu(y) \geq \frac{1}{3C_\mu} \|v\|_\infty \min(A, B/C_{dis}).$$

Using the Cauchy formula again, $A, B \geq \rho/3$. This gives $\|v\|_\infty \leq \frac{9C_\mu C_{dis}}{\rho} \|v\|_\infty^*$. $\square$

2.3 The result for unbounded $\tilde{h}$

The following Diophantine condition below plays the role (A2) in [7] (namely that there exists periodic points $y_1, y_2 \in Y$ such that the ratio $\varphi_0(y_1)/\varphi_0(y_2)$ is Diophantine):

(♣) There exist two periodic points $\tilde{y}_1, \tilde{y}_2 \in \tilde{Y}$ such that the ratio $\varphi(\tilde{y}_1)/\varphi(\tilde{y}_2)$ is Diophantine.

Proposition 2.2. *Every system satisfying conditions (♣) and (1)-(7) for the space* $(\mathcal{B}(\tilde{Y}_\rho), \| \ \|_{\mathcal{B}})$ *satisfies the conclusions of Proposition 1.1. and Proposition 1.2.*

The proof consists of verifying the conditions required for Proposition 1.1. Condition (H0) is supplied by [1, Lemma 4.4], and it does not rely on the Markov structure. Condition (H1) can freely be assumed since h is bounded away from zero. Using this Diophantine assumption (♣) condition (H3) follows as in [7, Proposition 3.5 (a)]. The verification of (H2), (H4) and (H6) takes some more work; this will be carried out in the following subsections.

2.4 Verifying (H4)

Since computing the norm of $R_{t,a}$ will involve integration over preimage curves in $\mathcal{G}$, the following property about the slope of multivalued curves $G \in \mathcal{G}$ is necessary.

Lemma 2.3. *For every $r \geq 1$, $Z \in \mathcal{P}_r$ and $G = \{(y, u_j(y)\}_{y \in Y, 0 \leq j < 2^r} \in \mathcal{G}_r$,*

$$\frac{1}{2^r |Z| C_{dis}} \leq \left| \frac{du_j(y)}{dy} \right| \leq \frac{C_{dis}}{2^r |Z|}, \qquad j = 0, \dots, 2^r - 1, \tag{2.14}$$

whenever $\frac{du_j(y)}{dy}$ is defined at $y \in Z$.

Proof. This is a consequence of distortion condition (2.2), which, combined with the Mean Value Theorem, implies that $|Y|/C_{dis} \leq |DF^r(y')||Z| \leq C_{dis}|Y|$. If $G = \Phi^{-r}(G_0)$ is parametrized as $\{(y', u_j(y'))\}_{y' \in Y, j=0,\dots,2^r-1}$, and $y = F^r(y')$ is used to parametrize $G_0 = \{(y, u(y))\}_{y \in Y} = \{(F^r(y'), 2^r u_j(y') \bmod 1)\}_{y' \in Z}$, then we have

$$\frac{1}{|Y|} = \left| \frac{du(y)}{dy} \right| = 2^r \left| \frac{du_j(y')}{dy'} \right| \left| \frac{dy'}{dy} \right|.$$

This gives

$$\left| \frac{du_j(y')}{dy'} \right| = \frac{|DF^r(y')|}{2^r |Y|} \leq \frac{C_{dis}}{2^r |Z|},$$

and the lower bound follows in the same way. □

Proposition 2.4. *Let $\mathcal{B}(\tilde{Y}_\rho)$ be the Banach space be equipped with the norm $\| \ \|_{\mathcal{B}} = | \ |_\theta^* + \| \ \|_\infty^*$ from (2.10) and (2.11). Assume (2.5) and tail condition (2.6), and let $0 < \epsilon' < \min\{1-1/p, 1-\gamma\}$ as in property (6). Then*

$$\|R_{t,a}\|_{\mathcal{B}} \ll t^{-(1+\beta-\epsilon')}.$$

Proof. We divide the 2-cylinders in $\tilde{Y}$ into three groups, and estimate $\|R_{t,a}v\|_\infty^*$, splitting the involved integrals according to these cases. The final estimate for $\|R_{t,a}v\|_\infty^*$ brings these three together in the form of the sum of convolutions. To estimate $|R_{t,a}v|_\theta^*$, we split the involved integrals according to the same three cases, leading again to a final sum of convolutions. However, since we need compare the integrals along different (parallel) multivalued curves, the way how these multivalued curves intersect $S_{t,a}$ requires a further subdivision into cases A, B and C.

Step I: Subdividing into Cases (1)-(3). Recall from (1.2) that $\varphi(y,u) = \varphi_0(y) + \psi(y,u)$, where

$$\psi(y,u) = \begin{cases} (2\tilde{h} \circ F(y) - \tilde{h}(y))u, & u \in [0, \frac{1}{2}); \\ (2\tilde{h} \circ F(y) - \tilde{h}(y))u - \tilde{h} \circ F(y), & u \in [\frac{1}{2}, 1). \end{cases} \tag{2.15}$$

A 1-cylinder Z with $\tau_F(Z) = n$ only contributes to the estimate for $R_{t,a}$ if $\psi(y,u) + h_n \approx t$ for some $y \in Z$, $u \in [0,1)$. On 2-cylinders $W \in \tilde{\mathcal{P}}_2$, $2\tilde{h}(Fy) - \tilde{h}(y)$ varies no more than $2C_{\varphi_0}^\gamma$ due to (2.5) and (H1), *i.e.,* $\tilde{h} = \varphi_0^\gamma$. It helps to split the set $\tilde{Y}$ into three regions, each coming with a certain range of "allowed" $n = \tau_F(Z)$ that contribute to the estimates for $R_{t,a}$ for particular ranges of the value of $2\tilde{h}(Fy) - \tilde{h}(y)$, and we first take the range $0 \le u < \frac{1}{2}$. Recall constant $C_{\varphi_0} > 2$ from property (3).

Case (1) $|2\tilde{h}(Fy) - \tilde{h}(y)| \le 4C_{\varphi_0}$. Here $|t - h_n| \le |\varphi - \varphi_0| + a + |\varphi_0 - h_n| \le 3C_{\varphi_0} + a$, so there is $C' = C'(a)$ such that $\lfloor \eta(t) - C' \rfloor \le n < \lfloor (\eta(t) + C' \rfloor$.

Case (2) $2\tilde{h}(Fy) - \tilde{h}(y) < -4C_{\varphi_0}$. Hence there is a region W_0 of $(y,u) \in W$ where $\psi(y,u) < -C_{\varphi_0}$, and there $t - h_n \le (\varphi - \varphi_0) + (\varphi_0 - h_n) < 0$. On the other hand, recalling from (H1) that $\tilde{h} = \varphi_0^\gamma$, we have the lower bound $t - h_n \ge (\varphi - \varphi_0) - C_{\varphi_0} - a \ge -\tilde{h}(y) - C_{\varphi_0} - a \ge -(h_n + C_{\varphi_0})^\gamma - C_{\varphi_0} - a \ge -2h_n^\gamma$. Therefore $h_n - 2h_n^\gamma \le t < h_n$, so there is C' such that $\eta(t) \le n \le \lfloor \eta(t) + C' t^\gamma \rfloor$.

Case (3) $2\tilde{h}(Fy) - \tilde{h}(y) > 4C_{\varphi_0}$. Since now $\psi(y,u)$ has no upper bound, the range of allowed n will be $1 \le n \le \eta(t)$.

For the range $\frac{1}{2} \le u < 1$, we can make similar computations, but the effect will be that we replace $\eta(t)$ in the above formulas by the larger value $\eta(t + \tilde{h}(Fy))$. This means that the estimates will improve compared to the range $0 \le u < \frac{1}{2}$, so we will omit the computations for $\frac{1}{2} \le u < 1$.

Step II: Estimates for $\|R_{t,a}\|_\infty^*$. By pointwise formula of the transfer operator gives

$$R_{t,a}v = \sum_{W \in \tilde{\mathcal{P}}} \frac{1}{2} e^{p(y'_W)} 1_{S_{t,a}}(y'_W, u'_W)\, v(y'_W, u'_W),$$

where $(y'_W, u'_W) = \Phi^{-1}(y,u) \cap W$. Each $W \in \tilde{\mathcal{P}}$ has the form $Z \times [0, \frac{1}{2})$ or $Z \times [\frac{1}{2}, 1)$ for $Z \in \mathcal{P}$ with $\tau_F(Z) = n$. Two such Ws are stacked vertically above the same Z.

This means for the $\| \, \|_\infty^*$-norm

$$\begin{aligned}
\|R_{t,a}v\|_\infty^* &= \sup_{G\in\mathcal{G}} \sum_{W\in\tilde{\mathcal{P}}} \int_Y \frac{1}{2} e^{p(y'_W)} 1_{S_{t,a}}(y'_W, \underline{u}(y'_W)) |v(y'_W, \underline{u}(y'_W))| \, d\mu(y) \\
&\le \sup_{G\in\mathcal{G}} \sum_{Z\in\mathcal{P}} \int_Y \frac{1}{2} e^{p(y'_Z)} \left(1_{S_{t,a}}(y'_Z, \underline{u}(y'_W)) |v(y'_Z \underline{u}(y'_Z))| \right. \\
&\qquad \left. + 1_{S_{t,a}}(y', \underline{u}(y'_Z) + \frac{1}{2}) |v(y'_Z, \underline{u}(y'_Z) + \frac{1}{2})| \right) d\mu(y) \qquad (2.16) \\
&\le \|v\|_\infty \sup_{G\in\mathcal{G}} \sum_{Z\in\mathcal{P}} \int_Z \frac{1}{2} \left(1_{S_{t,a}}(y', \underline{u}(y')) + 1_{S_{t,a}}(y', \underline{u}(y') + \frac{1}{2}) \right) d\mu(y'),
\end{aligned}$$

and by (2.12), we can bound $\|v\|_\infty \le C_\rho \|v\|_\infty^*$. To estimate $\int_Z 1_{S_{t,a}}(y', \underline{u}(y')) \, d\mu(y')$ (and the estimate of $\int_Z 1_{S_{t,a}}(y', \underline{u}_1(y') + \frac{1}{2}) \, d\mu(y')$ goes by the same argument), we use the announced case distinction:

Case (1) For those $Z' \in \mathcal{P}_2$ contained $Z \in \mathcal{P}$ with $\tau_F(Z) = n$ satisfying $|n - \eta(t)| \le C'$, it suffices to estimate $\sum_{Z'\in\mathcal{P}_2\cap\text{Case (1)}, Z'\subset Z} \int_{Z'} 1_{S_{t,a}}(y', \underline{u}(y')) \, d\mu(y') \le \mu(Z)$.

Case (2) We need to consider those regions $W_0 \subset Z' \times [0, \frac{1}{2})$, $Z' \in \mathcal{P}_2$ contained in Z on which $\inf_{(y,u)\in W_0} \psi(y,u) < -C_{\varphi_0}$. For fixed t and any such Z', we have $|t - h_n| \le |\varphi - \varphi_0| + |\varphi_0 - h_n| \le \frac{1}{2}|2\tilde{h}(Fy) - \tilde{h}(y)| + C_{\varphi_0}$. This gives $|2\tilde{h}(Fy) - \tilde{h}(y)| \ge |t - h_n|$. The sets $S_{t,a} \cap (Z' \times [0,1))$ are contained in horizontal strips of height $\le a|2\tilde{h}(Fy) - \tilde{h}(y)|^{-1} \le a|t - h_n|^{-1}$. Since the preimage curve G' of G has $r(G') \ge 2$, G' intersects this strip transversally with a slope $\ge 1/4C_{dis}|Z'|$ by (2.14). Therefore

$$\int_{\cup_{\text{Case 2}, Z'\subset Z} Z'} 1_{S_{t,a}}(y', \underline{u}(y')) \, d\mu(y') \le a|t - h_n|^{-1} \, 4C_{dis}\mu(Z).$$

Case (3) Consider those $Z' \in \mathcal{P}_2$ contained in $Z \in \mathcal{P}$ with $\tau_F(Z) = n$ on which $2\tilde{h}(Fy) - \tilde{h}(y) > 4C_{\varphi_0}$. For fixed t, we have $t < \varphi(y,u) = \varphi_0(y) + \psi(y,u) \le h_n + C_{\varphi_0} + \tilde{h}(Fy)$, so that $\frac{1}{2}(t - h_n) \le (t - h_n) - C_{\varphi_0} \le \tilde{h}(Fy)$.

Due to the distortion control of F of property (2) and the tail estimates of $\varphi_0 = \tilde{h}^{1/\gamma}$ given in property (4), we can find a constant $\tilde{C}$ such that

$$\begin{aligned}
\mu(y \in Z : k \le \tilde{h}(Fy) < k+1) &\ll \mu(y \in Y : k \le \tilde{h}(y) < k+1)\mu(Z) \\
&\ll \mu(y \in Y : k^{1/\gamma} \le \varphi_0(y) < (k+1)^{1/\gamma})\mu(Z) \\
&\le \tilde{C}(2k)^{-(1+\beta/\gamma)}\mu(Z). \qquad (2.17)
\end{aligned}$$

This implies that the 2-cylinders $Z' \in \mathcal{P}_2$, contained in Z, on which $k \le \tilde{h}(Fy) < k+1$ with $\lfloor \frac{1}{2}(t - h_n) \rfloor = k$, have combined measure $\le \tilde{C}|t - h_n|^{-(1+\beta/\gamma)}\mu(Z)$. Thus we obtain

$$\int_{\cup_{\text{Case 3}} Z'} 1_{S_{t,a}}(y', \underline{u}(y')) \, d\mu(y') \le \tilde{C}|t - h_n|^{-(\beta+1)/\gamma} \, \mu(Z).$$

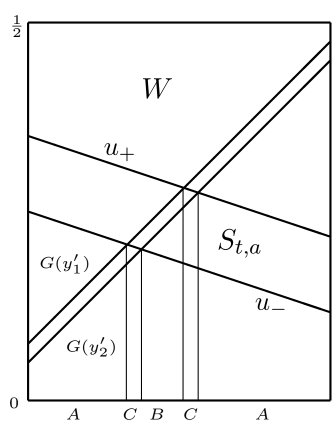
Combining Cases (1)-(3) and taking into account the allowed ranges of $n = \tau_F(Z)$, gives

$$\begin{aligned}\|R1_{S_{t,a}\cap(Z\times[0,1))}v\|_\infty^* \;&\le\; \big(1_{\{|n-\eta(t)|\le C'\}}\mu(Z)\\ &\qquad + 1_{\{\eta(t)<n<\eta(t)+C't^\gamma\}}C_{\varphi_0}|t-h_n|^{-1}\mu(Z)\\ &\qquad + 1_{\{1\le n<\eta(t)\}}\tilde{C}|t-h_n|^{-(1+\beta/\gamma)}\mu(Z)\big)\,\|v\|_\infty,\end{aligned}$$

and $\|v\|_\infty \le C_{dis}\|v\|_\infty^*$ by (2.12). Recall from (2.6) that $\mu(\cup_{\tau_F(Z)=n}Z) = \ell(n)n^{-(\beta+1)}$. Therefore, summing over all $Z\in\mathcal{P}$ gives the convolutions:

$$\begin{aligned}\|R_{t,a}v\|_\infty^* \;&\ll\; \left(\sum_{n=\lfloor\eta(t)-C'\rfloor}^{\lfloor\eta(t)+C'\rfloor}\ell(n)n^{-(\beta+1)} + \sum_{n=\eta(t)+1}^{\lfloor\eta(t)+C't^\gamma\rfloor}\ell(n)n^{-(\beta+1)}|t-h_n|^{-1}\right.\\ &\qquad\left. + \sum_{n=1}^{\eta(t)}\ell(n)n^{-(\beta+1)}|t-h_n|^{-(\beta+1)/\gamma}\right)\|v\|_\infty^*\\ &\ll\; \left(\tilde{\ell}(t)t^{-(\beta+1)} + \tilde{\ell}(t)t^{-(\beta+1)}\log t + \tilde{\ell}(t)t^{-(\beta+1)}\right)\|v\|_\infty^*.\end{aligned}$$

Step III: Estimates for $|R_{t,a}v|_\theta^*$. For the $|\,|_\theta^*$-seminorm, we need to compare the integral of v over preimage multivalued curves $G(y_1')$ and $G(y_2')$, which are the vertical translation of each other by $\frac{1}{2}u_{1,2}$. (Recall from (2.9) that $u_{1,2}$ is small if $s(y_1,y_2)$ is large.) We have to consider the intersections of these multivalued curves with $S_{t,a}$, and therefore it makes sense to subdivide the Cases (1)-(3) into subcases, see Figure 2.



The set $S_{t,a}\cap W$ is bounded above and below by curves

$$\begin{cases} u_+(y) = \frac{t+a-\varphi_0(y)}{2\tilde{h}(Fy)-\tilde{h}(y)},\\ u_-(y) = \frac{t-\varphi_0(y)}{2\tilde{h}(Fy)-\tilde{h}(y)}.\end{cases}$$

Figure 2: Schematic picture of cases A-C for $S_{t,a}$ intersecting a cylinder W.

Case A: Both $(y',\underline{u}_1(y'))$ and $(y',\underline{u}_2(y'))\notin S_{t,a}$. There is no contribution to the integral, so we can ignore this case.

Case B: Both $(y',\underline{u}_1(y'))$ and $(y',\underline{u}_2(y'))\in S_{t,a}$. In this case, by Lemmas 2.1 and 2.3,

$$|v(y',\underline{u}_1(y')) - v(y',\underline{u}_2(y'))| \le \|\frac{\partial v}{\partial u}\|_\infty\frac{u_{1,2}}{2} \le \frac{C_\rho C_{dis}}{2\rho}\|v\|_\infty^*\,|y_1-y_2| \tag{2.18}$$

provided the preimages $y_1', y_2'\in Z$. (Note that, assuming $y_1' < y_2'$, the bottom piece of the multivalued curve $G(y_1')$ may have to be paired to the top piece of the multivalued curve $G(y_2')$. Due to our assumption (2.7), this pairing doesn't create discontinuity problems.)

Case C: Only one of $(y', \underline{u}_1(y'))$ and $(y', \underline{u}_2(y')) \in S_{t,a}$. This applies to two intervals (one "on either side" of Case B) of length at most

$$|y_1' - y_2'| \le C_\mu \mu(Z)|y_1 - y_2|, \tag{2.19}$$

provided the preimages $y_1', y_2' \in Z$.

The $|\,|_\theta^*$-seminorm of $R_{t,a}v$ takes a form similar to (2.16). Changing coordinates $y \to y' = F_Z^{-1}(y)$ gives:

$$\begin{aligned} |R_{t,a}v|_\theta^* &= \sup_{y_1,y_2\in Y} \theta^{-s(y_1,y_2)} \sup_{G_1,G_2\in\mathcal{G}} \sum_{W\in\tilde{\mathcal{P}}} \int_Y \frac{1}{2} \Big| e^{p(y_W')} 1_{S_{t,a}}(y_W', \underline{u}_1(y_W'))v(y_W', \underline{u}_1(y_W')) \\ &\qquad - e^{p(y_W')} 1_{S_{t,a}}(y_W', \underline{u}_2(y_W'))v(y_W', \underline{u}_2(y_W')) \Big| \, d\mu(y) \\ &\le \sup_{y_1,y_2\in Y} \theta^{-s(y_1,y_2)} \sup_{G_1,G_2\in\mathcal{G}} \frac{1}{2} \sum_{k=0}^{1} \sum_{Z\in\tilde{\mathcal{P}}} \\ &\qquad \int_Z \left| 1_{S_{t,a}}(y', \underline{u}_1(y'))v(y', \underline{u}_1(y')) - 1_{S_{t,a}}(y', \underline{u}_2(y'))v(y', \underline{u}_2(y')) \right| \, d\mu(y'), \end{aligned}$$

where the sum over $k = 0, 1$ refers to the two cylinders W stacked above the same Z. (For simplicity of notation, we suppress this k-dependence in the parametrizations $\underline{u}$ of the multivalued curves.)

We will use the division into cases B and C for the $\|\,\|_\theta^*$-norm confined to each $Z \in \mathcal{P}$ separately.

Case (1) For those $Z' \in \mathcal{P}$ contained in $Z \in \mathcal{P}$ with $\tau_F(Z) = n$ and $|n - \eta(t)| \le C'$, we have by (2.18) and (2.19):

$$\begin{aligned} &\int_{\cup Z'\cap \text{Case B}} + \int_{cup Z'\cap \text{Case C}} \left| 1_{S_{t,a}}(y', \underline{u}_1(y'))v(y', \underline{u}_1(y')) - 1_{S_{t,a}}(y', \underline{u}_2(y'))v(y', \underline{u}_2(y')) \right| \, d\mu(y') \\ &\qquad \le \frac{C_\rho C_{dis}}{2\rho} \|v\|_\infty^* \, |y_1 - y_2| \mu(Z) + 2C_\mu \mu(Z)|y_1 - y_2| \|v\|_\infty \\ &\qquad \le \left(\frac{C_{dis}}{2\rho} + 2C_\mu\right) C_\rho \|v\|_\infty^* \theta^{s(y_1,y_2)} \mu(Z). \end{aligned}$$

Case (2) Again, we consider those regions $W_0 \subset Z' \times [0, \frac{1}{2})$, $Z' \in \mathcal{P}_2$ contained in Z, on which $\inf_{(y,u)\in W_0} \psi(y, u) < -C_{\varphi_0}$. As before the sets $S_{t,a} \cap (Z' \times [0, 1))$ are contained in horizontal strips of height $\le a|2\tilde{h}(Fy) - \tilde{h}(y)|^{-1} \le a|t - h_n|^{-1}$.

For the "Case B part" in Z', the multivalued curves $G(y_2')$ and $G(y_2')$ cross $S_{t,a} \cap (Z' \times [0, 1))$ with slope $\ge 1/(2C_{dis}|Z'|)$ by Lemma 2.3. This means that $Z' \cap$ Case B is an interval of length $\le 2C_{dis}|Z'|a|t - h_n|^{-1}$ and $|Z'| \le C_\mu \mu(Z')$. Thus by (2.18), the integral over Z' results in

$$\begin{aligned} \int_{Z'\cap \text{Case B}} |v(y', \underline{u}_1(y')) - v(y', \underline{u}_2(y'))| \, d\mu(y') &\le \int_{Z'\cap \text{Case B}} \frac{C_\rho C_{dis}}{2\rho} \|v\|_\infty^* |y_1 - y_2| \, d\mu(y') \\ &\le \frac{aC_\mu C_\rho C_{dis}}{\rho} |t - h_n|^{-1} \theta^{s(y_1,y_2)} \mu(Z'). \end{aligned}$$

For the "Case C part", the integration within Z' is over at most two separate curves with slope $\geq 1/(2C_{dis}|Z'|)$ inside $S_{t,a}$. Hence, we need to integrate $|v(y', \underline{u}_1(y'))|$ or $|v(y', \underline{u}_2(y'))|$ over intervals of length at most

$$\begin{aligned}\min\{C_\mu \mu(Z')|y_1 - y_2| \,,\, 2C_{dis}|Z'|\, a|t - h_n|^{-1}\} \\ &\leq C_\mu(1 + 2C_{dis})\mu(Z')|y_1 - y_2|^{\epsilon'} a|t - h_n|^{-(1-\epsilon')} \\ &\leq C_\mu(1 + 2C_{dis})\mu(Z')\theta^{s(y_1,y_2)} a|t - h_n|^{-(1-\epsilon')},\end{aligned}$$

where ϵ' is given in property (6). By (2.12),

$$\begin{aligned}\int_{Z' \cap \text{Case C}} &|1_{S_{t,a}}(y', \underline{u}_1(y'))v(y', \underline{u}_1(y')) - 1_{S_{t,a}}(y', \underline{u}_2(y'))v(y', \underline{u}_2(y'))|\, d\mu(y') \\ &\leq C_\mu C_\rho(1 + 2C_{dis})\|v\|_\infty^* a|t - h_n|^{-(1-\epsilon')}\theta^{s(y_1,y_2)}\mu(Z').\end{aligned}$$

Summing up over all $Z' \in \mathcal{P}_2$ of this type contained in Z of Case (2), we get

$$\begin{aligned}\int_Z &\left|1_{S_{t,a}}(y', \underline{u}_1(y'))v(y', \underline{u}_1(y')) - 1_{S_{t,a}}(y', \underline{u}_2(y'))v(y', \underline{u}_2(y'))\right|\, d\mu(y') \\ &\leq C_\mu C_\rho(1 + 2C_{dis})a|t - h_n|^{-(1-\epsilon')}\theta^{s(y_1,y_2)}\|v\|_\infty^* \mu(Z).\end{aligned}$$

Case (3) Consider those $Z' \in \mathcal{P}_2$ contained in $Z \in \mathcal{P}$ with $\tau_F(Z) = n$ on which $2\tilde{h}(Fy) - \tilde{h}(y) > 4C_{\varphi_0}$. For fixed t, we have again $t - h_n \leq \tilde{h}(Fy)$. As in the argument leading up to (2.17), the 2-cylinders Z' where $k \leq \tilde{h}(Fy) < k + 1$ have combined measure $\leq \tilde{C}|t - h_n|^{-(\beta+1)/\gamma}\mu(Z)$. For the "Case B part" of the integral, we therefore obtain by (2.18)

$$\begin{aligned}\int_{Z' \cap \text{Case B}} &|v(y', \underline{u}_1(y')) - v(y', \underline{u}_2(y'))|\, d\mu(y') \\ &\leq \int_{Z' \cap \text{Case B}} \frac{C_\rho C_{dis}}{2\rho}\|v\|_\infty^* |y_1 - y_2|\, d\mu(y') \\ &\leq \frac{C_\rho C_{dis}}{2\rho}\|v\|_\infty^* \tilde{C}|t - h_n|^{-(1+\beta/\gamma)}\theta^{s(y_1,y_2)}C_\mu \mu(Z).\end{aligned}$$

For the Case C part, we need to integrate $|v(y', \underline{u}_1(y'))|$ or $|v(y', \underline{u}_2(y'))|$ over at most two separate intervals of length at most

$$\begin{aligned}\min\{C_\mu \mu(Z')|y_1 - y_2| \,,\, \tilde{C}\mu(Z')|t - h_n|^{-(1+\beta/\gamma)}\} \\ &\leq \max\{C_\mu, \tilde{C}\}\mu(Z')|y_1 - y_2|^{\epsilon'}|t - h_n|^{-(1+\beta/\gamma)(1-\epsilon')} \\ &\leq \max\{C_\mu, \tilde{C}\}\mu(Z')\theta^{s(y_1,y_2)}|t - h_n|^{-(1+\beta/\gamma)(1-\epsilon')}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{Z' \cap \text{Case C}} &|1_{S_{t,a}}(y', \underline{u}_1(y'))v(y', \underline{u}_1(y')) - 1_{S_{t,a}}(y', \underline{u}_2(y'))v(y', \underline{u}_2(y'))|\, d\mu(y') \\ &\leq \max\{C_\mu, \tilde{C}\}C_\rho\|v\|_\infty^* |t - h_n|^{-(1+\beta/\gamma)(1-\epsilon')}\theta^{s(y_1,y_2)}\mu(Z).\end{aligned}$$

Finally, combining Cases (1)-(3) with all the constants $a, C_{dis}, \cdots, C_\rho$ replaced by the notation $\ll$, and taking into account the allowed ranges of $n = \tau_F(Z)$, we obtain

$$
\begin{aligned}
|R1_{S_{t,a}\cap(Z\times[0,1))}v|_\theta^* \ll \Big(& 1_{\{|n-\eta(t)|\le C'\}}\mu(Z) \\
& + 1_{\{\eta(t)<n<\lfloor\eta(t)+C't^\gamma\rfloor\}}C'|t-h_n|^{-(1-\epsilon')}\mu(Z) \\
& + 1_{\{1\le n\le\eta(t)\}}C'|t-h_n|^{-(1+\beta/\gamma)(1-\epsilon')}\mu(Z)\Big)\,\|v\|_\infty^*.
\end{aligned}
$$

Recall that $\mu(\cup_{\tau_F(Z)=n}Z) = \ell(n)n^{-(\beta+1)}$. Therefore, summing over all $Z \in \mathcal{P}$ gives:

$$
\begin{aligned}
|R_{t,a}v|_\theta^* &\ll \Bigg(\sum_{n=\lfloor\eta(t)-C'\rfloor}^{\lfloor\eta(t)+C'\rfloor} \ell(n)n^{-(\beta+1)} + \sum_{n=\eta(t)+1}^{\lfloor\eta(t)+C't^\gamma\rfloor} \ell(n)n^{-(\beta+1)}|t-h_n|^{-(1-\epsilon')} \\
&\qquad + \sum_{n=1}^{\eta(t)} \ell(n)n^{-(\beta+1)}|t-h_n|^{-(1+\beta/\gamma)(1-\epsilon')}\Bigg)\,\|v\|_\infty^* \\
&\ll \left(\tilde\ell(t)t^{-(\beta+1)} + \tilde\ell(t)t^{-(\beta+1-\gamma\epsilon')} + \tilde\ell(t)t^{-(1+\beta/\gamma)(1-\epsilon)')}\right)\|v\|_\infty^*.
\end{aligned}
$$

Combining the above estimates for $|\,|_\theta^*$ and $\|\,\|_\infty^*$ gives the bound $\|R_{t,a}\|_{\mathcal{B}} \ll t^{-(1+\beta-\epsilon')}$ as required. $\square$

2.5 Verifying (H2)

First we show that the twisted transfer operator $\hat R(s)v = R(e^{-s\varphi}v)$, $\Re s \ge 0$, satisfies the Lasota-Yorke inequality. The difficult part is the behavior of $\hat R$ under $|\,|_\theta^*$, and the discontinuities in the twist $e^{-s\varphi}$ that it comes with.

Lemma 2.5. *Assume that $\Re s \ge 0$. There exists constants $K_1, K_2 > 0$ such that*

$$
|\hat R^n(s)v|_\theta^* \le K_1\theta^n|v|_\theta^* + K_2\,|s|\,\|v\|_\infty^*, \quad \textit{and} \quad \|\hat R(s)v\|_\infty^* \le \|v\|_\infty^*, \tag{2.20}
$$

for all $n \in \mathbb{N}$ and $v \in \mathcal{B}(\tilde Y_\rho)$ satisfying (2.12).

Proof. Let $W = Z \times [j2^{-n}, (j+1)2^{-n}) \in \tilde{\mathcal{P}}_n$, with $Z \in \mathcal{P}_n$ defined in Section 2.2. Let multivalued curve $G \in \mathcal{G}$ with $r = r(G)$ be given. The translated multivalued curves $G(y_1), G(y_2)$ are parametrized as $(y, \underline{u}_1(y))$ and $(y, \underline{u}_2(y))$. For $(y, \underline{u}_j(y)) \subset \tilde Y$, $j = 1, 2$, and $W \in \tilde{\mathcal{P}}_n$, we will use $(y'_W, \underline{u}_j(y'_W))$ to denote the points in $\Phi^{-n}(y, \underline{u}_j(y)) \cap W$. Also let $\varphi_n = \sum_{j=0}^{n-1}\varphi\circ\Phi^j$ and analogously $\varphi_{0,n} = \sum_{j=0}^{n-1}\varphi_0\circ F^j$ and $\psi_n = \sum_{j=0}^{n-1}\psi\circ\Phi^j$, where $\psi = \varphi - \varphi_0$ as in (2.15).

With this notation, we obtain

$$
\begin{aligned}
|\hat{R}^n(s)v|^*_\theta &= \sup_{G_1,G_2\in\mathcal{G}} \theta^{-s(y_1,y_2)} \int_Y \sum_{W\in\tilde{\mathcal{P}}_n} \frac{1}{2^n} |e^{p_n(y'_W)-s\varphi_{0,n}(y'_W)}| \\
&\qquad \left| e^{-s\psi_n(y'_W,\underline{u}_1(y'_W))} v(y'_W,\underline{u}_1(y'_W)) - e^{-s\psi_n(y'_W,\underline{u}_2(y'_W))} v(y'_W,\underline{u}_2(y'_W)) \right| \, d\mu(y) \\
&\le \sup_{G_1,G_2\in\mathcal{G}} \theta^{-s(y_1,y_2)} \int_Y \sum_{W\in\tilde{\mathcal{P}}_n} \frac{1}{2^n} |e^{p_n(y'_W)}| \\
&\qquad \Big(|e^{-s\varphi_n(y'_W,\underline{u}_2(y'_W))}| \left| e^{-s(\psi_n(y'_W,\underline{u}_1(y'_W))-\psi_n(y'_W,\underline{u}_2(y'_W)))} - 1 \right| |v(y'_W,\underline{u}_1(y'_W))| \\
&\qquad\quad + |e^{-s\varphi_n(y'_W,\underline{u}_2(y'_W))}| \, |v(y'_W,\underline{u}_1(y'_W)) - v(y'_W,\underline{u}_2(y'_W))| \Big) \, d\mu(y) \\
&=: \sup_{G_1,G_2\in\mathcal{G}} \theta^{-s(y_1,y_2)} \, (I_1 + I_2).
\end{aligned}
$$

To estimate I_1, we majorize $|e^{-s\varphi_n(y'_W,\underline{u}_2(y'_W))}|$ by 1 (possible because $\Re s \ge 0$). Using the change of coordinates $y \to y' = y'_W$ (so $e^{p_n(y'_W)}\, d\mu(y) = d\mu(y')$) we obtain

$$
\begin{aligned}
I_1 &= \frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{Z\in\mathcal{P}_n} \int_Z \left| e^{-s(\psi_n(y',\underline{u}_1(y'))-\psi_n(y',\underline{u}_2(y')))} - 1 \right| \, |v(y',\underline{u}_1(y'))| \, d\mu(y') \\
&\le |s| \, \|v\|_\infty \frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{Z\in\mathcal{P}_n} \int_Z \left| \psi_n(y',\underline{u}_1(y')) - \psi_n(y',\underline{u}_2(y')) \right| \, d\mu(y'),
\end{aligned}
\tag{2.21}
$$

where the sum over k refers to the 2^n cylinders $W \in \tilde{\mathcal{P}}_n$ stacked over a single $Z \in \mathcal{P}_n$. (We suppress this k-dependence in our notation $\underline{u}_1(y')$ and $\underline{u}_2(y')$.)

To estimate this integral, we pair pieces Q_1 of $G(y'_1)$ with pieces Q_2 of $G(y'_2)$ if they are vertical translations of one another by $2^{-n}u_{1,2}$. The discontinuity of ψ_n (or rather the discontinuity of ψ appearing at $\{u = \frac{1}{2}\}$, where there is a jump of $\tilde{h} \circ F$) causes some complications, which we will deal with below. But if $\Phi^j(Q_1)$ and $\Phi^j(Q_2)$ are not separated by the line $\{u = \frac{1}{2}\}$, then

$$
|\psi \circ \Phi^j(y',\underline{u}_1(y')) - \psi \circ \Phi^j(y',\underline{u}_2(y'))| \le |2\tilde{h} \circ F^{j+1}(y') - \tilde{h} \circ F^j(y')| 2^{j-n} u_{1,2}.
$$

This gives an estimate or the "continuous part" of (2.21), using (2.8) with $r = r(G)$, as

$$
\begin{aligned}
&\frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{Z\in\mathcal{P}_n} \int_{Z\cap\text{continuous}} \left| \psi_n(y',\underline{u}_1(y')) - \psi_n(y',\underline{u}_2(y')) \right| \, d\mu(y') \\
&\quad \le 2^r \sum_{k=0}^{2^n-1} \sum_{Z\in\mathcal{P}_n} \sum_{j=0}^{n-1} \int_Z |2\tilde{h} \circ F^{j+1}(y') - \tilde{h} \circ F^j(y')| 2^{j-n} u_{1,2} \, d\mu(y') \\
&\quad \le 2^r \sum_{j=0}^{n-1} 2^{j-n} u_{1,2} \int_Y (2\tilde{h} \circ F^{j+1}(y') + \tilde{h} \circ F^j(y')) \, d\mu(y') \le \frac{3\|\tilde{h}\|_{L^1(\mu)}}{|Y|} \, \theta^{s(y_1,y_2)},
\end{aligned}
$$

by F-invariance of μ and property (a) to bound $u_{1,2} \le 2^{-r}|Y|^{-1}|y_1 - y_2| \le |Y|^{-1}\theta^{s(y_1,y_2)}$.

Discontinuities in $\psi_n(y,u)$ occur when $\Phi^j(y,u)$ lies on the horizontal line $\{u = \frac{1}{2}\}$ for some $j \le n$, and the jump in the value of $\psi \circ \Phi^j$ is $\tilde{h} \circ F(F^j(y))$. There is also a difference in value

between $\psi \circ \Phi^j(y,0)$ with $\psi \circ \Phi^j(y,1)$. Therefore, to estimate the "discontinuous part" of (2.21), we need to integrate over all pieces Q_1 of $G(y_1')$ and Q_2 of $G(y_2')$ that:
(i) touch a line $\{u = a2^{-(j+1)}\}$, $0 \le j < n$ and odd integer a, from opposite sides (because here the discontinuity line $\{u = \frac{1}{2}\}$ is reached after j iterates), or
(ii) Q_1 touches $\{u = 0\}$ while Q_2 touches $\{u = 1\}$ (because these are the only pieces of $G(y_1')$ and $G(y_2')$ that are not the vertical translations of each other by $2^{-n}u_{1,2}$).

There are precisely 2^k points $u = a2^{-(k+1)}$, a odd, such that $2^k u \bmod 1 = \frac{1}{2}$. For each of these, the pieces Q_1, Q_2, touching the line $\{u = a2^{-(k+1)}\}$ contribute to the discontinuous part for iterate $j = k, \dots, n-1$, namely for $n-k$ iterates.

Let $U = [y_1, y_2]$ and $U_j = F^{-j}(U)$; these are unions of intervals of combined measure $\mu(U)$. Since $\tilde h \in L^p(\mu)$, the Hölder inequality implies that the integral of $\tilde h \circ F$ over any set of measure $\mu(U)$ is at most $\le \|\tilde h\|_{L^p(\mu)} \mu(U)^{1-\frac{1}{p}} \le \|\tilde h\|_{L^p(\mu)} C_\mu^{1-\frac{1}{p}} |y_2 - y_1|^{1-\frac{1}{p}} \le \|\tilde h\|_{L^p(\mu)} C_\mu^{1-\frac{1}{p}} \theta^{s(y_1,y_2)}$ by the choice of θ in property 6. This gives

$$\begin{aligned} \int_{U_n} \tilde h \circ F^{j+1}\, d\mu(y') &= \int_Y (1_{U_{n-j}} \tilde h \circ F) \circ F^j(y')\, d\mu(y') \\ &= \int_{U_{n-j}} \tilde h \circ F(y)\, d\mu(y) \le \|\tilde h\|_{L^p(\mu)} C_\mu^{1-\frac{1}{p}} \theta^{s(y_1,y_2)}. \end{aligned}$$

Therefore, the "discontinuous part" of (2.21) is bounded as

$$\begin{aligned} &\frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{Z \in \mathcal{P}_n} \int_{Z \cap \text{discontinuous}} \left| \psi_n(y', \underline{u}_1(y')) - \psi_n(y', \underline{u}_2(y')) \right| \, d\mu(y') \\ &\quad \le \sum_{j=0}^{n} \frac{1}{2^n} ((n-j)2^j + n) \int_{U_n} \tilde h \circ F^{j+1}(y') \le 2 \|\tilde h\|_{L^p(\mu)} C_\mu^{1-\frac{1}{p}} \theta^{s(y_1,y_2)}. \end{aligned}$$

Combining the two, we find by (2.12)

$$I_1 \le |s|\, C_\rho \|v\|_\infty^* \left(3\|\tilde h|_{L^1(\mu)}/|Y| + 2\|\tilde h\|_{L^p(\mu)} C_\mu^{1-\frac{1}{p}} \right) \theta^{s(y_1,y_2)}.$$

To estimate I_2, we majorize $|e^{-s\varphi_n(y_W', \underline{u}_2(y_W'))}|$ by 1, and then we have the difference of the integrals of v taken over the preimage curves $\Phi^{-n}(G(y_1))$ and $\Phi^{-n}(G(y_2))$. By the definition of $|\ |_\theta^*$, this is less than $\theta^{n+s(y_1,y_2)} |v|_\theta^*$.

The $\|\ \|_\infty^*$-norm poses no problem:

$$\begin{aligned} \|\hat R(s) v\|_\infty^* &= \sup_{G \in \mathcal{G}} \sum_{W \in \tilde{\mathcal{P}}} \frac{1}{2} \int_Z |e^{p(y') - s\varphi(y', u(y'))}|\, |v(y', u(y'))|\, d\mu(y) \\ &\le \sup_{\Phi^{-1} G \in \mathcal{G}} \int_Y |v(y', \underline{u}(y'))|\, d\mu(y') \le \|v\|_\infty^*, \end{aligned}$$

because the sum of integrals over all $W \in \tilde{\mathcal{P}}$, after the change of coordinates $y \to y' = y_W'$, amounts to integrating over a single Φ-preimage multivalued curve as in (2.8). □

Remark 2.6. *For every $\theta \in (\frac{1}{2}, 1)$ and using (2.13), we have*

$$\sup_{u_1,u_2\in[0,1)} \frac{|v(y,u_1)-v(y,u_2)|}{\theta^{\tilde{s}((y,u_1),(y,u_2))}} \le \sup_{u_1,u_2\in[0,1)} \frac{|v(y,u_1)-v(y,u_2)|}{|u_1-u_2|} \le \|\frac{\partial v}{\partial u}\|_\infty \le \frac{1}{\rho}\|v\|_\infty,$$

uniformly in y, where the separation time $\tilde{s}((y,u_1),(y,u_2))$ is taken w.r.t. $\tilde{\mathcal{P}}$. Together with (2.12), this means that the norm $\|v\|_\theta = |v|_\theta + \|v\|_\infty$ is equivalent to $\|v\|_{\mathcal{B}}$.

The following can be proved directly for the norms $(\| \|_{\mathcal{B}}, \| \|_\infty^*)$ by means of the Arzela Ascoli Theorem. But passing to the equivalent pair of norms $(\| \|_\theta, \| \|_\infty)$, we can also refer to known results (for instance [7, Proposition 3.5(b)]) to conclude that the Theorem of Ionescu-Tulcea and Marinescu applies. That is, there is a uniform constant K such that

$$\|\hat{R}^n(s)(v - \int v\, d\mu_\Phi)\|_{\mathcal{B}} \le K\theta^n \|v\|_{\mathcal{B}}, \tag{2.22}$$

and in particular, $\hat{R}(s)$ acts quasi-compactly on $(\mathcal{B}, \| \|_{\mathcal{B}})$. Since $(\tilde{Y}, \Phi, \mu_\Phi)$ is ergodic, the eigenvalue 1 of $R = \hat{R}(0)$ is simple. This verifies (H2) ii).

2.6 Verifying (H6): the Dolgopyat type inequality

For the verification of hypothesis (H6) we refer to [7, Lemma 5.2]. However, let us sketch the argument for obtaining the weak form of the Dolgopyat type inequality. For details we refer to [5] (see also [7] for a different setting of the arguments in [5]).

For $b \in \mathbb{R}$, define $M_b : L^\infty(\mu_\Phi) \to L^\infty(\mu_\Phi)$, $M_b v = e^{ib\varphi} v \circ \Phi$. We say that there are *approximate eigenfunctions* on a subset $\mathcal{Z} \subset \tilde{Y}$ if there exist constants $\alpha > 0$ arbitrarily large, $\beta > 0$ and $C \ge 1$, and sequences $|b_k| \to \infty$, $\psi_k \in [0, 2\pi)$, θ-Hölder u_k with $|u_k| \equiv 1$, such that setting $n_k = [\beta \ln |b_k|]$,

$$|M_{b_k}^{n_k} u_k(\tilde{y}) - e^{i\psi_k} u_k(\tilde{y})| \le C|b_k|^{-\alpha},$$

for all $\tilde{y} \in \mathcal{Z}$ and all $k \ge 1$. A subset $\mathcal{Z}_0 \subset \tilde{Y}$ is called a *finite subsystem* if $\mathcal{Z}_0 = \bigcap_{n\ge 0} \Phi^{-n}\mathcal{Z}$ where $\mathcal{Z}$ is a finite union of partition elements $W \in \tilde{\mathcal{P}}$.

By [2, Section 13], the Diophantine condition (♣) ensures that there exists a finite subsystem such that there are no approximate eigenfunctions on $\mathcal{Z}_0$. Together with [5, Lemma 3.13], which can be applied to our setting because $\| \|_{\mathcal{B}}$ and $\| \|_\theta$ are equivalent (see Remark 2.6) and because the technical estimates in [7, Lemma 4.1] hold with exactly same proof since Φ is Gibbs Markov, this implies that (H6) holds.

References

[1] H. Bruin, D. Terhesiu, A renewal scheme for non-uniformly hyperbolic semiflows, Preprint 2015.

[2] D. Dolgopyat, Prevalence of rapid mixing in hyperbolic flows. *Ergodic Theory Dynam. Systems* **18** (1998) 1097–1114.

[3] S. Gouëzel, Sharp polynomial estimates for the decay of correlations. *Israel J. Math.* **139** (2004) 29–65.

[4] G. Keller, Generalized bounded variation and applications to piecewise monotonic transformations, Z. Wahrsch. Verw. Gebiete **69** (1985), 461–478.

[5] I. Melbourne, Rapid decay of correlations for nonuniformly hyperbolic flows. *Trans. Amer. Math. Soc.* **359** (2007) 2421–2441.

[6] I. Melbourne, D. Terhesiu, Operator renewal theory and mixing rates for dynamical systems with infinite measure, *Invent. Math.* **1** (2012) 61–110.

[7] I. Melbourne, D. Terhesiu, *Operator renewal theory for continuous time dynamical systems with finite and infinite measure*, Preprint 2014: http://arxiv.org/abs/1404.2508

[8] O. M. Sarig, Subexponential decay of correlations. *Invent. Math.* **150** (2002) 629–653.

[9] B. Saussol, Absolutely continuous invariant measures for multidimensional expanding maps. Israel J. Math. **116** (2000), 223–248.