

HITTING AND ESCAPING STATISTICS: MIXING, TARGETS AND HOLES

HENK BRUIN, MARK F. DEMERS, AND MIKE TODD

ABSTRACT. There is a natural connection between two types of recurrence law: hitting times to shrinking targets, and hitting times to a fixed target (usually seen as escape through a hole). We show that for systems which mix exponentially fast, one can move through a natural parameter space from one to the other. On the other hand, if the mixing is subexponential, there is a phase transition between the hitting times law and the escape law.

1. INTRODUCTION

This work is motivated by the natural connection between escape rates and hitting times. The existence of an exponential Hitting Time Statistics (HTS) law, which is a recurrence law to shrinking targets, is a rather soft condition: in all cases we are aware of, all one requires is mixing, with no rates necessary. However, under some mixing conditions, good error bounds can be derived (see e.g. [FFT2]) which mean that we can change the scaling in that law and still derive a non-degenerate limit law. If the mixing is exponential, the scaling can be changed to recover an escape rate to a fixed hole/target. In this paper we explore a parameter space which takes us between the escape rate case and the hitting time case. Under exponential mixing we can go between these laws in a non-degenerate way. A phase transition occurs when one leaves the hitting time setting and heads towards the escape case whenever the system is subexponentially mixing. In this paper we address such transitions in the case of stretched exponential, super-polynomial and polynomial rates of mixing.

1.1. Hitting times, escape rates, and between. Given a dynamical system $f : X \rightarrow X$ preserving an ergodic probability measure μ , one can consider first entry times to a sequence of subsets $(U_r)_r$ with U_r shrinking to a given point z as $r \rightarrow 0$. Letting τ_r be the *first hitting time* to U_r , i.e.,

$$\tau_r(x) := \inf \{n \geq 1 : f^n(x) \in U_r\},$$

one can ask how the quantity $\mu(\tau_r > t)$ depends asymptotically on both r and t (for a fixed z). To derive a HTS law, one scales the time via $t = s/\mu(U_r)$ for some $s \in \mathbb{R}^+$ and considers the limit

$$\lim_{r \rightarrow 0} \mu(\tau_r > s/\mu(U_r))$$

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For a large range of dynamical systems it is known that this limit is e^{-s} for μ -a.e. centre z . So we obtain an expression which is more convenient in this work:

$$\lim_{r \rightarrow 0} -\frac{1}{s} \log \mu(\tau_r > s/\mu(U_r)) = 1 \quad \text{for } \mu\text{-a.e. } z. \quad (1.1)$$

There is a wealth of literature on this topic, but here we just refer to the reviews [H] and [LF⁺, Chapter 5] and note that we only require very basic mixing properties for (1.1), for example for multimodal maps of the interval, if there is an absolutely continuous invariant measure (with no mixing requirement), this law holds [BT].

From the point of view of open systems, one declares U_r to be a (fixed) hole and considers any point entering U_r to be annihilated from the system. In contrast to hitting times (where $\tau_r(x) \geq 1$ a.s. also for $x \in U_r$), in an open system a point $x \in U_r$ is not allowed to exit U_r . Thus the escape time $e_r(x)$ satisfies $e_r(x) = \tau_r(x)$ if $x \notin U_r$ and $e_r(x) = 0$ if $x \in U_r$. However, an essential connection between the two is given by,

$$\{x \in X : \tau_r(x) = t\} = f^{-1}(\{x \in X : e_r(x) = t - 1\}), \quad \text{for all } t \geq 1.$$

Due to the invariance of μ , the escape rate can be defined by the following equivalent expressions,

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu(e_r > t - 1) = \lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu(\tau_r > t) \quad (1.2)$$

when this limit exists. If the limit exists, we label it $-\log \lambda_r$ for reasons that will become clear later and consider the ‘derivative of the escape rate,’ expressed as the limit,

$$\lim_{r \rightarrow 0} \frac{-\log \lambda_r}{\mu(U_r)} = 1 \quad \text{for } \mu\text{-a.e. } z, \quad (1.3)$$

which has been proved for certain exponentially mixing systems [BY, KL2]. We are not aware of examples where (1.2) holds (in the exponentially mixing setting), but (1.3) fails. Naturally if the system is subexponentially mixing, then (1.2) should be degenerate and so (1.3) fail, see [DF]. One expects (see e.g. [FP, FFT1]) that for periodic points z , the limit will be some number in $(0, 1)$ which can be expressed in terms of the relevant potential; if f is continuous, for all other points the limit should be 1.

Both of the limits (1.1) and (1.3) can be seen as special limiting cases of the expression

$$\frac{1}{\mu(U_r)} \frac{-1}{t} \log \mu(\tau_r > t), \quad (1.4)$$

where the open system perspective takes first the limit $t \rightarrow \infty$ then $r \rightarrow 0$, while the hitting time perspective takes the ‘diagonal limit’ $r \rightarrow 0$ with $t = s/\mu(U_r)$.

Once one views this expression in two-dimensional parameter space, one can naturally ask questions regarding convergence along various paths through this parameter space. Setting $t = s\mu(U_r)^{-\alpha}$ for some $\alpha \in (0, \infty)$, we formulate the generalised limit

$$L_{\alpha,s}(z) := \lim_{r \rightarrow 0} \frac{-1}{s\mu(U_r)^{1-\alpha}} \log \mu(\tau_r > s\mu(U_r)^{-\alpha}), \quad (1.5)$$

if the limit exists. With this formulation, the case $\alpha = 1$ coincides with the diagonal limit formulated above for hitting time statistics. Additionally, $\alpha = \infty$ can be thought of as coinciding

with the derivative of the escape rate¹ (where $t \rightarrow \infty$ as r is held fixed), while $\alpha = 0$ can be thought of as the reversed order of limits,

$$\lim_{t \rightarrow \infty} \lim_{r \rightarrow 0} \frac{-1}{t\mu(U_r)} \log \mu(\tau_r > t).$$

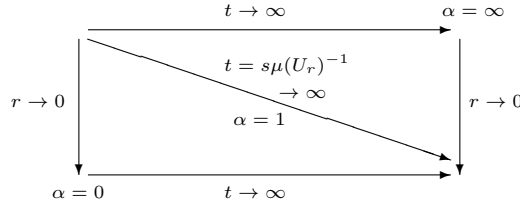


FIGURE 1. Different paths of taking the limit $r \rightarrow 0$, $t \rightarrow \infty$, with $t = s\mu(U_r)^{-\alpha}$.

Remark 1.1. For every $\alpha \in [0, 1)$, and supposing $\mu(\tau_r > t) > 0$ for all t , $L_{\alpha,s}(z) \in [0, 1]$, provided it exists. Indeed, for $t = s\mu(U_r)^{-\alpha}$, we have

$$\begin{aligned} 0 &\leq \frac{-\log \mu(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} = \frac{-\log(1 - \mu(\tau_r \leq t))}{s\mu(U_r)^{1-\alpha}} \\ &= \frac{-\log(1 - \mu(\cup_{j=0}^{t-1} f^{-j}(U_r)))}{s\mu(U_r)^{1-\alpha}} \leq \frac{-\log(1 - t\mu(U_r))}{s\mu(U_r)^{1-\alpha}} \\ &= \frac{-\log(1 - s\mu(U_r)^{1-\alpha})}{s\mu(U_r)^{1-\alpha}} \rightarrow 1 \quad \text{as } \mu(U_r) \rightarrow 0. \end{aligned}$$

Therefore, any limit point belongs to $[0, 1]$. The calculation above also implies that, when the limit exists, for $\alpha < 1$,

$$\lim_{r \rightarrow 0} \frac{-\log \mu(\tau_r > s\mu(U_r)^{-\alpha})}{s\mu(U_r)^{1-\alpha}} = \lim_{r \rightarrow 0} \frac{\mu(\tau_r \leq s\mu(U_r)^{-\alpha})}{s\mu(U_r)^{1-\alpha}}. \quad (1.6)$$

1.2. Brief summary of results. Our main results are, roughly speaking, that if the system behaves well and is exponentially mixing, then $L_{\alpha,s}(z)$ exists for all α ; it can be written in terms of the periodic behavior if z is periodic, and $L_{\alpha,s}(z) = 1$ otherwise (Theorem 2.1). On the other hand, if the system is slower than stretched exponentially mixing then $L_{\alpha,s}(z) = 0$ for $\alpha > 1$. If the system is (exactly) stretched exponentially mixing then there exists an $\alpha_0 > 1$ depending on the mixing rate so that we have the same result as for the exponential case if $\alpha < \alpha_0$, and $L_{\alpha,s}(z) = 0$ for $\alpha > \alpha_0$ (Theorem 3.2). The latter results employ an inducing argument and a large deviations law (either exponential, stretched exponential or polynomial). Our examples using inducing schemes require good large deviations of the inducing time, with various types of tail.

We remark that the existence of $L_{\alpha,s}(z)$ for $\alpha \neq 1$ is more delicate than for $\alpha = 1$ and gives additional information about the distribution of τ_r . For example, in the generic case, when $\alpha = 1$, one obtains that $\mu(\tau_r > s\mu(U_r)^{-1}) \rightarrow e^{-s}$ as $r \rightarrow 0$, but the rate of convergence does not appear. By contrast, when $\alpha \neq 1$, the limit of $\mu(\tau_r > s\mu(U_r)^{-\alpha})$ is always either 1 (for $\alpha < 1$) or

¹In the course of this project we became aware of related work in progress by Pollicott and Urbański. We thank Mark Pollicott for discussions.

0 (for $\alpha > 1$), and $L_{\alpha,s}(z)$ captures the exponential rate at which this convergence occurs. This rate provides information about the tail distribution of τ_r for small r . Again using the generic case as an example, when $\alpha > 1$, $L_{\alpha,s}(z) = 1$ implies $\mu(\tau_r > s\mu(U_r)^{-\alpha}) = e^{-(1\pm\varepsilon)s\mu(U_r)^{1-\alpha}}$; when $\alpha < 1$, $L_{\alpha,s}(z) = 1$ implies $\mu(\tau_r \leq s\mu(U_r)^{-\alpha}) = (1 \pm \varepsilon)s\mu(U_r)^{1-\alpha}$, due to (1.6).

The paper is organized as follows. In Section 2, we consider interval maps with good spectral properties; namely, that an associated family of transfer operators has a spectral gap. Our results are formulated abstractly, but in Section 2.6 we give specific examples including Lasota-Yorke maps, Gibbs-Markov maps and the Gauss map. In Section 3 we consider systems where a well-chosen first return map has good properties and show how the tail of the first return time affects the limits $L_{\alpha,s}$. Again we formulate our results abstractly and then provide examples in Section 4 to a variety of maps, including generalized Farey maps, several classes of unimodal maps and Young towers. In the appendix we (re)prove two technical results used in Sections 2 and 3.

Notation. We will use the following notational conventions throughout the paper without further mention. $A = C^{\pm 1}B$ means there exists $C \geq 1$ such that $C^{-1}B \leq A \leq CB$; similarly, $A = (1 \pm \varepsilon)B$ means $(1 - \varepsilon)B \leq A \leq (1 + \varepsilon)B$. We write $A \sim B$ if $\lim A/B = 1$, where the parameter in which the limit is taken is clear by context (usually it is $r \rightarrow 0$).

2. EXPONENTIALLY MIXING CASE

In this section, we consider a piecewise continuous map of the unit interval $f : I \rightarrow I$, with countably many intervals of monotonicity. Our assumptions will be general enough to allow both traditional piecewise expanding maps as well as more general Gibbs-Markov maps with contracting potentials. We will then prove our results regarding $L_{\alpha,s}(z)$, defined in (1.5), with respect to equilibrium states for these potentials. We will make assumptions on the map ((F1)–(F4) below as well as **(U)**, and where appropriate **(P)**) which imply the conditions of Rychlik [R] as well as giving a form of the Lasota-Yorke inequality needed in Proposition 2.5.

Assume that there exists a countable, disjoint collection of intervals $\mathcal{Z} = \{Z_i\}_i$, $Z_i \subset I$, such that f is continuous and strictly monotonic on each Z_i . We set $D = I \setminus \cup_i \text{int}(Z_i)$.

We assume that there exists a (nonatomic) Borel probability measure m_φ such that $m_\varphi(D) = 0$, which is conformal with respect to a potential $\varphi : I \rightarrow \mathbb{R}$, i.e., $dm_\varphi/d(m_\varphi \circ f) = e^\varphi$. The associated transfer operator acting on $L^1(m_\varphi)$ is

$$\mathcal{L}_\varphi \psi(x) = \sum_{y \in f^{-1}x} \psi(y) e^{\varphi(y)}, \quad \forall \psi \in L^1(m_\varphi).$$

We will study the action of \mathcal{L}_φ on functions of bounded variation. To that end, recall that the variation of a function ψ on an interval J is defined by

$$\bigvee_J \psi = \sup \left\{ \sum_{i=0}^{k-1} |\psi(x_{i+1}) - \psi(x_i)| : x_0 < x_1 < \dots < x_k, x_i \in J, \forall i \leq k \right\},$$

where the supremum is taken over all finite sets $\{x_i\}_i \subset J$.

Let $S_n\varphi = \sum_{i=0}^{n-1} \varphi \circ f^i$. We set $\varphi|_D = -\infty$ and assume the potential φ satisfies the following regularity properties:

- (F1) $\exists C_d > 0$ s.t. $|e^{S_n\varphi(x) - S_n\varphi(y)} - 1| \leq C_d |f^n x - f^n y|$ whenever $f^i x, f^i y$ lie in the same element of \mathcal{Z} for $i = 0, 1, \dots, n-1$;
- (F2) $\sum_{Z \in \mathcal{Z}} \sup_Z e^\varphi < \infty$;
- (F3) $\exists n_0 \in \mathbb{N}$ such that $\sup_I e^{S_{n_0}\varphi} < \inf_{I \setminus D} \mathcal{L}_\varphi^{n_0} 1$;
- (F4) for each interval $J \subset I \setminus D$, $\exists N = N(J)$ s.t. $\inf_{I \setminus D} \mathcal{L}_\varphi^N 1_J > 0$, where 1_J is the indicator function of the set J .

Due to the existence of the conformal measure m_φ , we have $\int \mathcal{L}_\varphi^n 1 dm_\varphi = \int 1 dm_\varphi = 1$, so that $\inf_{I \setminus D} \mathcal{L}_\varphi^n 1 \leq 1$ for each $n \in \mathbb{N}$. Thus by (F3), $\sup_I e^{S_{n_0}\varphi} < 1$. Then since $\sup_I e^{S_n\varphi}$ is submultiplicative,

$$\exists n_1 \in \mathbb{N} \text{ such that } (2 + 2C_d) \sup_I e^{S_{n_1}\varphi} < 1, \quad (2.1)$$

where C_d is from (F1).

Now fix $z \in I$ and for $r_0 > 0$, define $(U_r)_{r \in [0, r_0]}$ to be a nested sequence of intervals such that $\cap_r U_r = \{z\}$. From the point of view of open systems, for each r , we define the map with hole U_r and its iterates by, $f_r^n = f^n|_{I_r^{n-1}}$, where $I_r^0 = I \setminus U_r$ and $I_r^n = \cap_{i=0}^n f^{-i}(I \setminus U_r)$.

Let \mathcal{I}_r^n denote the intervals of monotonicity for f_r^n . We assume the following uniform large images condition for $f_r^{n_1}$ on the sequence $(U_r)_r$.

- (U) There exists $c_0 > 0$ such that

$$\inf_{r \in [0, r_0]} \inf \{m_\varphi(f_r^{n_1} J) : J \in \mathcal{I}_r^{n_1}\} \geq c_0,$$

where n_1 is from (2.1).

As we shall show in Section 2.1, under assumptions (F1)-(F4), \mathcal{L}_φ admits a unique invariant measure μ_φ , absolutely continuous with respect to m_φ , whose density g_0 is of bounded variation and is bounded away from 0. Note that μ_φ can also be characterised as an equilibrium state for φ . That is, for the variational pressure $P(\varphi) := \sup \{h(\mu) + \int \varphi d\mu\}$ where the supremum is taken over all f -invariant probability measures, μ_φ is an equilibrium state since it satisfies $h(\mu_\varphi) + \int \varphi d\mu_\varphi = P(\varphi)$. Moreover, since we can also think of $P(\varphi)$ as the log of the leading eigenvalue of \mathcal{L}_φ , our assumptions here give $P(\varphi) = 0$.

In the case when z is periodic of prime period p , we shall need the following additional condition.

- (P) The density g_0 is continuous at z and f^p is monotonic at z .

Let

$$I_{cont} := \{z \in I : f^k \text{ is continuous at } z \text{ for all } k \in \mathbb{N}\}. \quad (2.2)$$

That $\mu_\varphi(I_{cont}) = 1$ follows from the assumption that $m_\varphi(D) = 0$. The main result of this section is the following theorem.

Theorem 2.1. *Let (f, φ) satisfy (F1)-(F4). Fix $z \in I_{\text{cont}}$ and let $(U_r)_{r \in [0, r_0]}$ be a nested sequence of sets such that $\bigcap_r U_r = \{z\}$, satisfying **(U)**, and in the periodic case, **(P)** as well.*

Then, for any $s \in \mathbb{R}^+$ and $\alpha \in [0, \infty]$, taking $L_{\alpha, s}(z)$ with respect to the invariant measure μ_φ , we have

$$L_{\alpha, s}(z) = \begin{cases} 1, & \text{if } z \text{ is not periodic} \\ 1 - e^{S_p \varphi(z)}, & \text{if } z \text{ is } p\text{-periodic,} \end{cases}$$

where p -periodic means that the prime period of z is p .

Remark 2.2. *If f is continuous, then $I = I_{\text{cont}}$, while in the context of hitting time statistics, the case $z \in I \setminus I_{\text{cont}} \neq \emptyset$ is addressed in [AFV, Section 3.3].*

Remark 2.3. *As will be clear from the proof of Theorem 2.1, the case $\alpha = 0$ holds in great generality: The proof in Section 2.4 requires neither (F2)-(F4) nor **(U)**. In the non-periodic case we require only that $z \in I_{\text{cont}}$. In the periodic case, we require (F1), **(P)** and the fact that μ_φ is absolutely continuous with respect to m_φ with density bounded away from 0 at z . Alternatively, if it is known that $e^{S_p \varphi}$ is continuous at z , then (F1) is not needed.*

2.1. Preliminaries. We begin by establishing some easy facts about the potential φ . Let \mathcal{Z}^n denote the maximal intervals on which f^n is continuous and monotonic.

Lemma 2.4. *Assuming (F1)-(F4), for all $n \geq 0$, the following hold:*

- (a) $\sum_{Z \in \mathcal{Z}^n} e^{S_n \varphi} < \infty$;
- (b) for each $Z \in \mathcal{Z}$, $\bigvee_Z e^\varphi \leq C_d \sup_Z e^\varphi$;
- (c) $\bigvee_I e^{S_n \varphi} < \infty$.

Proof. (a) follows from a standard inductive argument using (F2).

(b) follows from (F1) since for any set $\{x_i\}_{i=1}^k \subset Z$, $|e^{\varphi(x_{i+1})} - e^{\varphi(x_i)}| \leq C_d e^{\varphi(x_i)}$.

For $n = 0$, (c) follows from (b) and (F2). Note that setting $\varphi|_D = -\infty$ only adds a term bounded by the series in (F2) to the variation. For $n \geq 1$, the argument again follows from a standard induction. \square

Potentials satisfying the above properties in addition to (F3) are called *contracting potentials* in the literature (see for example, [R, LSV1]), while (F4) is called the *covering property*. However, we require (F1) in order to obtain the stronger form of Lasota-Yorke inequalities in Proposition 2.5, which we shall need to apply perturbation theory to the open systems (f_r, U_r) , considering U_r as a hole.² We will prove Theorem 2.1 using the fact that the transfer operators associated with both the closed and open systems have spectral gaps and their spectral projectors vary in some uniform way with the size of the hole.

Let \mathcal{B} be the set of functions of bounded variation on I equipped with the variation norm $\|\psi\| = \bigvee_I \psi + |\psi|_1$, where $|\cdot|_1$ denotes the L^1 -norm with respect to the conformal measure m_φ .

² For a way to relax condition (F1) by requiring only a Hölder bound on distortion, see the application of Theorem 2.1 to the Gauss map in Section 2.6.3.

Using Lemma 2.4 and (2.1), the operator $\mathcal{L}_\varphi^{n_1}$ satisfies the assumptions of [R, Theorem 1], so that $\mathcal{L}_\varphi^{n_1}$ is quasi-compact as an operator on \mathcal{B} . Now using the decomposition in [R, Theorem 3] and the covering property (F4), it follows that $\mathcal{L}_\varphi^{n_1}$ has a simple eigenvalue at 1, and no other eigenvalue can have modulus 1, i.e., $\mathcal{L}_\varphi^{n_1}$ has a spectral gap. Using again Lemma 2.4(b),(c), since \mathcal{L}_φ is a bounded operator on \mathcal{B} , it also has a spectral gap. This will be the starting point from which we will perturb.

2.2. Notation and Initial Results for Open Systems. In this section, we first summarize standard notation for open systems that we will use throughout the paper. We then proceed to prove the existence of a uniform spectral gap for a family of associated transfer operators.

Recall that if we regard U_r as a hole, then the set of points that has not entered U_r by time n is denoted $\mathring{I}_r^n = \bigcap_{i=0}^n f^{-i}(I \setminus U_r)$, and the map corresponding to the open system is simply the restriction $\mathring{f}_r^n := f^n|_{\mathring{I}_r^{n-1}}$. Notice that by definition of the escape time e_r (see Section 1.1), we have $\{e_r > n\} = \mathring{I}_r^n$.

The transfer operator for the open system and its iterates are defined on $L^1(m_\varphi)$ by

$$\mathring{\mathcal{L}}_{\varphi, U_r}^n \psi(x) = \sum_{y \in \mathring{f}_r^{-n} x} \psi(y) e^{S_n \varphi(y)} = \mathcal{L}_\varphi^n(\psi 1_{\mathring{I}_r^{n-1}}), \quad (2.3)$$

for all $n \geq 1$. Due to the conformality of the measure m_φ , we have the following useful identity,

$$\int_I \mathring{\mathcal{L}}_{\varphi, U_r}^n \psi dm_\varphi = \int_I \mathcal{L}_\varphi^n(\psi 1_{\mathring{I}_r^{n-1}}) dm_\varphi = \int_{\mathring{I}_r^{n-1}} \psi dm_\varphi. \quad (2.4)$$

The importance of the above relation is the connection it provides between the escape rate with respect to the measure ψdm_φ and the spectral radius of $\mathring{\mathcal{L}}_{\varphi, U_r}^n$ acting on functions of bounded variation (see Proposition 2.5).

Since we fix the potential φ , for ease of notation and to emphasize the relationships among the operators, in what follows we will denote $\mathring{\mathcal{L}}_r := \mathring{\mathcal{L}}_{\varphi, U_r}$ and $\mathcal{L}_0 := \mathcal{L}_\varphi$. Similarly, we denote by g_0 the invariant density for \mathcal{L}_0 , m_0 the conformal measure, and $\mu_0 = g_0 m_0$ the invariant measure for the closed system.

Due to **(U)**, we have the following set of uniform Lasota-Yorke inequalities for this family of operators.

Proposition 2.5. *There exists $C_0 > 0$ and $\sigma < 1$ such that for any $\psi \in \mathcal{B}$, $r \in [0, r_0]$ and all $n \geq 0$,*

$$\begin{aligned} \|\mathring{\mathcal{L}}_r^n \psi\| &\leq C_0 \sigma^{-n} \|\psi\| + C_0 \int_{\mathring{I}_r^{n-1}} |\psi| dm_0, \\ |\mathring{\mathcal{L}}_r^n \psi|_1 &\leq \int_{\mathring{I}_r^{n-1}} |\psi| dm_0. \end{aligned}$$

The proof is by now fairly standard, even in this generalized context. Since our assumptions and estimates necessarily differ from those appearing in the literature for closed systems (given that we must show uniformity of the constants C_0 and σ in the sequence (U_r) as well as the fact that we require decay in the L^1 term), we include the proof for completeness in the appendix.

It follows from Proposition 2.5, the compactness of the unit ball of \mathcal{B} in $L^1(m_0)$, and the conformality of m_0 that the spectral radius of $\mathring{\mathcal{L}}_r$ acting in \mathcal{B} is at most one while its essential spectral radius is bounded by $\sigma^{-1} < 1$. Thus $\mathring{\mathcal{L}}_r$ is quasi-compact as an operator on \mathcal{B} , as is \mathcal{L}_0 . In addition, defining the following perturbative norm,

$$\| \mathcal{L}_0 - \mathring{\mathcal{L}}_r \| = \sup\{ |\mathcal{L}_0\psi - \mathring{\mathcal{L}}_r\psi|_1 : \|\psi\| \leq 1 \},$$

we have the following bound.

Lemma 2.6. $\| \mathcal{L}_0 - \mathring{\mathcal{L}}_r \| \leq m_0(U_r) \leq C_1 \mu_0(U_r)$, where $C_1^{-1} = \text{essinf } g_0$.

Proof. The proof is immediate since if $\psi \in \mathcal{B}$ with $\|\psi\| \leq 1$, we use the fact that m_0 is φ -conformal to estimate,

$$\int |(\mathcal{L}_0 - \mathring{\mathcal{L}}_r)\psi| dm_0 = \int |\mathcal{L}_0(1_{U_r}\psi)| dm_0 \leq |\psi|_\infty m_0(U_r),$$

and the fact that $\text{essinf } g_0 > 0$ follows from (F4). \square

Corollary 2.7. *There exists $r_1 \in (0, r_0]$ such that for all $r \in [0, r_1]$, the operators $\mathring{\mathcal{L}}_r$ have a uniform spectral gap on \mathcal{B} . In particular, for $r > 0$, there exist $\lambda_r < 1$, and linear operators Π_r, R_r , such that*

$$\mathring{\mathcal{L}}_r = \lambda_r \Pi_r + R_r,$$

$\Pi_r^2 = \Pi_r$, $\Pi_r R_r = R_r \Pi_r = 0$ and the spectral radius of R_r is at most $\rho < \lambda_{r_1}$. The range of Π_r is the span of a function $g_r \in \mathcal{B}$, satisfying $\mathring{\mathcal{L}}_r g_r = \lambda_r g_r$, and normalized so that $\int g_r dm_0 = 1$.

The above decomposition also holds for $r = 0$ with $\lambda_0 = 1$.

Proof. \mathcal{L}_0 has a spectral gap by [R] and the discussion following Lemma 2.4. It follows from Proposition 2.5, Lemma 2.6 and [KL1, Corollary 1] that the spectra and spectral projectors of $\mathring{\mathcal{L}}_r$ and \mathcal{L}_0 outside the disk of radius σ vary continuously in $\mu_0(U_r)$. Thus for r sufficiently small, $\mathring{\mathcal{L}}_r$ inherits a spectral gap from \mathcal{L}_0 , and by continuity, the spectral gap is uniform in r , yielding the existence of $\rho < \lambda_{r_1}$ in the statement of the corollary. Notice that by monotonicity of the sequence $(U_r)_{r \in [0, r_0]}$, $\lambda_{r_1} \leq \lambda_r \leq \lambda_0 = 1$ for all $r \in [0, r_1]$. \square

We proceed to the proof of Theorem 2.1, first proving the special cases $\alpha = \infty$ and $\alpha = 0$, and then turning to the general case $\alpha \in (0, \infty)$.

2.3. Proof of Theorem 2.1: The case $\alpha = \infty$. To address the case corresponding to $\alpha = \infty$, we must compute the double limit,

$$\lim_{r \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{\mu_0(U_r)} \frac{-1}{t} \log \mu_0(\tau_r > t).$$

For fixed $r \in (0, r_1]$, the spectral gap provided by Corollary 2.7 implies that the escape rate with respect to μ_0 is $-\log \lambda_r$, i.e.,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mu_0(\tau_r > t) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \mu_0(\mathring{I}_r^t) = \log \lambda_r,$$

where we have used (1.2) as well as the fact that $\{e_r > t\} = \mathring{I}_r^t$. (Indeed, the escape rate is $-\log \lambda_r$ with respect to the measure ψm_0 for any density $\psi \in \mathcal{B}$ that is bounded away from 0.)

In order to show the limit $r \rightarrow 0$ converges to the claimed value, we will use the results of [KL2]. To do this we must check the necessary conditions given there, listed as (A1)-(A7). In our setting, (A1)-(A3) are immediately satisfied by the existence of a uniform spectral gap for the operators $\mathring{\mathcal{L}}_r$ and the accompanying spectral decomposition given by Corollary 2.7.

(A4) requires that we normalize $m_0(g_r) = 1$, which we have done, and that there exists $C_2 > 0$ such that $\sup_{r \in [0, r_1]} \|g_r\| \leq C_2$, i.e., the conditionally invariant densities are uniformly bounded in \mathcal{B} . This follows from the uniform Lasota-Yorke inequalities given by Proposition 2.5 applied to g_r :

$$\lambda_r^n \|g_r\| = \|\mathring{\mathcal{L}}_r^n g_r\| \leq C_0 \sigma^n \|g_r\| + C_0 \int_{\mathring{I}_r^n} g_r dm_0 = C_0 \sigma^n \|g_r\| + C_0 \lambda_r^n,$$

where we have used the fact that $\int_{\mathring{I}_r^n} g_r dm_0 = \int \mathring{\mathcal{L}}_r^n g_r dm_0$, by conformality. Since $\sigma < \lambda_r$ in the spectral gap regime, we let $n \rightarrow \infty$ and conclude that $\|g_r\| \leq C_0$ independently of $r \in [0, r_1]$.

(A5) requires that $\eta_r := \|m_0(\mathcal{L}_0 - \mathring{\mathcal{L}}_r)\| \rightarrow 0$ as $r \rightarrow 0$, where $\|\cdot\|$ is the norm of the linear functional $m_0(\mathcal{L}_0 - \mathring{\mathcal{L}}_r) : \mathcal{B} \rightarrow \mathbb{R}$. This is precisely Lemma 2.6, since if $\psi \in \mathcal{B}$, we have

$$|m_0((\mathcal{L}_0 - \mathring{\mathcal{L}}_r)(\psi))| \leq \int |(\mathcal{L}_0 - \mathring{\mathcal{L}}_r)\psi| dm_0 \leq \|\psi\| m_0(U_r),$$

so that $\eta_r = m_0(U_r)$.

(A6) requires³ $\eta_r \cdot \|(\mathcal{L}_0 - \mathring{\mathcal{L}}_r)g_0\| \leq C_3 \mu_0(U_r)$, for some $C_3 > 0$. This is satisfied since (as noted in Lemma 2.6), $\text{essinf } g_0 = C_1^{-1} > 0$. Thus,

$$\begin{aligned} \eta_r \cdot \|(\mathcal{L}_0 - \mathring{\mathcal{L}}_r)g_0\| &= m_0(U_r) \|\mathcal{L}_0(1_{U_r}g_0)\| \leq C_1 \mu_0(U_r) \|\mathcal{L}_0\| \|1_{U_r}g_0\| \\ &\leq C_1 \|\mathcal{L}_0\| (\|g_0\| + 2|g_0|_\infty) \mu_0(U_r) \leq 3C_1 C_0 \|\mathcal{L}_0\| \mu_0(U_r), \end{aligned}$$

as required.

Finally, (A7) requires that the limit

$$q_k := \lim_{r \rightarrow 0} q_{k,r} := \lim_{r \rightarrow 0} \frac{m_0((\mathcal{L}_0 - \mathring{\mathcal{L}}_r) \mathring{\mathcal{L}}_r^k (\mathcal{L}_0 - \mathring{\mathcal{L}}_r)(g_0))}{\mu_0(U_r)},$$

exists for each integer $k \geq 0$. Notice that by conformality and using the fact that $\mathcal{L}_0 - \mathring{\mathcal{L}}_r = \mathcal{L}_0(1_{U_r} \cdot)$, we have

$$m_0((\mathcal{L}_0 - \mathring{\mathcal{L}}_r) \mathring{\mathcal{L}}_r^k (\mathcal{L}_0 - \mathring{\mathcal{L}}_r)(g_0)) = \int 1_{U_r} \circ f^{k+1} \cdot 1_{\mathring{I}_r^{k-1}} \circ f \cdot 1_{U_r} \cdot g_0 dm_0.$$

The product of indicator functions in the above expression is equivalent to the indicator function of the set

$$E_r^k = \{x \in U_r : f^i(x) \notin U_r, i = 1, \dots, k, \text{ and } f^{k+1}(x) \in U_r\}.$$

So $q_{k,r} = \frac{\mu_0(E_r^k)}{\mu_0(U_r)}$.

³[KL2] actually states this bound, and subsequent ones, in terms of a more general quantity, Δ_r ; however, in the present context, $\Delta_r = \int (\mathcal{L}_0 - \mathring{\mathcal{L}}_r)g_0 dm_0 = \int_{U_r} g_0 dm_0 = \mu_0(U_r)$, and we will use this simpler expression in what follows.

If z is not periodic, recall that since $z \in I_{cont}$, f^k is continuous at z for each $k \in \mathbb{N}$, so for fixed k , the set E_r^k is empty for all r sufficiently small. Thus $q_k = 0$ for all $k \geq 0$. On the other hand, if z is periodic with prime period p , then for sufficiently small r , E_r^k is empty except when $k = p - 1$. In this case, we use the monotonicity and continuity of f^p at z (assumption **(P)**) to conclude that $f^p(E_r^{p-1}) = U_r$. Let f_1 denote this branch of f^p . Now the continuity of g_0 at z and (F1) yield,

$$q_{p-1} = \lim_{r \rightarrow 0} \frac{1}{\mu_0(U_r)} \int_{E_r^{p-1}} g_0 dm_0 = \lim_{r \rightarrow 0} \frac{1}{\mu_0(U_r)} \int_{U_r} e^{S_p \varphi \circ f_1^{-p}} g_0 \circ f_1^{-p} dm_0 = e^{S_p \varphi(z)},$$

where we have used the fact that $f_1^{-p}(z) = z$.

Having verified conditions (A1)-(A7) in our setting, we conclude by [KL2, Theorem 2.1], that for $z \in I_{cont}$,

$$\lim_{r \rightarrow 0} \frac{-1}{\mu_0(U_r)} \log \lambda_r = \begin{cases} 1, & \text{if } z \text{ is not periodic} \\ 1 - e^{S_p \varphi(z)}, & \text{if } z \text{ is } p\text{-periodic.} \end{cases} \quad (2.5)$$

2.4. Proof of Theorem 2.1: The case $\alpha = 0$. Since we will not need the measures μ_r in this section, for simplicity, we will denote μ_0 simply by μ and m_0 by m .

We fix t and consider the limit $\lim_{r \rightarrow 0} \mu(U_r)^{-1} \log \mu(\tau_r > t)$. Note that the set $\{\tau_r \leq t\} = \cup_{j=0}^t f^{-j}(U_r)$.

Case 1: Nonperiodic z . Assume that $z \in I_{cont}$ is not a periodic point for f . Then we may choose r sufficiently small that the sets $f^{-j}(U_r)$, $j = 0, \dots, t$, are pairwise disjoint. Thus,

$$\begin{aligned} \lim_{r \rightarrow 0} \mu(U_r)^{-1} \log \mu(\tau_r > t) &= \lim_{r \rightarrow 0} \mu(U_r)^{-1} \log(1 - \mu(\tau_r \leq t)) \\ &= \lim_{r \rightarrow 0} \mu(U_r)^{-1} \log(1 - (t+1)\mu(U_r)) = -(t+1). \end{aligned}$$

Proceeding to the second limit, we complete the proof of this case,

$$\lim_{t \rightarrow \infty} \lim_{r \rightarrow 0} -t^{-1} \mu(U_r)^{-1} \log \mu(\tau_r > t) = \lim_{t \rightarrow \infty} t^{-1}(t+1) = 1.$$

Case 2: Periodic z . Fix z of prime period p for f , satisfying **(P)**. Choose r sufficiently small that the sets $f^{-i}(U_r)$, $i = 0, \dots, p-1$ are pairwise disjoint. This choice forces $f^{-i}(U_r) \cap f^{-j}(U_r) = \emptyset$ except when $i - j$ is a multiple of p .

Suppose $t \in \mathbb{N}$ satisfies $t = (k+1)p - 1$ for some $k \geq 0$. Then

$$\{\tau_r \leq t\} = \cup_{i=0}^k \cup_{j=0}^{p-1} f^{-ip-j}(U_r).$$

Note that by the above observation regarding when two pre-images of U_r may intersect, we conclude that the sets in the union above are disjoint for distinct j . Thus,

$$\mu(\tau_r \leq t) = \sum_{j=0}^{p-1} \mu(\cup_{i=0}^k f^{-ip-j}(U_r)) = p\mu(\cup_{i=0}^k f^{-ip}(U_r)).$$

To estimate the measure of the remaining set, we prove the following lemma.

Lemma 2.8. *Let z be a point of continuity of g_0 of prime period p . For $\varepsilon > 0$, let U_r be a sufficiently small neighbourhood of z with $\text{diam}(U_r) < \varepsilon$ such that f^p is monotonic on U_r and for each $x \in U_r$, $|g_0(x) - g_0(z)| \leq \varepsilon$. If k is such that $U_r, U_r \cap f^{-p}(U_r), \dots, U_r \cap f^{-kp}(U_r)$ forms a decreasing sequence of sets, then*

$$\mu(\cup_{i=0}^k f^{-ip}(U_r)) = \mu(U_r)(k + 1 - ke^{S_p\varphi(z)}(1 \pm \bar{C}\varepsilon)),$$

where $\bar{C} = C_d + C_1$, C_d is from (F1) and C_1 is from Lemma 2.6.

Proof. Write $V_r = U_r \cap f^{-p}(U_r)$. The proof goes by induction on k . For $k = 1$, we have

$$\begin{aligned} \mu(U_r \cup f^{-p}(U_r)) &= \mu(U_r) + \mu(f^{-p}(U_r)) - \mu(V_r) \\ &= 2\mu(U_r) - \mu(U_r \cap f^{-p}(U_r)). \end{aligned} \tag{2.6}$$

Letting f_1^p denote the branch of f^p mapping V_r onto U_r monotonically,

$$\begin{aligned} \mu(V_r) &= \int_{U_r \cap f^{-p}(U_r)} g_0 dm = \int_{U_r} g_0 \circ f_1^{-p} e^{S_n\varphi \circ f_1^{-p}} dm \\ &= \mu(U_r)e^{S_p\varphi(z)} + \int_{U_r} g_0 \circ f_1^{-p} e^{S_p\varphi \circ f_1^{-p}} (1 - e^{S_p\varphi(z) - S_p\varphi \circ f_1^{-p}}) dm \\ &\quad + e^{S_p\varphi(z)} \int_{U_r} (g_0 \circ f_1^{-p} - g_0) dm \end{aligned}$$

Using (F1), the first integral on the right hand side is bounded by

$$e^{S_p\varphi(z)} C_d \text{diam}(U_r) \int_{U_r} g_0 \circ f_1^{-p} e^{S_p\varphi \circ f_1^{-p}} dm \leq e^{S_p\varphi(z)} C_d \text{diam}(U_r) \mu(U_r),$$

where we have changed variables again for the last inequality. The second integral on the right hand side is bounded by,

$$e^{S_p\varphi(z)} \varepsilon m(U_r) \leq e^{S_p\varphi(z)} C_1 \varepsilon \mu(U_r),$$

where C_1 is from Lemma 2.6. Putting these estimates together and using the fact that $\text{diam}(U_r) < \varepsilon$, we obtain,

$$\mu(V_r) = (1 \pm \bar{C}\varepsilon) e^{S_p\varphi(z)} \mu(U_r), \tag{2.7}$$

where $\bar{C} = C_d + C_1$.

Plugging this into (2.6) yields the lemma for $k = 1$.

Now suppose the statement holds for k and consider the set,

$$\cup_{i=0}^{k+1} f^{-ip}(U_r) = f^{-(k+1)p}(U_r) \cup (\cup_{i=0}^k f^{-ip}(U_r)) =: f^{-(k+1)p}(U_r) \cup A_k.$$

We claim that any intersection between $f^{-(k+1)p}(U_r)$ and A_k necessarily belongs to $f^{-kp}(U_r)$. To see this, suppose $x \in f^{-(k+1)p}(U_r) \cap f^{-jp}(U_r)$ for some $j \leq k$. Then $f^{jp}(x) \in U_r \cap f^{-(k+1-j)p}(U_r)$, which necessarily remains in U_r for the next $k+1-j$ iterates of f^p , due to the nested property of the sets $U_r \cap f^{-ip}(U_r)$. In particular, $f^{(k-j)p}(f^{jp}x) \in V_r$. Thus $x \in f^{-kp}(U_r)$.

Using this fact about intersection as well as (2.7), we now estimate,

$$\begin{aligned} \mu(\cup_{i=0}^{k+1} f^{-ip}(U_r)) &= \mu(f^{-(k+1)p}(U_r)) + \mu(A_k) - \mu(f^{-(k+1)p}(U_r) \cap f^{-kp}(U_r)) \\ &= \mu(U_r) + \mu(A_k) - \mu(U_r \cap f^{-p}(U_r)) \\ &= \mu(U_r) + \mu(A_k) - (1 \pm \bar{C}\varepsilon)\mu(U_r)e^{S_p\varphi(z)}, \end{aligned}$$

and the lemma is proved using the inductive hypothesis on $\mu(A_k)$. \square

Using the lemma, we may estimate

$$\begin{aligned} \frac{1}{\mu(U_r)} \log \mu(\tau_r > t) &= \frac{1}{\mu(U_r)} \log \left(1 - p\mu(\cup_{i=0}^k f^{-ip}(U_r)) \right) \\ &= \frac{1}{\mu(U_r)} \log \left(1 - \mu(U_r)p \left[k+1 - k(1 \pm \varepsilon)e^{S_p\varphi(z)} \right] \right) \\ &\xrightarrow{r \rightarrow 0} - \left[pk + p - pk(1 \pm \bar{C}\varepsilon)e^{S_p\varphi(z)} \right]. \end{aligned}$$

Now dividing by $-t$ and taking the limit as $t \rightarrow \infty$ completes the proof of the periodic case, up to an error $\pm \varepsilon e^{S_p\varphi(z)}$. Since ε was arbitrary, the case is proved for t of the form $(k+1)p - 1$. For more general $t = kp + \ell$, for some $\ell = 0, \dots, p-1$, we have

$$\mu(\tau_r > (k+1)p - 1) \leq \mu(\tau_r > t) \leq \mu(\tau_r > kp - 1),$$

and since the upper and lower bounds yield the same limit as $k \rightarrow \infty$, the limit for general t exists and has the same value.

2.5. Proof of Theorem 2.1: The case $\alpha \in (0, \infty)$. Fix $z \in I$, $\alpha \in (0, \infty)$, and a sequence of intervals $(U_r)_{r \in r_0}$ satisfying **(U)**. If z is periodic, we also assume **(P)**. Let $t = s\mu_0(U_r)^{-\alpha}$ for some $s \in \mathbb{R}^+$. We must consider the limit,

$$\lim_{r \rightarrow 0} \frac{1}{s\mu_0(U_r)^{1-\alpha}} \log \mu_0(\tau_r > s\mu_0(U_r)^{-\alpha}).$$

As in Section 2.1, there exists $r_1 > 0$ such that all associated transfer operators $\hat{\mathcal{L}}_r^\circ$ have a uniform spectral gap on \mathcal{B} for all $r \in [0, r_1]$.

To simplify notation, set $k_r = \lfloor s\mu_0(U_r)^{-\alpha} \rfloor$. Notice that,

$$\begin{aligned} \mu_0(\tau_r > k_r) &= \int_{I_r^{k_r}} g_0 dm_0 = \int \hat{\mathcal{L}}_r^{\circ k_r+1} g_0 dm_0 \\ &= \lambda_r^{k_r+1} \int \lambda_r^{-k_r-1} \hat{\mathcal{L}}_r^{\circ k_r+1} (g_0 - g_r) dm_0 + \lambda_r^{k_r+1} \int g_r dm_0, \end{aligned}$$

where g_r is the unique normalized conditionally invariant density corresponding to λ_r from Corollary 2.7. Thus

$$\log \mu_0(\tau_r > k_r) = (k_r + 1) \log \lambda_r + \log \left(1 + \int \lambda_r^{-k_r-1} \hat{\mathcal{L}}_r^{\circ k_r+1} (g_0 - g_r) dm_0 \right). \quad (2.8)$$

Notice that the first term above, when divided by $s\mu_0(U_r)^{1-\alpha}$, is simply $\mu_0(U_r)^{-1} \log \lambda_r$ (up to integer part) and thus converges as $r \rightarrow 0$ to the required limit by (2.5), which depends on z .

It remains to show that the second term in (2.8) converges to zero after division by $\mu_0(U_r)^{1-\alpha}$. Using Corollary 2.7, we may decompose the transfer operator as $\mathring{\mathcal{L}}_r = \lambda_r \Pi_r + R_r$, where as before, Π_r is the projection onto the eigenspace spanned by g_r and the spectral radius of R_r is strictly less than λ_r . Thus defining $\Pi_r g_0 = c_r g_r$ for some $c_r > 0$, we have

$$\lambda_r^{-k_r-1} \mathring{\mathcal{L}}_r^{k_r+1}(g_0 - g_r) = (c_r - 1)g_r + \lambda_r^{-k_r-1} R_r^{k_r+1} g_0, \quad (2.9)$$

where we have used the facts, $\Pi_r^2 = \Pi_r$, $\Pi_r g_r = g_r$ and $R_r g_r = 0$. Integrating, we have

$$\log \left(1 + \int \lambda_r^{-k_r-1} \mathring{\mathcal{L}}_r^{k_r+1}(g_0 - g_r) dm_0 \right) = \log \left(c_r + \int \lambda_r^{-k_r-1} R_r^{k_r+1} g_0 dm_0 \right).$$

Now since the operators $\mathring{\mathcal{L}}_r$ have a uniform spectral gap for r close to 0, there exists $\beta > 0$ such that the spectral radius of $\lambda_r^{-1} R_r$ in \mathcal{B} is less than $e^{-\beta}$ for all r sufficiently small. Since the variation norm dominates the L^∞ norm, we have,

$$\left| \int \lambda_r^{-k_r-1} R_r^{k_r+1} g_0 dm_0 \right| \leq \| \lambda_r^{-k_r-1} R_r^{k_r+1} g_0 \| \leq C e^{-\beta(k_r+1)} \leq C e^{-\beta s \mu_0(U_r)^{-\alpha}},$$

for some fixed $C > 0$, and this quantity is super-exponentially small in $\mu_0(U_r)$. Moreover, since by [KL1, Corollary 1], the spectral projectors Π_r of $\mathring{\mathcal{L}}_r$ vary by at most $-\mu_0(U_r) \log \mu_0(U_r)$ for small r and $\Pi_0 g_0 = g_0$, i.e., $c_0 = 1$, we have $|1 - c_r| \leq -C \mu_0(U_r) \log \mu_0(U_r)$, for some uniform $C > 0$.

Using these estimates in the second term of (2.8) and dividing by $s \mu_0(U_r)^{1-\alpha}$, the relevant expression becomes,

$$\lim_{r \rightarrow 0} \frac{1}{s \mu_0(U_r)^{1-\alpha}} \log \left(1 + \mathcal{O}(-\mu_0(U_r) \log \mu_0(U_r)) \right).$$

For $\alpha \geq 1$, it suffices to note that $\log(1 + \mathcal{O}(-\mu_0(U_r) \log \mu_0(U_r)))$ converges to 0 as $r \rightarrow 0$ to conclude that the above limit vanishes. For $\alpha \in (0, 1)$, we note that in addition $\frac{\mu_0(U_r) \log \mu_0(U_r)}{\mu_0(U_r)^{1-\alpha}} \rightarrow 0$ as $r \rightarrow 0$, which completes the proof of Theorem 2.1.

2.6. Examples. In this section, we provide examples of several classes of maps and potentials for which our assumptions (F1)-(F4) of Section 2 hold. More general examples, including the existence of a conformal measure for contracting potentials, can be constructed using [LSV1].

2.6.1. Lasota-Yorke maps of the interval with $\varphi = -\log |Df|$. Such maps are assumed to admit a finite partition \mathcal{Z} of I into intervals on which f is differentiable and $|Df| \geq \sigma^{-1} > 1$. f is assumed to be C^2 on the closure of each $Z \in \mathcal{Z}$.

The conformal measure m is Lebesgue measure on I , and (F1)-(F3) are standard consequences of uniform expansion, the existence of $D^2 f$ and the finiteness of the partition.

Since the potential is bounded, condition (F4) can be guaranteed by the equivalent condition that for each interval J , there exists $n(J)$, such that $f^{n(J)}(J) = I \bmod 0$.

Thus as long as we choose a sequence of holes satisfying **(U)**, i.e., guaranteeing large images for a fixed number of iterates, Theorem 2.1 holds for this class of maps.

2.6.2. *Mixing Gibbs-Markov maps with large images.* Assume that $f(Z)$ is a union of elements of \mathcal{Z} for each $Z \in \mathcal{Z}$, where \mathcal{Z} is the countable partition defined at the beginning of Section 2. Thus \mathcal{Z} is a Markov partition for f .

We assume that f satisfies the big images and pre-images⁴ (BIP) property: there exists a finite set $\{Z_j\}_{j \in \mathcal{J}} \subset \mathcal{Z}$ such that $\forall Z \in \mathcal{Z}, \exists j, k \in \mathcal{J}$ such that $f(Z_j) \supseteq Z$ and $f(Z) \supseteq Z_k$. We also assume that $|Df| \geq \sigma^{-1} > 1$ on each $Z \in \mathcal{Z}$.

We assume that φ is a potential which is Lipschitz continuous on each $Z \in \mathcal{Z}$, and admits a non-atomic conformal probability measure m_φ with $m_\varphi(I \setminus \cup_{Z \in \mathcal{Z}} Z) = 0$.

Then (F1) follows immediately from the regularity of φ and the expansion of f , and we have a Gibbs-Markov map. Condition (F2) follows from the existence of m_φ and (BIP) since by (F1), $\sup_Z e^\varphi \leq (1 + C_d)m_\varphi(Z)/m_\varphi(f(Z))$:

$$\sum_{Z \in \mathcal{Z}} \sup_Z e^\varphi \leq (1 + C_d) \sum_{Z \in \mathcal{Z}} \frac{m_\varphi(Z)}{m_\varphi(f(Z))} \leq (1 + C_d)c_0^{-1},$$

where $c_0 = \inf_{Z \in \mathcal{Z}} m_\varphi(f(Z)) > 0$ by (BIP).

(F4) follows from mixing plus (BIP). For (F3), we use the fact that for maps satisfying our assumptions, the transfer operator \mathcal{L}_φ acting on functions which are Lipschitz on each element of \mathcal{Z} is known to have a spectral gap. Since $\mathcal{L}_\varphi^n 1$ converges to an invariant density that is bounded away from 0 (by (F4)), the expression on the right side of (F3) is bounded away from 0 for all n large enough. On the other hand, the expression on the left side of (F3) must tend to 0 by conformality and (F1), since for $Z \in \mathcal{Z}^n$,

$$\sup_Z e^{S_n \varphi} \leq (1 + C_d) \frac{m_\varphi(Z)}{m_\varphi(f^n Z)} \leq (1 + C_d) \frac{m(Z)}{c_0},$$

and the diameter of \mathcal{Z}^n must tend to 0 by the expansivity of f .

Having verified (F1)-(F4), we may apply Theorem 2.1 to this class of Gibbs-Markov maps. Note that we can always arrange for **(U)** to be satisfied as long as we do not choose $\cap_r U_r$ to be an accumulation point of the endpoints of the intervals in \mathcal{Z}^{n_1} .

2.6.3. *Gauss map, $f(x) = 1/x \bmod 1$.* In this case, $\varphi = -\log |Df|$, m_φ is Lebesgue measure on $[0, 1]$, the invariant density is $g_0 = \frac{1}{\ln 2} \frac{1}{1+x}$, and f is continuously differentiable on each element of the partition $\mathcal{Z} = \{Z_j\}_{j=1}^\infty$, $Z_j = (1/(j+1), 1/j)$.

For this potential, (F1) fails. However this system is well known to satisfy Rychlik's conditions and moreover the potential satisfies the weaker (Hölder) distortion control given by the following lemma.

Lemma 2.9. $\exists C_d > 0$ s.t. $|e^{S_n \varphi(x) - S_n \varphi(y)} - 1| \leq C_d |f^n x - f^n y|^{1/2}$, whenever $f^i x, f^i y$ lie in the same element of \mathcal{Z} for $i = 0, 1, \dots, n-1$.

Before proving the lemma, we will verify the other conditions and show that in this case, Lemma 2.9 suffices to prove Proposition 2.5, so that the conclusions of Theorem 2.1 hold.

⁴This is automatic if f is full-branched.

(F2) is immediate since $\sup_{Z_j} e^\varphi \leq Cj^{-2}$, $\forall j \geq 1$.

Notice that $|e^\varphi|_\infty \leq 1$, while $|e^{S_{2^j}\varphi}|_\infty < 1$. Thus the expression on the left side of (F3) decreases exponentially in n , while $\mathcal{L}_\varphi^n 1$ converges to g_0 , which is bounded away from 0 on I . Thus (F3) holds.

(F4) holds since f is full-branched, and the potential satisfies the distortion control given by Lemma 2.9.

We verify also that the items of Lemma 2.4 hold: (a) holds by induction on (F2); (b) holds with $C_d = 1$ since e^φ is monotonic on each Z_j , so that $\bigvee_{Z_j} e^\varphi \leq \sup_{Z_j} e^\varphi$; (c) holds by induction on (b), using (a).

Next we show that the operators \mathcal{L}_r satisfy the uniform Lasota-Yorke inequalities of Proposition 2.5 under assumption **(U)**. The assumption (F1) is used in precisely two places in the proof of the proposition: in equations (A.2) and (A.3). For (A.2), the Hölder distortion control given by Lemma 2.9 suffices to give precisely the same bound. For (A.3), we use $\bigvee_{J_i} e^{S_{n_1}\varphi} \leq \sup_{J_i} e^{S_{n_1}\varphi}$ by the monotonicity of $e^{S_{n_1}\varphi}$ on each J_i .

With these estimates, the contracting term in (A.4) becomes $4|e^{S_{n_1}\varphi}|_\infty \bigvee_I \psi$ (it is the same expression, but with $C_d = 1$). Thus we need only choose n_1 such that $4|e^{S_{n_1}\varphi}|_\infty < 1$, replacing (2.1), in order to prove the required Lasota-Yorke inequalities under assumption **(U)**. Note that since f is full-branched, we can arrange for **(U)** to be satisfied as long as $\{z\} = \bigcap_r U_r$ is not chosen to be an endpoint of \mathcal{Z}^{n_1} .

Turning to the proof of Theorem 2.1, condition (F1) is used directly in one additional place: the proof of Lemma 2.8. In that case, using the Hölder bound given by Lemma 2.9, we need only replace $\text{diam}(U_r)$ by $\sqrt{\text{diam}(U_r)}$ and choose U_r sufficiently small that $\sqrt{\text{diam}(U_r)} < \varepsilon$. Then the rest of the proof of Lemma 2.6 goes through without changes.

With these minor changes to the proof, the conclusions of Theorem 2.1 apply to the Gauss map.

Proof of Lemma 2.9. Let x, y be as in the statement of the lemma, and let $f^i x \in Z_{j_i}$. The following bounds are elementary, yet essential to what follows,

$$\sup_{Z_j} \frac{|Df^2|}{|Df|} \leq Cj \quad \text{while} \quad \text{diam}(Z_j) \leq Cj^{-2}.$$

Using these estimates, one may complete the standard (Hölder) distortion estimate,

$$\begin{aligned} \left| \log \frac{Df^n(x)}{Df^n(y)} \right| &\leq \sum_{i=0}^{n-1} \left| \log |Df(f^i x)| - \log |Df(f^i y)| \right| \\ &\leq \sum_{i=0}^{n-1} \sup_{Z_{j_i}} \frac{|Df^2|}{|Df|} |f^i x - f^i y| \leq \sum_{i=0}^{n-1} C |f^i x - f^i y|^{1/2} \\ &\leq C |f^n x - f^n y|^{1/2} \sum_{i=0}^{n-1} |e^{S_{n-i}\varphi}|_\infty^{1/2}. \end{aligned}$$

The final sum converges exponentially in i because $|e^\varphi|_\infty \leq 1$ and $|e^{S_2\varphi}|_\infty < 1$. \square

3. RESULTS VIA INDUCING

In this section, we consider some cases in which the map $f : I \rightarrow I$ and potential φ do not satisfy (F1)-(F4) of Section 2. In such cases, a common strategy is to consider an induced map to a subset of I with stronger statistical properties. This is the situation we shall address in this section: explicit examples will be given in the following section.

We begin with a map $f : I \rightarrow I$, a conformal measure m_φ with potential φ , and an invariant probability measure μ_φ , absolutely continuous with respect to m_φ . We will fix the potential and simply denote this measure by μ in this section.

Fixing a sequence of sets $(U_r)_{r \in [0, r_0]}$, we assume that we can select an interval Y with $\mu(Y) > 0$ and $U_r \subset Y \subset I$, such that the first return map $F = f^{R_Y} : Y \rightarrow Y$ and the induced potential $\Phi = \sum_{i=0}^{R_Y-1} \varphi \circ f^i$ satisfy (F1)-(F4).

Let $\mu_Y := \frac{1}{\mu(Y)}\mu|_Y$ be the F -invariant probability measure, and $\tau_{Y,r}(y) = \min\{u \geq 1 : F^u(y) \in U_r\}$ be the first hitting time for the set U_r , which we sometimes refer to as the hole. Let $R_{Y,u}(y) = \sum_{i=0}^{u-1} R_Y \circ F^i(y)$ be the u^{th} return time to Y .

Remark 3.1. *We will assume for simplicity that the hole is always in Y . This is not much of a restriction because it is generically possible, once the location of the hole is known (and it is not at an indifferent fixed point or a recurrent critical point), to select a set Y with good return map containing the hole.*

For μ_Y -a.e. $y \in Y$, we have $R_{Y,u}/u \rightarrow 1/\mu(Y)$, but for our purposes we need specific estimates for the large deviations $\mu_Y(A_u)$ for the set

$$A_u = A_{Y,u,\varepsilon} := \{y \in Y : \exists n \geq u \text{ such that } |R_{Y,n} - n/\mu(Y)| > n\varepsilon\}.$$

Following (2.2), we define Y_{cont} to be the set of points in Y at which F^k is continuous for all $k \in \mathbb{N}$.

Theorem 3.2. *Suppose $f : I \rightarrow I$ is as above and there exists $Y \subset I$ with $z \in Y_{\text{cont}}$ such that the first return map $F = f^{R_Y} : Y \rightarrow Y$ satisfies the assumptions of Theorem 2.1.*

- (1) *If for any small $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that $\mu_Y(A_u) \leq e^{-c(\varepsilon)u}$ for all large u , then for each $\alpha \in (0, \infty]$,*

$$L_{\alpha,s}(z) = \begin{cases} 1, & \text{if } z \text{ is not periodic,} \\ 1 - e^{S_p\varphi(z)}, & \text{if } z \text{ is } p\text{-periodic for } f. \end{cases} \quad (3.1)$$

- (2) *If there exist $\gamma \in (0, 1)$ such that for any small $\varepsilon > 0$, there exist $C, c(\varepsilon) > 0$ such that $\mu_Y(A_u) \leq Ce^{-c(\varepsilon)u^\gamma}$ for all large u , then (3.1) holds for each $\alpha < \frac{1}{1-\gamma}$.*
- (3) *If there exists $\gamma \in (0, 1)$ and $C, c > 0$ such that $\mu_Y(R_Y \geq u) \geq Ce^{-cu^\gamma}$ for all large u , then $L_{\alpha,s}(z) = 0$ for $\alpha > \frac{1}{1-\gamma}$ and each $z \in I$.*

- (4) If both $\mu_Y(A_u)$ and $\mu_Y(R_Y \geq u)$ decay superpolynomially in u , but more slowly than any stretched exponential, then (3.1) holds if $\alpha \leq 1$ and $L_{\alpha,s}(z) = 0$ if $\alpha > 1$ for each $z \in I_{\text{cont}}$.

Remark 3.3. One expects, as in the examples of Section 4, that the decay of $\mu_Y(R_Y \geq u)$ matches that of $\mu_Y(A_u)$, so (2) and (3) in this theorem can be seen as complementary cases.

Remark 3.4. Theorem 3.2 excludes the case $\alpha = 0$ since as already noted in Remark 2.3, the limit holds in this case under general conditions which do not require a spectral gap. Thus it is not necessary to pass to the induced map F in this case; one simply needs to verify the conditions listed in Remark 2.3.

3.1. Proof of Theorem 3.2. We first prove the theorem for the quantity

$$L_{Y,\alpha,s}(z) := \lim_{r \rightarrow 0} \frac{-1}{s\mu(U_r)^{1-\alpha}} \log \mu_Y(\tau_r > s\mu(U_r)^{-\alpha}).$$

Proposition 3.5. Under the conditions of Theorem 3.2, all parts of the theorem hold with $L_{Y,\alpha,s}(z)$ replacing $L_{\alpha,s}(z)$. In particular, in cases (1) and (2),

$$L_{Y,\alpha,s}(z) = \begin{cases} 1, & \text{if } z \text{ is not periodic,} \\ 1 - e^{S_p\varphi(z)}, & \text{if } z \text{ is } p\text{-periodic for } f. \end{cases} \quad (3.2)$$

Proof. We will assume throughout that $\alpha \neq 1$ since this case is straightforward and proved elsewhere. Fix some small $\varepsilon > 0$, and assume that the hole U_r is contained inside one domain of F . For notational simplicity, here we will assume that the centre z of our U_r is non-periodic, but the periodic case is then immediate. We remark only that if z is periodic for f with period p and in the domain of F , then z is periodic for F with period $q \leq p$, and $e^{S_q\Phi(z)} = e^{S_p\varphi(z)}$.

If $y \in A_u^c$ and $t = u/\mu(Y)$, then $\tau_r(y) > t$ implies that $\tau_{Y,r}(y) > u/(1 + \varepsilon\mu(Y))$ and is implied by $\tau_{Y,r}(y) > u/(1 - \varepsilon\mu(Y))$. Since Theorem 2.1 applies to (F, Y, μ_Y) , there exist values $\theta^+(v, r), \theta^-(v, r)$ so that

$$\theta^-(v, r)e^{-v\mu_Y(U_r)^{1-\alpha}} \leq \mu_Y(\tau_{Y,r} > v\mu_Y(U_r)^{-\alpha}) \leq \theta^+(v, r)e^{-v\mu_Y(U_r)^{1-\alpha}},$$

where $\lim_{r \rightarrow 0} \frac{\log \theta^\pm(v, r)}{v\mu(U_r)^{1-\alpha}} = 0$.

We compute for the path $t = s\mu(U_r)^{-\alpha}$, so

$$u = s\mu(Y)\mu(U_r)^{-\alpha} = s\mu(Y)^{1-\alpha}\mu_Y(U_r)^{-\alpha}.$$

We write $\theta^-(r) = \theta^-(s(1 - \varepsilon\mu(Y))^{-1}\mu(Y)^{1-\alpha}, r)$ and $\theta^+(r) = \theta^+(s(1 + \varepsilon\mu(Y))^{-1}\mu(Y)^{1-\alpha}, r)$ to shorten notation. Also we abbreviate

$$G^\pm = \{y \in Y : \tau_{Y,r}(y) > s(1 \pm \varepsilon\mu(Y))^{-1}\mu(Y)^{1-\alpha}\mu_Y(U_r)^{-\alpha}\}.$$

First we bound $\mu_Y(\tau_r > t)$ from above, since $\{\tau_r > t \wedge A_u^c\} \subset G^+$,

$$\begin{aligned} \mu_Y(\tau_r > t) &= \mu_Y(\tau_r > t \wedge A_u^c) + \mu_Y(\tau_r > t \wedge A_u) \\ &\leq \mu_Y(G^+) + \mu_Y(A_u) \\ &\leq \theta^+(r) e^{-s(1+\varepsilon\mu(Y))^{-1}\mu(Y)^{1-\alpha}\mu(U_r)^{1-\alpha}} + \mu_Y(A_u) \\ &= \theta^+(r) e^{-s(1+\varepsilon\mu(Y))^{-1}\mu(U_r)^{1-\alpha}} + \mu_Y(A_u). \end{aligned} \tag{3.3}$$

Similarly, we bound $\mu_Y(\tau_r > t)$ from below, using G^- :

$$\begin{aligned} \mu_Y(\tau_r > t \wedge A_u^c) &\geq \mu_Y(\tau_{Y,r} > u(1 - \varepsilon\mu(Y))^{-1} \wedge A_u^c) \\ &\geq \mu_Y(\tau_{Y,r} > s(1 - \varepsilon\mu(Y))^{-1}\mu(Y)\mu(U_r)^{-\alpha}) - \mu_Y(G^- \wedge A_u) \\ &\geq \theta^-(r) e^{-s(1-\varepsilon\mu(Y))^{-1}\mu(U_r)^{1-\alpha}} - \mu_Y(G^- \wedge A_u), \end{aligned}$$

and therefore,

$$\begin{aligned} \mu_Y(\tau_r > t) &= \mu_Y(\tau_r > t \wedge A_u^c) + \mu_Y(\tau_r > t \wedge A_u) \\ &\geq \theta^-(r) e^{-s(1-\varepsilon\mu(Y))^{-1}\mu(U_r)^{1-\alpha}} + \mu_Y(\tau_r > t \wedge A_u) - \mu_Y(G^- \wedge A_u) \\ &\geq \theta^-(r) e^{-s(1-\varepsilon\mu(Y))^{-1}\mu(U_r)^{1-\alpha}} - \mu_Y(A_u). \end{aligned} \tag{3.4}$$

Now to find the limit in (3.2), we use first (3.3) to bound $L_{Y,\alpha,s}$ from below (taking a minus sign, so the inequality flips):

$$\begin{aligned} \frac{-\log \mu_Y(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} &\geq -\frac{\log \left(\theta^+(r) e^{-s(1+\varepsilon\mu(Y))^{-1}\mu(U_r)^{1-\alpha}} + \mu_Y(A_u) \right)}{s\mu(U_r)^{1-\alpha}} \\ &= -\frac{\log \theta^+(r)}{s\mu(U_r)^{1-\alpha}} + \frac{1}{1 + \varepsilon\mu(Y)} - \frac{\log \left(1 + \frac{e^{s(1+\varepsilon\mu(Y))^{-1}\mu(U_r)^{1-\alpha}}}{\theta^+(r)} \mu_Y(A_u) \right)}{s\mu(U_r)^{1-\alpha}}. \end{aligned} \tag{3.5}$$

The first term converges to zero as $r \rightarrow 0$ by assumption, so we focus on the final term.

Case I: $\alpha \in (0, 1)$. In this case since $\theta^+(r) = O(e^{s\mu(U_r)^{1-\alpha+\delta}})$ for any $\delta > 0$, and $\mu(U_r)^{1-\alpha} \rightarrow 0$ as $r \rightarrow 0$, hence $\frac{e^{-s(1+\varepsilon\mu(Y))^{-1}\mu(U_r)^{1-\alpha}}}{\theta^+(r)} = O(1)$ and we see that the final term of (3.5) is of order $\frac{\mu_Y(A_u)}{\mu(U_r)^{1-\alpha}}$. Assuming that $\mu_Y(A_u) \leq Cu^{-\beta}$ for some $C, \beta > 0$, we have,

$$\mu_Y(A_u) \leq C(s\mu(Y)\mu(U_r)^{-\alpha})^{-\beta},$$

so that $L_{Y,\alpha,s}(z) \geq 1/(1 + \varepsilon\mu(Y))$ if $\alpha - 1 + \alpha\beta > 0$, i.e., $\alpha > \frac{1}{1+\beta}$. So the lower bound for the non-degenerate part (i.e., the “ $\alpha \leq 1$ ” part) of (4) follows along with (2) and (1) for the $\alpha \in (0, 1)$ case since ε was arbitrary. The upper bound follows immediately from Remark 1.1.

Case II: $\alpha \in (1, \infty)$. Here we focus on the stretched exponential case since all remaining parts of this proposition then follow. To complete the proof of (2), we again refer to (3.5). Suppose that there exist $C, c(\varepsilon) > 0$ and $\gamma \in (0, 1)$ such that $\mu_Y(A_u) \leq Ce^{-c(\varepsilon)u^\gamma}$. Then for (3.2) to hold it is sufficient that the decay of $\mu_Y(A_u)$, which is $Ce^{-c(s\mu(U_r)^{-\alpha})^\gamma}$, is faster than

$e^{-s(1+\varepsilon\mu(Y))^{-1}\mu(U_r)^{1-\alpha}}$. So we require that $\alpha < \frac{1}{1-\gamma}$. The upper bound follows similarly, using (3.4) in place of (3.3), completing (2).

To prove (3) and the degenerate (i.e., “ $\alpha > 1$ ”) part of (4), we assume that there exist $C, c > 0$, $\gamma \in (0, 1)$ such that $\mu_Y(R_Y \geq t) \geq Ce^{-ct^\gamma}$. Fix $\alpha > \frac{1}{1-\gamma}$. Then using the fact that $\{\tau_r > t\} \supset \{R_Y \geq t\}$, we estimate

$$\begin{aligned} \frac{-\log \mu_Y(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} &\leq \frac{-\log \mu_Y(R_Y \geq t)}{s\mu(U_r)^{1-\alpha}} \leq \frac{-\log(Ce^{-ct^\gamma})}{s\mu(U_r)^{1-\alpha}} \\ &\leq \frac{-\log C}{s\mu(U_r)^{1-\alpha}} + \frac{cs^\gamma \mu(U_r)^{-\alpha\gamma}}{s\mu(U_r)^{1-\alpha}} \end{aligned}$$

and both terms tend to 0 with r since $\alpha > 1/(1-\gamma)$. Note that this estimate easily extends from the measure μ_Y to the measure μ , so that $L_{\alpha,s}(z) = 0$ for all $z \in I$.

Case III: $\alpha = \infty$. For this case, we compute first the limit $t \rightarrow \infty$ and then $r \rightarrow 0$ in the expression given by (1.4).

Fix $\varepsilon > 0$ and define A_u as before. In analogy to the previous two cases, set

$$G^\pm = \{y \in Y : \tau_{Y,r}(y) > u/(1 \pm \varepsilon\mu(Y))\},$$

where $u = t\mu(Y)$. Notice then that $\{\tau_r > t \wedge A_u^c\} \subset G^+$ as before. Thus as in (3.3),

$$\mu_Y(\tau_r > t) \leq \mu_Y(G^+) + \mu_Y(A_u).$$

Following (3.4), we obtain,

$$\mu_Y(\tau_r > t) \geq \mu_Y(G^-) - \mu_Y(A_u).$$

To prove the exponential case (1), assume that there exists $c(\varepsilon) > 0$ such that $\mu_Y(A_u) \leq Ce^{-c(\varepsilon)u}$. Since F satisfies the assumptions of Theorem 2.1, we only consider r so small such that all associated transfer operators \mathcal{L}_r have a uniform spectral gap. Let Λ_r denote the leading eigenvalue of $\mathring{\mathcal{L}}_r$ and choose r_1 so small $\Lambda_r^{(1-\varepsilon\mu(Y))^{-1}} > e^{-c(\varepsilon)}$ for all $r < r_1$. By Corollary 2.7,

$$C^{-1}\Lambda_r^{u(1\pm\varepsilon\mu(Y))^{-1}} \leq \mu_Y(G^\pm) \leq C\Lambda_r^{u(1\pm\varepsilon\mu(Y))^{-1}},$$

for some $C > 0$, independent of t , but possibly depending on r .

Thus on the one hand we derive a lower bound,

$$\begin{aligned} \lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu_Y(\tau_r > t) &\geq \lim_{t \rightarrow \infty} -\frac{1}{t} \log \left(C\Lambda_r^{t\mu(Y)/(1+\varepsilon\mu(Y))} + Ce^{-c(\varepsilon)t\mu(Y)} \right) \\ &= \frac{-\mu(Y) \log \Lambda_r}{1 + \varepsilon\mu(Y)}. \end{aligned}$$

On the other hand, the analogous upper bound holds,

$$\begin{aligned} \lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu_Y(\tau_r > t) &\leq \lim_{t \rightarrow \infty} -\frac{1}{t} \log \left(C^{-1}\Lambda_r^{t\mu(Y)/(1-\varepsilon\mu(Y))} - Ce^{-c(\varepsilon)t\mu(Y)} \right) \\ &= \frac{-\mu(Y) \log \Lambda_r}{1 - \varepsilon\mu(Y)}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this yields

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu_Y(\tau_r > t) = \lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu(\tau_r > t \wedge Y) = -\mu(Y) \log \Lambda_r. \quad (3.6)$$

Now using (2.5) applied to the induced map F , we conclude

$$\lim_{r \rightarrow 0} \lim_{t \rightarrow \infty} -\frac{1}{t\mu(U_r)} \log \mu_Y(\tau_r > t) = \lim_{r \rightarrow 0} \frac{-\log \Lambda_r}{\mu_Y(U_r)} = 1$$

in the generic case, and $1 - e^{S_p \varphi(z)}$ in the periodic case.

For the remaining items (2)-(4) of the proposition, it suffices to show that $L_{Y,\alpha,s}(z) = 0$ when $\alpha = \infty$ under the assumption that $\mu_Y(R_Y \geq t) \geq Ce^{-ct^\gamma}$ for some $\gamma \in (0, 1)$. This is a trivial estimate since in this case the escape rate is 0, i.e.,

$$0 \leq \lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu_Y(\tau_r > t) \leq \lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu_Y(R_Y \geq t) \leq \lim_{t \rightarrow \infty} ct^{\gamma-1} = 0.$$

It follows immediately that $L_{Y,\alpha,s}(z) = 0$ for all $z \in Y$. \square

Proof of Theorem 3.2. We will apply Proposition 3.5 to convert the results for $L_{Y,\alpha,s}$ to $L_{\alpha,s}$. For this, we turn to an extended system implied by the existence of the first return map F . We will refer to this as a Rokhlin tower (our map F defines what is nearly a Young tower, see [Y], except that we do not require that F have a Markov structure). Define

$$\Delta = \{(y, n) \in Y \times \mathbb{N} : n < R_Y(y)\}.$$

The ℓ th level of the tower is $\Delta_\ell = \{(y, n) \in \Delta : n = \ell\}$ and the dynamics is defined by $f_\Delta(y, n) = (y, n+1)$ if $n < R_Y(y) - 1$ and $f_\Delta(y, R_Y(y) - 1) = (F(y), 0)$. The first return map to the base of the tower $\Delta_0 = Y$ is again $F = f^{R_Y}$.

The assumptions on F imply that μ_Y is an invariant probability measure on $Y = \Delta_0$, which induces an f_Δ -invariant probability measure μ_Δ on Δ : Define $\mu_\Delta|_{\Delta_\ell} = c(f_\Delta)_* \mu_Y|_{f_\Delta^{-\ell}(\Delta_\ell)}$, where $c = \mu_Y(R_Y)^{-1}$ is the normalizing constant. Letting $\pi : \Delta \rightarrow I$ denote the natural projection, $\pi(x, \ell) = f^\ell(x)$, we have $\pi \circ f_\Delta = f \circ \pi$ and $\pi_* \mu_\Delta = \mu$.

Letting $\Delta^{(n)} = \cup_{\ell=0}^n \Delta_\ell$ denote the n first levels of the tower, we observe that this gives us a sequence of induced maps $F_n : \Delta^{(n)} \rightarrow \Delta^{(n)}$ each satisfying the conditions of Theorem 2.1. (In fact, using the assumption on F , the potential for $(F_n)^n$ is contracting.) The projection $\pi(\Delta^{(n)})$ gives a sequence of sets exhausting the space: $\mu(\pi(\Delta^{(n)})) \rightarrow 1$ as $n \rightarrow \infty$. We will carry out the proof for $\pi(\Delta^{(n)})$ in place of Y , but calling it Y again and suppressing the index n .

Case I: $\alpha \in (0, 1)$. We will use the facts

- (a) $\mu(\tau_r \leq t \wedge Y) \leq \mu(\tau_r \leq t)$
- (b) given $\gamma \in \mathbb{R}$, for x small, $\log(1 + \gamma x) \sim \gamma \log(1 + x)$

Then for $\alpha \in (0, 1)$, $t = s\mu(U_r)^{-\alpha}$,

$$\begin{aligned} -\frac{\log \mu_Y(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} &= -\frac{\log(1 - \mu_Y(\tau_r \leq t))}{s\mu(U_r)^{1-\alpha}} = -\frac{\log(1 - \mu(Y)^{-1}\mu(\tau_r \leq t \wedge Y))}{s\mu(U_r)^{1-\alpha}} \\ &\leq -\frac{\log(1 - \mu(Y)^{-1}\mu(\tau_r \leq t))}{s\mu(U_r)^{1-\alpha}} \sim -\frac{1}{\mu(Y)} \frac{\log(1 - \mu(\tau_r \leq t))}{s\mu(U_r)^{1-\alpha}} \\ &= -\frac{1}{\mu(Y)} \frac{\log \mu(\tau_r > t)}{s\mu(U_r)^{1-\alpha}}. \end{aligned}$$

Here we used (a) in the ‘ \leq step’ and (b) in the ‘ \sim step’. So choosing $Y = \pi(\Delta^{(n)})$ exhausting our phase space, we deduce

$$\liminf_{r \rightarrow 0} \frac{-\log \mu(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} \geq \begin{cases} 1, & \text{if } z \text{ is not periodic,} \\ 1 - e^{S_p \varphi(z)}, & \text{if } z \text{ is } p\text{-periodic for } f. \end{cases} \quad (3.7)$$

For non-periodic z , Remark 1.1 gives the upper bound as 1 too, so $L_{\alpha,s}(z) = 1$.

For the periodic case we adapt Remark 1.1 and use a result of [FFT2]. First recall $V_r := U_r \cap f^{-p}(U_r)$ from the proof of Lemma 2.8 and let $V'_r := U_r \setminus f^{-p}(U_r)$. For all small r , this will be a topological annulus around z . By conditions **(P)** and (F1) (see (2.7)),

$$\lim_{r \rightarrow 0} \frac{\mu(V'_r)}{\mu(U_r)} = \lim_{r \rightarrow 0} \frac{m_\varphi(V'_r)}{m_\varphi(U_r)} = 1 - e^{S_p \varphi(z)}. \quad (3.8)$$

We set $\tau'_r := \inf\{n \geq 1 : f^n(x) \in V'_r\}$. Now [FFT2, Proposition 2.7] (with $B = U_r$ and $A = V'_r$) implies that

$$\mu(\tau'_r > n) - \mu(\tau_r > n) \leq \sum_{j=1}^p \mu(\tau'_r > n \wedge f^{-n+j}(V_r)) \leq p\mu(V_r) < p\mu(U_r)$$

for all large n . So we now proceed as in Remark 1.1:

$$\begin{aligned} 0 &\leq \frac{-\log \mu(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} < \frac{-\log(\mu(\tau'_r > t) - p\mu(U_r))}{s\mu(U_r)^{1-\alpha}} \\ &= \frac{-\log(1 - \mu(\tau'_r \leq t) - p\mu(U_r))}{s\mu(U_r)^{1-\alpha}} = \frac{-\log\left(1 - \mu\left(\bigcup_{j=0}^{t-1} f^{-j}(V'_r)\right) - p\mu(U_r)\right)}{s\mu(U_r)^{1-\alpha}} \\ &\leq \frac{-\log(1 - t\mu(V'_r) - p\mu(U_r))}{s\mu(U_r)^{1-\alpha}} = \frac{-\log(1 - s\mu(V'_r)\mu(U_r)^{-\alpha} - p\mu(U_r))}{s\mu(U_r)^{1-\alpha}}. \end{aligned}$$

So by (3.8), the upper bound above converges to $1 - e^{S_p \varphi(z)}$ as $\mu(U_r) \rightarrow 0$, so we conclude that $L_{\alpha,s}(z) = 1 - e^{S_p \varphi(z)}$.

Case II: $\alpha \in (1, \infty)$.

For $\alpha > 1$, we obtain the following upper bound:

$$\begin{aligned} -\frac{\log \mu_Y(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} &= \frac{\log \mu(Y)}{s\mu(U_r)^{1-\alpha}} - \frac{\log \mu(\tau_r > t \wedge Y)}{s\mu(U_r)^{1-\alpha}} \\ &\sim -\frac{\log \mu(\tau_r > t \wedge Y)}{s\mu(U_r)^{1-\alpha}} \geq -\frac{\log \mu(\tau_r > t)}{s\mu(U_r)^{1-\alpha}}. \end{aligned}$$

So we conclude that $L_{\alpha,s}(z) \leq L_{Y,\alpha,s}(z)$. Note the above shows $L_{Y,\alpha,s}(z) = 0$ implies $L_{\alpha,s}(z) = 0$ so that items (3) and (4) of the theorem hold for $\alpha > 1$.

To prove items (1) and (2) of the theorem, we also need a lower bound on $L_{\alpha,s}(z)$. For this, recall that the measure μ can be expressed in terms of μ_Y by,

$$\mu(A) = \frac{1}{\int R_Y d\mu_Y} \sum_{k=0}^{\infty} \sum_{i=0}^k \mu_Y(f^{-i}(A) \cap Y_k),$$

where $Y_k = \{R_Y = k\}$, and A is any measurable set. Applying this expression to $A = \{\tau_r > t\}$, we note that $f^{-i}(\tau_r > t) \cap Y_k = \{\tau_r > t + i\} \cap Y_k$ since $U_r \subset Y$. Then reversing order of summation, we obtain,

$$\mu(\tau_r > t) = \frac{1}{\int R_Y d\mu_Y} \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \mu_Y(\tau_r > t + i \wedge Y_k) \leq \mu(Y) \sum_{i=0}^{\infty} \mu_Y(\tau_r > t + i). \quad (3.9)$$

To proceed, we prove a slight extension of our estimates in Section 2.5. Let $\mathring{\mathcal{L}}_r$ denote the punctured transfer operator for F with potential $\Phi = S_{R_Y}\varphi$ and hole U_r as defined in (2.3). By assumption on F and Corollary 2.7, $\mathring{\mathcal{L}}_r = \Lambda_r \Pi_r + R_r$ has a uniform spectral gap, i.e., there exists $\beta > 0$ such that the spectral radius of $\Lambda_r^{-1}R_r$ is less than $e^{-\beta}$ for all r sufficiently small.

Lemma 3.6. *For all $r > 0$ sufficiently small and any $n \in \mathbb{N}$ such that $e^{-\beta n} < \mu_Y(U_r) \log \mu_Y(U_r)$, we have*

$$\mu_Y(\tau_{Y,r} > n) = \Lambda_r^n [1 + \mathcal{O}(\mu_Y(U_r) \log \mu_Y(U_r))].$$

Proof. Noting that (2.8) is valid for all iterates of F , we write

$$\mu(\tau_{Y,r} > n) = \Lambda_r^n \left[1 + \int_Y \Lambda_r^{-n} \mathring{\mathcal{L}}_r^n (g_0 - g_r) dm \right],$$

where g_0 and g_r are the normalized eigenfunctions for \mathcal{L}_0 and $\mathring{\mathcal{L}}_r$, respectively. Following (2.9), we note that the error term above can be split into two terms, one bounded by $Ce^{-\beta n}$ and the other by $-C\mu_Y(U_r) \log \mu_Y(U_r)$. By assumption on n , the error is of order $\mu_Y(U_r) \log \mu_Y(U_r)$. \square

Now fix $\varepsilon > 0$ and define $A_u = A_{Y,u,\varepsilon}$ as before. Recall that if $\tau_r(y) > n$ and $y \in A_{n\mu(Y)}^c$, then $\tau_{Y,r} > n\mu(Y)/(1 + \varepsilon\mu(Y))$. We assume that there exist $C, c(\varepsilon), \gamma > 0$ such that $\mu_Y(A_u) \leq Ce^{-c(\varepsilon)u^\gamma}$, and require that $\alpha < \frac{1}{1-\gamma}$.

For the sake of brevity, set $\vartheta = L_{Y,\alpha,s}(z)$, and by (2.5), we may choose r sufficiently small so that $\Lambda_r \leq e^{-(1-\varepsilon)\mu_Y(U_r)\vartheta}$. Setting $n = t + i$, $\rho_r = \mu_Y(U_r) \log \mu_Y(U_r)$, and using Lemma 3.6, we estimate each term in (3.9) by

$$\begin{aligned} \mu_Y(\tau_r > t + i) &\leq \mu_Y(\tau_r > t + i \wedge A_{(t+i)\mu(Y)}^c) + \mu_Y(A_{(t+i)\mu(Y)}) \\ &\leq \mu_Y(\tau_{Y,r} > (t+i)\mu(Y)/(1 + \varepsilon\mu(Y))) + \mu_Y(A_{(t+i)\mu(Y)}) \\ &\leq \Lambda_r^{(t+i)\mu(Y)/(1 + \varepsilon\mu(Y))} [1 + \mathcal{O}(\rho_r)] + \mu_Y(A_{(t+i)\mu(Y)}) \\ &\leq e^{-(1-\varepsilon)\mu(U_r)\vartheta(t+i)/(1 + \varepsilon\mu(Y))} [1 + \mathcal{O}(\rho_r)] + Ce^{-c(\varepsilon)(t+i)^\gamma \mu(Y)^\gamma}. \end{aligned}$$

To estimate (3.9), we must sum both terms above over i . Recalling that $t = s\mu(U_r)^{-\alpha}$, the sum over the first term is bounded by,

$$\begin{aligned} \sum_{i \geq 0} e^{-(1-\varepsilon)\mu(U_r)\vartheta(t+i)/(1+\varepsilon\mu(Y))} [1 + \mathcal{O}(\rho_r)] &= \frac{[1 + \mathcal{O}(\rho_r)]e^{-(1-\varepsilon)\vartheta s\mu(U_r)^{1-\alpha}/(1+\varepsilon\mu(Y))}}{1 - e^{-(1-\varepsilon)\vartheta\mu(U_r)/(1+\varepsilon\mu(Y))}} \\ &\leq \frac{2e^{-(1-\varepsilon)\vartheta s\mu(U_r)^{1-\alpha}/(1+\varepsilon\mu(Y))}}{(1-\varepsilon)\vartheta\mu(U_r)}, \end{aligned}$$

for ε and r sufficiently small. The sum over the second term is (recalling that $c = c(\varepsilon)$),

$$\sum_{i \geq 0} C e^{-c(t+i)^\gamma \mu(Y)^\gamma} \leq C \int_0^\infty e^{-c(t+x)^\gamma \mu(Y)^\gamma} dx = \frac{C}{c^{1/\gamma} \mu(Y)^\gamma} \int_{ct^\gamma \mu(Y)^\gamma}^\infty e^{-y} y^{\frac{1}{\gamma}-1} dy,$$

where we have changed variables, $y = c(t+x)^\gamma \mu(Y)^\gamma$. Setting $n = \lceil \frac{1}{\gamma} - 1 \rceil$, we have $y^{\frac{1}{\gamma}-1} \leq y^n$, so making this substitution and integrating by parts n times, yields

$$\sum_{i \geq 0} C e^{-c(t+i)^\gamma \mu(Y)^\gamma} \leq \frac{C e^{-ct^\gamma \mu(Y)^\gamma}}{c^{1/\gamma} \mu(Y)^\gamma} \sum_{k=0}^n \frac{n!}{k!} (ct^\gamma \mu(Y)^\gamma)^{n-k} \leq \frac{e C n! t e^{-ct^\gamma \mu(Y)^\gamma}}{\gamma c^{\frac{1}{\gamma}-n} \mu(Y)^{1-\gamma n}}.$$

Putting these estimates together with (3.9), we have,

$$\mu(\tau_r > t) \leq \mu(Y) \left[\frac{2e^{-(1-\varepsilon)\vartheta s\mu(U_r)^{1-\alpha}/(1+\varepsilon\mu(Y))}}{(1-\varepsilon)\vartheta\mu(U_r)} + C' t e^{-c(s\mu(Y))^\gamma \mu(U_r)^{-\alpha\gamma}} \right],$$

so that

$$-\log \mu(\tau_r > t) \geq \log \left(\frac{(1-\varepsilon)\vartheta\mu(U_r)}{2\mu(Y)} \right) + \frac{(1-\varepsilon)\vartheta s\mu(U_r)^{1-\alpha}}{1+\varepsilon\mu(Y)} - \log[1 + B_r],$$

where

$$B_r = \frac{C' s\mu(U_r)^{1-\alpha} (1-\varepsilon)\vartheta}{2} e^{-c(s\mu(Y))^\gamma \mu(U_r)^{-\alpha\gamma} + (1-\varepsilon)\vartheta s\mu(U_r)^{1-\alpha}/(1+\varepsilon\mu(Y))}.$$

Note that $B_r \rightarrow 0$ as $r \rightarrow 0$ since $\alpha < \frac{1}{1-\gamma}$. Thus dividing by $s\mu(U_r)^{1-\alpha}$ and recalling that $\alpha > 1$, we have,

$$\lim_{r \rightarrow 0} \frac{-\log \mu(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} \geq \vartheta \frac{1-\varepsilon}{1+\varepsilon\mu(Y)},$$

which is the required lower bound since $\varepsilon > 0$ was arbitrary. Thus $L_{\alpha,s}(z) = L_{Y,\alpha,s}(z)$ and items (1) and (2) of the theorem are proved for this case.

Case III: $\alpha = \infty$. First we note that an upper bound similar to the one derived in Case II holds:

$$\begin{aligned} \lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu_Y(\tau_r > t) &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \mu(Y) - \frac{1}{t} \log \mu(\tau_r > t \wedge Y) \\ &\geq \lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu(\tau_r > t). \end{aligned} \tag{3.10}$$

To prove items (2)-(4) of the Theorem, we must show $L_{\infty,s}(z) = 0$ under the assumption $\mu_Y(R_Y \geq t) \geq C e^{ct^\gamma}$ for some $\gamma \in (0, 1)$. This follows from Case III in the proof of Proposition 3.5 since then $L_{Y,\infty,s}(z) = 0$. Due to the upper bound above, $L_{\infty,s}(z) = 0$ as well.

To prove item (1) of the theorem, fix $\varepsilon > 0$ and assume there exists $c(\varepsilon) > 0$ such that $\mu_Y(A_u) \leq C e^{-c(\varepsilon)u}$. As in Case II, we take r_0 so small that the transfer operators \mathring{L}_r associated with the induced map F have a uniform spectral gap for all $r \in [0, r_0]$ and denote their leading eigenvalues by Λ_r . Using (2.5), we choose $r_1 < r_0$ so small that $e^{-(1-\varepsilon)\mu_Y(U_r)\vartheta} \geq \Lambda_r \geq e^{-c(\varepsilon)(1+\varepsilon\mu(Y))/2}$ for all $r < r_1$.

By (3.6) in the proof of Proposition 3.5(1), and (3.10), we have

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu(\tau_r > t) \leq -\mu(Y) \log \Lambda_r$$

To prove the corresponding lower bound, we follow (3.9) and the estimates in Case II of the present proof (with $\gamma = 1$). In particular, using Lemma 3.6,

$$\begin{aligned} \mu_Y(\tau_r > t + i) &\leq \mu_Y(\tau_r > t + i \wedge A_{(t+i)\mu(Y)}^c) + \mu_Y(A_{(t+i)\mu(Y)}) \\ &\leq \Lambda_r^{(t+i)\mu(Y)/(1+\varepsilon\mu(Y))} [1 + \mathcal{O}(\rho_r)] + C e^{-c(\varepsilon)(t+i)\mu(Y)}. \end{aligned}$$

Summing over i , we obtain

$$\mu(\tau_r > t) \leq \mu(Y) \left[\frac{2\Lambda_r^{t\mu(Y)/(1+\varepsilon\mu(Y))}}{(1-\varepsilon)\vartheta\mu(U_r)} + C e^{-c\mu(Y)t} \right].$$

And finally,

$$-\log \mu(\tau_r > t) \geq \log \frac{(1-\varepsilon)\vartheta\mu(U_r)}{2\mu(Y)} - \frac{t\mu(Y) \log \Lambda_r}{1+\varepsilon\mu(Y)} - \log[1 + B_r],$$

where $B_r \leq \frac{C(1-\varepsilon)\vartheta\mu(U_r)}{2} e^{-c\mu(Y)t/2}$, by choice of r_1 . Now dividing by t and taking $t \rightarrow \infty$ yields

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \log \mu(\tau_r > t) \geq -\frac{\mu(Y) \log \Lambda_r}{1+\varepsilon\mu(Y)}.$$

Since $\varepsilon > 0$ was arbitrary, our upper and lower bounds match. Thus using again (2.5), we have

$$\lim_{r \rightarrow 0} \lim_{t \rightarrow \infty} \frac{-\log \mu(\tau_r > t)}{t\mu(U_r)} = \lim_{r \rightarrow 0} \frac{-\mu(Y) \log \Lambda_r}{\mu(U_r)} = \lim_{r \rightarrow 0} \frac{-\log \Lambda_r}{\mu_Y(U_r)} = 1$$

in the generic case, and $1 - e^{S_p \varphi(z)}$ in the periodic case. Thus $L_{\infty, s}(z) = L_{Y, \infty, s}(z)$ as required. \square

3.2. Remarks about the Polynomial Case. Theorem 3.2 gives optimal results when the induced system has deviations that are superpolynomial and when the decay rate of A_u matches that of $\{R_Y \geq u\}$. However, it gives only partial results if the induced system has only polynomial deviations, i.e., $\mu_Y(A_u) \approx u^{-\beta}$ and $\mu_Y(R_Y \geq u) \geq u^{-\beta-1}$. In particular, the proofs of Proposition 3.5(4) and Theorem 3.2(4) yields in the generic case,

$$L_{\alpha, s}(z) = L_{Y, \alpha, s}(z) = 0 \quad \text{if } \alpha > 1, \quad \text{and } L_{\alpha, s}(z) = L_{Y, \alpha, s}(z) = 1 \quad \text{if } \frac{1}{1+\beta} < \alpha \leq 1.$$

It might appear that by improving our upper and lower bounds in (3.3) and (3.4), we might extend our results to the case $\alpha \leq 1/(1+\beta)$, but a closer look reveals there is a real obstruction to using inducing arguments to evaluate the required limits in the polynomial case. In particular, there is a nontrivial dependence between the sets $\{\tau_r > t\}$ and A_u which makes the polynomial case particularly delicate from this point of view.

To illustrate this point, consider the class of Manneville-Pomeau or LSV maps on the unit interval, defined by

$$f(x) = \begin{cases} x + 2^\gamma x^{1+\gamma}, & \text{for } x \in [0, 1/2), \\ 2x - 1, & \text{for } x \in [1/2, 1]. \end{cases}$$

When $\gamma \in (0, 1)$, these maps preserve an invariant probability measure μ , absolutely continuous with respect to Lebesgue, with density $g \approx x^{-\gamma}$ for x near 0 [Y, LSV2].

Set $Y = [1/2, 1]$ and let $(U_r)_{r \in [0, r_0]} \subset (1/2 + \delta, 1)$, for some $\delta > 0$. For $k \geq 0$, let $a_k = f_L^{-k}(1/2)$, where f_L is the left branch of f . Set $J_0 = Y$ and $J_k = [a_k, a_{k-1})$ for $k \geq 1$. Note that $\tau_Y = k + 1$ on $f_R^{-1}(J_k)$, where f_R is the right branch of f .

We claim $\{\tau_r > t \wedge A_u\} \supseteq \bigcup_{k \geq t} f_R^{-1}(J_k)$. Note that $\bigcup_{k \geq t} f_R^{-1}(J_k) = \{R_Y > t\}$, and that $\{\tau_r > t\} \supset \{R_Y > t\}$ since $U_r \subset Y$. Moreover, if $R_Y(x) > t$, then $\tau_{Y,u}(x) > u - 1 + t$, and for $u = \mu(Y)t$, we have

$$\frac{1}{u} \tau_{Y,u}(x) > 1 - \frac{1}{u} + \frac{1}{\mu(Y)} \implies \frac{1}{u} \tau_{Y,u}(x) - \frac{1}{\mu(Y)} > 1 - \frac{1}{u},$$

so that $x \in A_u$ for all $u \geq 2$ and $\varepsilon < 1/2$. Thus $\{R_Y > t\} \subset A_u$, completing the proof of the claim.

Using well-known estimates [LSV2] on the spacing of a_k , $a_k \approx k^{-1/\gamma}$,

$$\mu_Y(\tau_r > t \wedge A_u) \geq ct^{-1/\gamma} = cs^{-1/\gamma} \mu(U_r)^{\alpha/\gamma}, \quad (3.11)$$

for some uniform constant $c > 0$, where as usual we have set $t = s\mu(U_r)^{-\alpha}$. Using this lower bound, we may split up the relevant expression in the limit defining $L_{Y,\alpha,s}$ as follows,

$$\frac{-\log \mu_Y(\tau_r > t)}{s\mu(U_r)^{1-\alpha}} = \frac{-\log \mu_Y(\tau_r > t \wedge A_u^c)}{s\mu(U_r)^{1-\alpha}} - \frac{\log[1 + \frac{\mu_Y(\tau_r > t \wedge A_u)}{\mu_Y(\tau_r > t \wedge A_u^c)}]}{s\mu(U_r)^{1-\alpha}}. \quad (3.12)$$

To use the results for the induced map, one would expect that the first term above tends to the desired limit, while the second term above acts as an error term and tends to 0 as $r \rightarrow 0$. However, using (3.11), we see that the ‘error’ term is bounded below by

$$\frac{\mu_Y(\tau_r > t \wedge A_u)}{s\mu(U_r)^{1-\alpha}} \geq c' \mu(U_r)^{-1+\alpha+\alpha/\gamma} \xrightarrow{r \rightarrow 0} \infty,$$

whenever $\alpha < \gamma/(1 + \gamma)$.

By Remark 1.1, we know all limit points of $L_{Y,\alpha,s}$ lie in $[0, 1]$, so in the range $\alpha < \gamma/(1 + \gamma)$, the limit relies on cancellation between two diverging terms in (3.12). This implies that what we would like to consider to be an error term does not function as one for small α .

4. APPLICATIONS OF INDUCING

Theorem 3.2 applies whenever we have a system (I, f, μ) with an inducing scheme (X, F, ν) where $F = f^\tau$ and τ is the first return time to X where, moreover, $\nu(A_u)$ is known to satisfy a suitable large deviations principle. At present such large deviations principles are known in quite specific cases. We mention several examples here.

4.1. Generalized Farey maps. In the i.i.d. case it has been shown that the large deviation rate of an unbounded observable ψ matches the tail of the observable. For example for $\bar{\psi} = \int \psi d\nu$, $\gamma \in (0, 1)$ and $c > 0$,

$$\nu(\psi > n) \leq ce^{-n^\gamma} \implies \lim_{n \rightarrow \infty} \frac{1}{n^\gamma} \log \nu(S_n \psi > \varepsilon + \bar{\psi}) = -\varepsilon^\gamma,$$

where $S_n \psi$ is the n -th ergodic sum of these observables, see [GRR]. Similarly, if the tail of an observable is polynomial of order β , then the deviations are polynomial of order $\beta - 1$; and for exponential, the orders match exactly [G].

An application of Theorem 3.2 is to generalized Farey maps as in [KMS]. Here one chooses a countable partition $\{A_n\}_n$ of $(0, 1]$ by left-open, right-closed intervals labelled in decreasing order in the interval with length of A_n equal to a_n for each n . Then for $t_n := \sum_{k=n}^{\infty} a_k$ and $x \in [0, 1]$,

$$f(x) = \begin{cases} (1-x)/a_1 & \text{if } x \in A_1, \\ a_{n-1}(x - t_{n+1})/a_n + t_n & \text{if } x \in A_n, n \geq 2, \\ 0 & \text{if } x = 0. \end{cases}$$

Lebesgue measure is invariant for this map and taking a first return map to the interval A_1 gives us an induced map satisfying the conditions of Theorem 2.5. Since the branches are linear, the map behaves in an i.i.d. manner so that Lebesgue measure is a Markov measure for the map.

Moreover, one can choose the intervals $\{A_n\}_n$ in such a way that any of the tail decay conditions given by $(t_n)_n$ apply to our observable R_Y . By the results above these match the large deviations, so we may also apply the appropriate items of Theorem 3.2. We observe that the only points z which this theorem does not apply to directly are $\cup_{n \geq 0} f^{-n}0$. It is straightforward to adapt the theorem slightly to cover all elements of this set except 0.

4.2. Maps with exponential tails. If we start with an interval map $f : I \rightarrow I$ and can find a well-behaved first return map to an interval $Y \subset I$ with exponential tails, then Theorem 3.2(1) holds. That is, we require the first return map $F = f^{R_Y}$ to be a full-branched Gibbs-Markov map where the induced measure μ_Y has $\mu_Y(n-1 \leq R_Y < n) \leq Ce^{-\beta n}$ for some constants $C, \beta > 0$. By Section 2.6.2, F satisfies (F1)-(F4) of Section 2.

The fact that a full-branched Gibbs-Markov map has large deviations for observables with exponential tails appears to be essentially folklore. Yuri [Yu] quotes such a result, but the setting is slightly different and the proof there is not given explicitly, so for completeness, we provide the proof in Appendix B. Since R_Y has exponential tails, it follows from Proposition B.1 and Corollary B.2 that R_Y satisfies a (local) exponential large deviations estimate and thus Theorem 3.2(1) applies to the original system $f : I \rightarrow I$.

We remark that by this argument, Theorem 3.2(1) applies to the tower map (f_Δ, Δ) whenever one can construct a Young tower [Y] over an interval as described in the proof of Theorem 3.2.

In order to develop a specific class of examples, for the remainder of this section, we make the following standing assumptions. We assume that $f : I \rightarrow I$ is a C^2 , topologically mixing unimodal map with critical point c and $\text{orb}(c) = \{f^n(c) : n \geq 1\}$ nowhere dense. (We note that we can drop the topologically mixing and unimodal assumptions, but this makes our statements

more involved.) Then one can find an interval Y , compactly contained in $I \setminus \overline{\text{orb}(c)}$, such that (Y, F) is full branched (see [MeS, Chapter 4] for details), where F is the first return map to Y . Moreover we assume F has bounded distortion, e.g. f has negative Schwarzian derivative; then (Y, F) is Gibbs-Markov. Finally, we assume that our measure is an equilibrium state for some φ and discuss when our induced system has (F1)–(F4) and the return time has exponential tails so that we can conclude that Theorem 3.2(1) holds.

4.2.1. *Collet-Eckmann case.* If f satisfies the Collet-Eckmann condition (i.e., $|Df^n(c)|$ grows exponentially, and this case includes Misiurewicz maps, i.e., c is not recurrent nor attracted to a stable periodic orbit, provided f is non-flat at c), then for $\varphi = -t \log |Df|$, there is a unique equilibrium state for each t in a neighbourhood of $[0, 1]$, see for example [PR]. Moreover, it can be deduced (e.g. from [PR]) that (Y, F) satisfies the conditions (F1)–(F4) for Theorem 2.1 to hold for the induced version of μ_t (note that the conformal measure is w.r.t. the normalised potential $\varphi - P(\varphi)$), and that the return time has exponential tails. By Corollary B.2, R_Y enjoys exponential large deviations with respect to the equilibrium measure μ_t , so that Theorem 3.2(1) holds for each t in a neighborhood of $[0, 1]$.

4.2.2. *Non-Collet Eckmann case.* If f fails the Collet-Eckmann condition, then for the potential $\varphi = -t \log |Df|$, there is still a unique equilibrium state for $t \in (t_0, 1)$ for some $t_0 < 0$, again see for example [PR]. Moreover, (Y, F) satisfies the conditions for Theorem 2.1 to hold for the induced version of μ_t , and the return time has exponential tails, so Theorem 3.2(1) holds. By contrast, for $t = 1$, even if there is an equilibrium state for $-\log |Df|$, it will have sub-exponential mixing, so Theorem 3.2(1) will fail.

4.2.3. *Hölder potentials.* If φ is a Hölder potential, then our results hold more generally: for Theorem 3.2(1) to hold for the equilibrium state we only need the potential to be *hyperbolic*, i.e., $\sup_{x \in I} \frac{1}{n} S_n \varphi(x) < P(\varphi)$ for some n , where $P(\varphi)$ denotes the variational pressure. As shown in [LR] this is automatic if we merely assume

$$|Df^n(f(c))| \rightarrow \infty.$$

APPENDIX A. PROOF OF PROPOSITION 2.5

The L^1 bound on $\mathring{\mathcal{L}}_r^n \psi$ in Proposition 2.5 follows directly from (2.4), so we focus on proving the required bound on the variation of $\mathring{\mathcal{L}}_r^n \psi$.

For $r \in [0, r_0]$, let $\mathcal{I}_r^n = \{J_i\}_i = \{(a_i, b_i)\}$ denote the intervals of monotonicity for f_r^n and set $K_i = f^n(J_i)$. Then for $\psi \in \mathcal{B}$ and $n \geq 0$, we estimate,

$$\begin{aligned} \bigvee_I \mathring{\mathcal{L}}_r^n \psi &\leq \sum_i \bigvee_{J_i} \psi e^{S_n \varphi} + \psi(a_i) e^{S_n \varphi(a_i)} + \psi(b_i) e^{S_n \varphi(b_i)} \\ &\leq \sum_i 2 \bigvee_{J_i} \psi e^{S_n \varphi} + \frac{1}{m_0(J_i)} \int_{J_i} \psi e^{S_n \varphi} dm_0 \end{aligned} \tag{A.1}$$

For the second term in (A.1), we note that by conformality and the bounded distortion property (F1), we have for each $x \in J_i$,

$$e^{S_n \varphi(x)} \cdot \frac{m_0(K_i)}{m_0(J_i)} \leq 1 + C_d. \quad (\text{A.2})$$

For the first term in (A.1), we split

$$\bigvee_{J_i} \psi e^{S_n \varphi} \leq \sup_{J_i} e^{S_n \varphi} \bigvee_{J_i} \psi + \sup_{J_i} |\psi| \bigvee_{J_i} e^{S_n \varphi} \leq \sup_{J_i} e^{S_n \varphi} \bigvee_{J_i} \psi + \sup_{J_i} |\psi| C_d \sup_{J_i} e^{S_n \varphi}, \quad (\text{A.3})$$

where we have used Lemma 2.4(b) to bound $\bigvee_{J_i} e^{S_n \varphi}$. Using the bound $\sup_{J_i} |\psi| \leq \bigvee_{J_i} \psi + (m_0(J_i))^{-1} \int_{J_i} |\psi| dm_0$, we put these estimates together in (A.1) and use (A.2) to obtain,

$$\begin{aligned} \bigvee_I \mathring{\mathcal{L}}_r^n \psi &\leq \sum_i (2 + 2C_d) \sup_{J_i} e^{S_n \varphi} \bigvee_{J_i} \psi + \frac{(1 + C_d)(1 + 2C_d)}{m_0(K_i)} \int_{J_i} |\psi| dm_0 \\ &\leq (2 + 2C_d) |e^{S_n \varphi}|_\infty \bigvee_I \psi + \inf_i \frac{(1 + C_d)(1 + 2C_d)}{m_0(K_i)} \int_{J_i} |\psi| dm_0. \end{aligned} \quad (\text{A.4})$$

Applying (A.4) when $n = n_1$, setting $\bar{\sigma} := (2 + 2C_d) |e^{S_{n_1} \varphi}|_\infty < 1$, and using **(U)** yields,

$$\bigvee_I \mathring{\mathcal{L}}_r^{n_1} \psi \leq \bar{\sigma} \bigvee_I \psi + \frac{(1 + 2C_d)^2}{c_0} \int_{J_r^{n_1}} |\psi| dm_0,$$

and since $\bigvee_I e^{S_n \varphi} < \infty$ for each n , this relation can be iterated to complete the proof of Proposition 2.5 with $\sigma = \bar{\sigma}^{1/n_1}$.

APPENDIX B. EXPONENTIAL DEVIATIONS

In this section, we prove the fact that full-branched Gibbs-Markov maps have exponential large deviations for observables with exponential tails.

Let $P_G(\phi)$ denote the Gurevich pressure of ϕ (see [S1]). Note that as in [S1, Theorem 2], this is equal to the variational definition of pressure given in Section 2.

Proposition B.1. *Let F be a full-branched Gibbs-Markov map and ϕ, ψ weakly Hölder continuous potentials. If there exists $\delta > 0$ such that $P_G(\phi + t\psi) < \infty$ for each $|t| < \delta$ (or equivalently that $|\mathcal{L}_{\phi+t\psi} 1|_\infty < \infty$), then ψ enjoys exponential large deviations for μ the equilibrium state for ϕ .*

Proof. First we note that the assumptions on ϕ imply: (i) ϕ has finite Gurevich pressure $P_G(\phi)$ [S1, Theorem 1]; (ii) ϕ is positive recurrent [S3, Corollary 2]; and (iii) there exists a finite conformal Borel measure m_ϕ , positive on cylinders, such that $\frac{dm}{dm \circ F} = e^{\phi - P_G(\phi)}$ [S1, Theorem 4, Proposition 3].

Under these conditions, the associated transfer operator \mathcal{L}_ϕ acting on the space of Hölder continuous functions⁵ has a spectral gap. It then follows from [CS, Theorem 2.1], that ϕ is strongly positive recurrent. (We refer the reader to [CS] for the relevant definition.)

⁵Hölder continuity here is defined using the same constant θ as for the potential ϕ , i.e., weak Hölder continuity of ϕ means that there are $\theta \in (0, 1)$ and $C \geq 1$ so that $\sup_{C_n^i \in \mathcal{P}_n} \sup\{|\phi(x) - \phi(y)| : x, y \in C_n^i\} \leq C\theta^n$ for all

Strong positive recurrence implies that if ψ is a weakly Hölder continuous function such that $P_G(\phi + t\psi) < \infty$ for all $|t| < \delta$ and some $\delta > 0$, then $t \mapsto P_G(\phi + t\psi)$ is analytic in t [S2, Theorem 3]. Moreover, $\phi + t\psi$ is positive recurrent for each $|t| < \delta$ and has a Gibbs measure μ_t which is moreover the unique equilibrium state for $\phi + t\psi$. Denote by $\mu = \mu_0$ the Gibbs measure for ϕ . Without loss of generality, in what follows we assume $P_G(\phi) = \mu(\psi) = 0$ and that 0 is a local minimum for $t \mapsto P_G(\phi + t\psi)$.

Now define $J_n^+(\varepsilon)$ to be the collection of n -cylinders containing a point x so that $S_n\psi(x) > n\varepsilon$; similarly, let $J_n^-(\varepsilon)$ denote the collection of n -cylinders containing x such that $S_n\psi(x) < -n\varepsilon$. We first consider $J_n^+(\varepsilon)$. Since $\frac{d}{dt}P_G(\phi + t\psi)|_{t=t_0} = \mu_{t_0}(\psi)$ for $|t_0| < \delta$, and by continuity of the derivative for ε small enough we can find $q > 0$ so that $\mu_q(\psi) = \varepsilon$.

Then strict convexity of pressure implies that $P_G(q\psi + \phi) - q\varepsilon < 0$ (a slightly more sophisticated argument allows us to express this in terms of the Helmholtz free energy, but we do not require this here).

Let \mathcal{P}_n denote the set of n -cylinders and for $C_n^i \in \mathcal{P}_n$, let x_n^i be the fixed point of F^n in C_n^i . So we compute, using the Gibbs property (here the constant C covers the Gibbs constant and distortion constants):

$$\begin{aligned} \mu(S_n\psi > n\varepsilon) &\leq \sum_{C_n^i \in J_n^+(\varepsilon)} \mu(C_n^i) \leq C \sum_{C_n^i \in J_n^+(\varepsilon)} e^{S_n\phi(x_n^i)} \\ &\leq C^2 \sum_{C_n^i \in J_n^+(\varepsilon)} e^{q(S_n(\psi-\varepsilon))(x_n^i) + S_n\phi(x_n^i)} \\ &\leq C^2 e^{-nq\varepsilon} \sum_{C_n^i \in \mathcal{P}_n} e^{S_n(\phi+q\psi)(x_n^i)} \end{aligned}$$

Taking logs, dividing by n and taking limits we obtain

$$\limsup_n \frac{1}{n} \log \mu(S_n\psi > n\varepsilon) \leq P(\phi + q\psi) - q\varepsilon < 0$$

as required. A similar argument, with $q < 0$, applies to $J_n^-(\varepsilon)$. \square

Corollary B.2. *Under the assumptions of Proposition B.1, suppose that ψ is weakly Hölder continuous with exponential tails, i.e., $\mu(n-1 < |\psi| \leq n) \leq e^{-\beta n}$, for some $\beta > 0$. Then ψ enjoys exponential local large deviations with respect to μ .*

Proof. Letting $\{x_j\}_j$ be the collection of all fixed points of F , and ψ_j be the maximum value $|\psi|$ takes on the 1-cylinder X_j , by the Gibbs property,

$$\begin{aligned} \sum_j e^{(\phi+t\psi)(x_j)} &= \sum_{n \geq 1} \sum_{n-1 < \psi_j \leq n} e^{(\phi+t\psi)(x_j)} \leq C^2 \sum_n e^{n|t|} \mu(n-1 < |\psi| \leq n) \\ &\leq C^2 \sum_n e^{n(|t|-\beta)} < \infty \end{aligned}$$

n , where \mathcal{P}_n is the set of n -cylinders for F . We study the transfer operator on the class of functions $f : X \rightarrow \mathbb{R}$ sharing the same property as ϕ .

provided $|t| < \beta$. Standard theory shows that this implies that $P_G(\phi + t\psi) < \infty$, so that ψ satisfies the hypotheses of Proposition B.1. \square

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HENK BRUIN, FACULTY OF MATHEMATICS, VIENNA UNIVERSITY, OSKAR MORGENSTERNPLATZ 1, VIENNA 1090, AUSTRIA

Email address: `henk.bruin@univie.ac.at`

URL: `http://www.mat.univie.ac.at/~bruin/`

MARK F. DEMERS, DEPARTMENT OF MATHEMATICS, FAIRFIELD UNIVERSITY, FAIRFIELD, CT 06824, USA

Email address: `mdemers@fairfield.edu`

URL: `http://faculty.fairfield.edu/mdemers`

MIKE TODD, MATHEMATICAL INSTITUTE, UNIVERSITY OF ST ANDREWS, NORTH HAUGH, ST ANDREWS, KY16 9SS, SCOTLAND

Email address: `m.todd@st-andrews.ac.uk`

URL: `http://www.mcs.st-and.ac.uk/~miket/`