Interval Translation Maps with Weakly Mixing Attractors

Henk Bruin*, Silvia Radinger*

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Abstract

We study linear recurrence and weak mixing of a two-parameter family of interval translation maps $T_{\alpha,\beta}$ for the subset of parameter space where $T_{\alpha,\beta}$ has a Cantor attractor. For this class, there is a procedure similar to the Rauzy induction which acts as a dynamical system G on parameter space, which was used previously to decide whether $T_{\alpha,\beta}$ has an attracting Cantor set, and if so, whether $T_{\alpha,\beta}$ is uniquely ergodic. In this paper we use properties of G to decide whether $T_{\alpha,\beta}$ is linearly recurrent or weak mixing.

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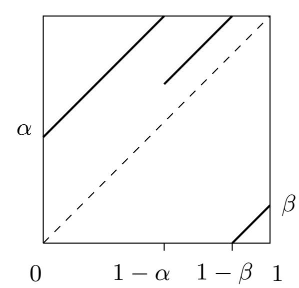
1 Introduction

In [13] the following family of interval translation maps (ITMs) was introduced:

$$T_{\alpha,\beta}(x) = \begin{cases} x+\alpha, & x \in [0, 1-\alpha), \\ x+\beta, & x \in [1-\alpha, 1-\beta), \\ x-1+\beta, & x \in [1-\beta, 1]. \end{cases}$$



on the parameter space $U = \{(\alpha,\beta) : 0 < \beta \le \alpha \le 1\}$. Viewed as circle map, $T_{\alpha,\beta}$ is a piecewise rotation on two pieces; it was studied in greater generality in [1, 11].

ITMs come in: (i) finite type if reduced to a set of subintervals, it is an interval exchange transformation (IET), and (ii) if the only compact invariant set are Cantor sets, and (iii) mixtures of the two. A conjecture by Boshernitzan & Kornfeld [8] suggests that infinite type occurs only for a set of zero Lebesgue measure in parameter space, and for the family $T_{\alpha,\beta}$ his indeed holds, see [13, Theorem 6]. Determining the type for this family goes via renormalization consisting of the first return map to $[1-\alpha, 1]$ and rescaling to unit size, in analogy to Rauzy induction for IETs. This transforms $T_{\alpha,\beta}$ into $T_{\alpha',\beta'}$ where

$$(\alpha', \beta') = G(\alpha,\beta) = \left(\frac{\beta}{\alpha}\,,\ \frac{\beta-1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor\right). \tag{1}$$

*Faculty of Mathematics, University of Vienna, Oskar Morgensternplatz 1, 1090 Vienna, Austria; *henk.bruin@univie.ac.at*, *silvia.radinger@univie.ac.at*. We gratefully acknowledge the support of the Austrian Exchange Service (WTZ Project PL 15/2022). We also thank Reem Yassawi for the fruitful conversations.

Note that $G(U) = U \cup L$ for $L = \{(\alpha, \beta) : \alpha - 1 \leq \beta \leq 0 \leq \alpha \leq 1\}$ and exactly the parameters in $\Omega_\infty := \{(\alpha, \beta) : G^n(\alpha, \beta) \in U^\circ$ for all $n \geq 0\}$ have the property that $T_{\alpha,\beta}$ is of infinite type, i.e., $\Omega := \bigcap_{n \geq 0} \overline{T^n_{\alpha,\beta}([0, 1])}$ is a Cantor set on which $T_{\alpha,\beta}$ acts as a minimal endomorphism. See Figure 1 for the parameter set associated to type infinity ITMs.

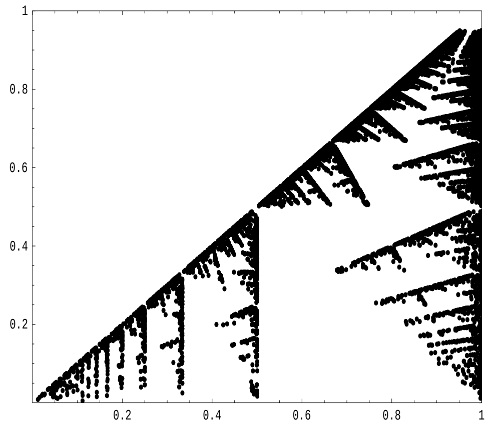


Figure 1: Approximation of the set Ω_∞ of type infinity parameters with 10,000 pixels.

Symbolically, $T_{\alpha,\beta}$ is described by an S-adic subshift (X, σ), based on a sequence of substitutions χ_{k_i}, $k_i \in \mathbb{N} = \{1, 2, 3, \dots\}$, and we call $T_{\alpha,\beta}$ linearly recurrent if the subshift (X, σ) is, i.e., there is L such that for every $x \in X$, every subword w reappears in x with gap $\leq L|w|$. Theorem 2.6 gives a precise condition on the sequence $(k_i)_{i\in\mathbb{N}}$ that results in linear recurrence.

Unique ergodicity of $T_{\alpha,\beta}$ fails if the sequence (k_i) increases exponentially fast, see [13, Theorem 11]. In this non-generic situation, there are two ergodic measures. We can also conclude from its representation as S-adic shifts, that $T_{\alpha,\beta}$ is never strongly mixing, see e.g. [12, Theorem 6.79].

Instead, the other property we investigate in this paper is weakly mixing of $T_{\alpha,\beta}$. General results in weak mixing for interval exchange transformations were obtained by Nogueira & Rudolph [19] (generic IETs), Sinaĭ & Ulcigrai [20] ("periodic" IETs) and Avila & Forni [3] (Lebesgue typical IETs). Recent extensions of these results can be found in e.g. [2, 4, 5]. However, the relative simplicity of the family $\{T_{\alpha,\beta}\}$ allows us to use methods from linear algebra (rather than results of Veech and Teichmüller theory) combined with a general condition on Bratteli-Vershik systems for existence of eigenvalues of the Koopman operator. This condition goes back to Host [17], and worked out in more generals settings in [7, 9, 10, 14, 15]. We prove that all "periodic" ITMs in our class (i.e., those for which the corresponding sequence $(k_i)_{i\in\mathbb{N}}$ is (pre-)periodic) as well as "typical" (in a sense made precise in Theorem 4.12) ITMs in our class are weakly mixing.

The paper is structured as follows. In Section 2 we characterize the linear recurrent ITMs by giving an if and only if condition on the index sequence $(k_i)_{i\in\mathbb{N}}$ of the substitutions in the S-adic representation. When the S-adic representation is viewed as non-autonomous sequences of toral automorphisms $(A_k)_{k\geq 1}$, the condition for the existence of an eigenvalue $e^{2\pi i \xi}$ of the Koopman operator is close (although not equivalent) to the vector $\vec{\xi} := (\xi, \xi, \xi) \in \mathbb{T}^3$ belonging to the stable space $W^s(\vec{0})$ of $(A_k)_{k\geq 1}$. In Section 3, we study the Lyapunov exponents of long concatenations $A_1 \cdots A_n$ and show that $W^s(\vec{0})$ is one-dimensional, so that the absence of eigenvalues becomes the generic situation, as studied in Section 4. Section 4.2 gives algebraic reasons why ITMs with (pre-)periodic sequences $(k_i)_{i\in\mathbb{N}}$ are weak mixing. Section 4.3 investigates the stable direction further, showing that it is uniquely determined by $(k_i)_{i\in\mathbb{N}}$. In Section 4.4 we show that $\vec{\xi} \in W^s(\vec{0})$ is a necessary, although not a sufficient, condition for the existence of continuous eigenvalues. Finally, in Section 4.5 we show that if $\liminf_i k_i < \infty$, then the ITM is uniquely ergodic. This implies that for any G-invariant probability measure ν on Ω_∞, the

ITM $T_{\alpha,\beta}$ is uniquely ergodic for ν-a.e. parameter (α, β). If in addition to $\liminf_i k_i < \infty$ we have $\vec{\xi} \notin W^s(\vec{0})$, then the Koopman operator has no measurable eigenvalues.

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2 Linear recurrence

We use symbolic dynamics w.r.t. the partition $\{[0, 1-\alpha), [1-\alpha, 1-\beta), [1-\beta, 1]\}$, with symbols $1, 2, 3$, respectively. One renormalization step is given by the substitution

$$\chi_k : \begin{cases} 1 \to 2 \\ 2 \to 31^k \\ 3 \to 31^{k-1} \end{cases} \qquad \text{for } k = \left\lfloor \frac{1}{\alpha} \right\rfloor \in \mathbb{N}. \tag{2}$$

The associate matrix and its inverse are

$$A_k = \begin{pmatrix} 0 & k & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A_k^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1-k \\ -1 & 0 & k \end{pmatrix}.$$

Note that $\det(A_k) = \det(A_k^{-1}) = -1$ and the characteristic polynomial of A_k is $P_k(\lambda) = \lambda^3 - \lambda^2 - k\lambda + 1$.

Every ITM of infinite type in this family is uniquely characterised by a sequence $(k_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that

$$k_{2i} > 1 \text{ for infinitely many } i \in \mathbb{N} \text{ and } k_{2i-1} > 1 \text{ for infinitely many } i \in \mathbb{N}. \tag{3}$$

Otherwise $G^n(\alpha, \beta)$ will eventually belong to the upper and right boundary alternatingly, and this behaviour belongs to the finite type case.

The itinerary of the point $1 \in [0, 1]$ is then

$$\rho = \lim_{i\to\infty} \chi_{k_1} \circ \chi_{k_2} \circ \chi_{k_3} \circ \cdots \circ \chi_{k_i}(3).$$

Let X be the closure of $\{\sigma^n(\rho)\}_{n\in\mathbb{N}}$ where σ is the left-shift. For such sequences, the attractor Ω of the ITM is a Cantor set, on which the action is isomorphic to an S-adic shift (X, σ) with associated matrices A_{k_i}. In [13] conditions are given under which $(\Omega, T_{\alpha,\beta})$ is uniquely ergodic, see also Corollary 4.15. For the first result on linear recurrence for our family of ITMs, we need certain properties for the S-adic shift.

A S-adic subshift based on substitutions $\chi_i : \mathcal{A}_i \to \mathcal{A}_{i-1}^+$ is called *primitive* if for all $m \in \mathbb{N}$ there exists $n \geq m$ such that for all $a \in \mathcal{A}_n$

$$\chi_m \circ \cdots \circ \chi_n(a) \text{ contains every } b \in \mathcal{A}_{m-1}.$$

The corresponding shift space X is the shift orbit closure of the set of accumulation points of $\{\chi_1 \circ \cdots \circ \chi_n(a) : n \in \mathbb{N}, a \in \mathcal{A}_n\}$. A subshift is *aperiodic* if there exists no $x \in X$ such that $\sigma^k(x) = x$ for some $k \in \mathbb{N}$.

A substitution is called *left-proper*, if for all letters $a \in \mathcal{A}$ the word $\chi(a)$ has the same starting letter. While the substitutions χ_{k_i} based on an ITM of infinite type are not left-proper, any telescoping $\chi_k \circ \chi_{k_{i+1}}$ is, because

$$\chi_{k_i} \circ \chi_{k_{i+1}} : \begin{cases} 1 \to 31^{k_i} \\ 2 \to 31^{k_i-1}2^{k_{i+1}} \\ 3 \to 31^{k_i-1}2^{k_{i+1}-1}. \end{cases}$$

Thus ρ, the itinerary of the point $1 \in [0, 1]$, is the unique one-sided fixed point under $(\chi_{k_i})_{i\in\mathbb{N}}$.

Next we need to show that our S-adic shift (X, σ) is aperiodic. In the literature one can find some results in this direction, e.g., [6, Lemma 3.3] requires that the substitutions are unimodular, primitive and proper, where we only have left-proper. The next lemma uses a weak form a recognizability, which does hold in our case.

Definition 2.1. *Let (X, σ) be a subshift based on the substitution χ. We call χ* combinatorially recognizable *if there is $N \in \mathbb{N}$ such that for every word w in X of length $|w| \geq N$ and every $v = v_1 v_2 v_3 \dots v_n$ such that w appears twice in $\chi(v)$, then the first appearance of w starts at $\chi(v_1 \cdots v_i)$ if and only if the second appearance of w starts at $\chi(v_1 \cdots v_j)$ for some $0 \leq i < j < n$. (Here $i = 0$ means that $\chi(v)$ starts with w.)*

The S-adic subshift (X, σ) based on substitutions $(\chi_i)_{i \in \mathbb{N}}$ is recognizable if each χ_i is recognizable (not necessarily with the same N).

Lemma 2.2. *Let (X, σ) be an injective, combinatorially recognizable S-adic subshift based on substitutions $\chi_i : \mathcal{A}_i \to \mathcal{A}_{i-1}^+$ such that for every $n \in \mathbb{N}$ there is $m > n$ such that*

$$|\chi_n \circ \cdots \circ \chi_m(a)| > 1 \qquad \text{for every } a \in \mathcal{A}_m. \tag{4}$$

Then (X, σ) is aperiodic.

Proof. Assume by contradiction that (X, σ) has a periodic element $x = \sigma^k \circ \chi(y) \in X$ with period p. It follows that $x = w^\infty$ with w a word of length p. We can pick k in such a way that w^∞ starts with $\chi_1(a)$ for some $a \in \mathcal{A}_1$. As χ_{k_1} is recognizable and injective, every new occurrence of w must start with $\chi_1(a)$. In fact, there is a unique word $a \cdots b \in \mathcal{A}_1^+$ such that $\chi_1(a \cdots b) = w$ and

$$\chi_1^{-1}(x) = \chi_1^{-1}(w)^\infty = (a \cdots b)^\infty.$$

This unique $\chi_1^{-1}(x)$ is again periodic with period $p_1 \leq p$. Using (4) and taking preimages $\chi_m^{-1} \circ \cdots \circ \chi_1^{-1}(x)$ decreases the period: $p_m < p_1$. Since we can repeat this argument, even when the period is reduced to 1, we get a contradiction. Hence all elements of (X, σ) are aperiodic. □

Proposition 2.3. *The S-adic subshift (X, σ), based on substitutions $(\chi_{k_i})_{i \in \mathbb{N}}$ from an ITM of infinite type, is primitive, combinatorially recognizable and aperiodic.*

Proof. **Primitivity.** We prove that for any $i \in \mathbb{N}$ there exists n_i such that the product of matrices associate to $\chi_{k_i}, \dots, \chi_{k_{i+n_i+1}}$ is strictly positive.

We write A_k for matrices with $k > 1$. For odd or even length blocks of substitutions with $k_i = 1$ the product of associated matrices are

$$A^{2r-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ r-1 & r & 1 \end{pmatrix}, \quad A^{2r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & r & 1 \end{pmatrix}.$$

Let $i \in \mathbb{N}$ be such that $k_i > 1$, as the ITM is of infinity type there exists an odd m_i such that $k_{i+m_i} > 1$.

In between k_i and k_{i+m_i} the positions of odd distance to k_i are always equal to one, the even positions may take any value $k \geq 1$, i.e.,

$$(k_i, 1, k_{i+2}, 1, \dots, 1, k_{i+2j}, 1, \dots, 1, k_{i+2n}, k_{i+m_i}).$$

Thus we can write the multiplication of matrices from k_i to k_{i+m_i} in the following way

$$A_{k_i} \left(\prod_{(k,r)} A^{2r-1} A_k \right) A^{2s} \; A_{k_{i+m_i}}, \tag{5}$$

where $s \geq 0$ and (k, r) are the values of $k_j > 1$ for $i < j < i + m_i$ and $2r - 1$ is the length of the chain of ones between the previous $k > 1$ and k_j with $r > 0$.

If the product in (5) is empty, it is the identity matrix. If it contains only one pair (k, r), it has the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ r-1 & r & 1 \end{pmatrix} \begin{pmatrix} 0 & k & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & k-1 \\ r & (r-1)k+1 & (r-1)(k-1)+1 \end{pmatrix}.$$

Otherwise, if the product contains multiple such factors, we get

$$\prod_{(k,r)} A^{2r-1} A_k = \prod_{(k,r)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix},$$

where the $*$ represent integers greater than zero. As the identity matrix is minimal in each entry, we use it as the lowest possible bound for each entry in the matrix multiplication with $s > 0$

$$A_{k_i} \left(\prod_{(k,r)} A^{2r-1} A_k \right) A^{2s} \geq \begin{pmatrix} 0 & k_i & k_i - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & s & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ 1 & 0 & 0 \\ * & * & * \end{pmatrix}$$

In case $s = 0$, the result is A_{k_i}, which is the entry-wise lower possibility. Thus

$$\begin{aligned} A_{k_i} \left(\prod_{(k,r)} A^{2r-1} A_k \right) A^{2s} A_{k_{i+m_i}} &\geq \begin{pmatrix} 0 & k_i & k_i - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & k_{i+m_i} & k_{i+m_i} - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix}. \end{aligned}$$

We see that the multiplication from k_i to k_{i+m_i} does not result in a full matrix. By multiplying one more step, independently of the value of k_{i+m_i+1} (it can be 1), we get

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix} A_{k_{i+m_i+1}} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} 0 & k_{i+m_i+1} & k_{i+m_i+1} - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

and thus we have a full matrix by multiplying $m_i + 2$ steps together.

For an arbitrary position j with $k_j \geq 1$ there exists $n_j \geq 0$ such that there is $j \leq i \leq j + n_j$ with $k_i > 1$ and m_i odd with $k_{i+m_i} > 1$. Thus the matrix associated to the substitution $\chi_{k_j} \circ \cdots \circ \chi_{k_{i+m_i+1}}$ is strictly positive. Therefore the S-adic shift is primitive.

Combinatorial Recognizability. Let (X_i, σ) be the S-adic subshift based on $(\chi_{k_j})_{j \geq i}$, we show that substitution χ_{k_i} is combinatorially recognizable. Take $x \in X_i$, there is a unique way to decompose x into blocks $\chi_{k_i}(a)$ for $a \in \mathcal{A}$. Any 2 in x is its own block $\chi_{k_i}(1)$. Further every 3 in x starts a new block and depending on the number of 1s directly following, we can determine if the block is $\chi_{k_i}(2)$ or $\chi_{k_i}(3)$. For example

$$\begin{aligned} x &= \ldots \mid 2 \mid 3\ 1\ 1 \mid 3\ 1 \mid 2 \mid 2 \mid 3\ 1\ 1 \mid \ldots \\ &= \ldots \chi_2(1)\ \chi_2(2)\ \chi_2(3)\ \chi_2(1)\ \chi_2(1)\ \chi_2(2) \ldots \end{aligned}$$

For any word w with $|w|1$ and v such that w appears twice in $\chi_{k_i}(v)$, if w starts with letters 2 or 3 we know that there exists $i < j$ with

$$\chi_{k_i}(v) = \chi_{k_i}(v_1 \cdots v_i) w \cdots = \chi_{k_i}(v_1 \cdots v_j) w \cdots.$$

If w starts with letter 1, then there exists no such i, j.

Thus the substitution χ_{k_i} is combinatorially recognizable and therefore the S-adic subshift based by substitutions $(\chi_{k_i})_{i\in\mathbb{N}}$ is combinatorially recognizable.

Aperiodicity. By Lemma 2.2 there exist no periodic elements in (X, σ). □

From this proof, it is clear that a general method of telescoping the sequence A_{k_n} to obtain full matrices $\tilde{A}_i$ is the following:

$$\tilde{A}_i = \underbrace{A_1 \cdots A_1}_{r_{i,1}} \cdot A_{k_{i,1}} \cdot \underbrace{A_1 \cdots A_1}_{r_{i,2}} \cdot \cdots \cdot A_{k_{i,m}} \cdot \underbrace{A_1 \cdots A_1}_{r_{i,m+1}} \cdot A_{k_{i,m+1}} \cdot A_{k_{i,m+2}}, \tag{6}$$

where $k_{i,j} \geq 2$ for $1 \leq j \leq m+1$, $k_{i,m+2} \geq 1$ and $r_{i,j}$ odd for $2 \leq j \leq m$, $r_{i,1} \geq 0$ and $r_{i,m+1}$ even.

Remark 2.4. *If the lengths of gaps between $k_i > 1$ and $k_{i+m_i} > 1$ with m_i odd, is bounded for all i, then the subshift is strongly primitive, i.e., there exists a constant $M > \max_i(m_i)$ such that the matrix associated to the substitution $\chi_{k_i} \circ \cdots \circ \chi_{k_{i+2M}}$ is strictly positive for all i.*

Lemma 2.5. *Let (X, σ) be a primitive non-periodic S-adic shift, based on substitutions $\chi_i : \mathcal{A}_i \to \mathcal{A}_{i-1}^+$. If for every $k \in \mathbb{N}$ there are $i < j$ and $a \in \mathcal{A}_{i-1}$, $b \in \mathcal{A}_j$ such that a^k is a subword of $\chi_i \circ \cdots \circ \chi_j(b)$, then (X, σ) is not linearly recurrent.*

Proof. Let $L \in \mathbb{N}$ be arbitrary and find $i < j$ and $b \in \mathcal{A}_j$ such that $\chi_i \circ \cdots \circ \chi_j(b)$ has a^{L+2} as subword. Set $w = \chi_1 \circ \cdots \circ \chi_{i-1}(a)$; then X contains a sequence x having w^{L+2} as subword. Since x is not periodic, the word-complexity $p_x(|w|) > |w|$, and hence there is a word v of length $|v| = |w|$ that is not a subword of w^{L+2}. By primitivity, v and w^{L+2} appear infinitely often in x. But then v must have reappearances with gap $\geq L|v|$. Hence (X, σ) is not L-linearly recurrent. As L was arbitrary, the lemma follows. □

By Theorem 5.4 in [7], a S-adic subshift based on recognizable substitutions is linearly recurrent if and only if we can telescope the substitutions $(\chi_{k_i})_{i\geq 1}$ into finitely many, left-proper substitutions with strictly positive transition matrices.

Theorem 2.6. *The subshift (X, σ) associated to an ITM $(\Omega, T_{\alpha,\beta})$ of infinite type is linearly recurrent if and only if $(k_i)_{i\in\mathbb{N}}$ is bounded and the sets $\{i : k_{2i} > 1\}$ and $\{i : k_{2i-1} > 1\}$ have bounded gaps.*

Proof. Let M and N be the maximal gap sizes of the sets $\{i : k_{2i-1} > 1\}$ and $\{i : k_{2i} > 1\}$ respectively. Every M-gap (i.e., $(k_n)_{n=i+1}^{M}$ for any $i \in \mathbb{N}$) contains at least one $k_{2i-1} > 1$ and every N-gap at least one $k_{2i} > 1$. Then there exists $\tilde{M} > \max\{M, N\}$ where for any $k_i \geq 1$ there exists $k_{i+2j-1} > 1$ and $k_{i+2l} > 1$ for $j, l < \tilde{M}$. Thus as in the proof of Proposition 2.3 for any $i \in \mathbb{N}$ multiplying $2\tilde{M} + 1$ matrices together from k_i to $k_{i+2\tilde{M}}$ will result in a full matrix. As there are only finitely many values $k_j \in \mathbb{N}$ can take, the set of full associated matrices is finite and the subshift (X, σ) is linearly recurrent.

On the other hand if $(k_n)_n$ unbounded we have that for any $L \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $k_n > L$ and therefore

$$\chi_{k_n}(2) = 31^{k_n}$$

contains the subword 1^{k_n}.

Similarly if the gaps of $\{i : k_i > 1\}$ are unbounded, for any $L \in \mathbb{N}$ there exists a gap of length $r_n > 2L$ and

$$\chi_1^{r_n}(2) = \begin{cases} 3^{l_n}1 & \text{for } r_n = 2l_n - 1, \\ 3^{l_n}2 & \text{for } r_n = 2l_n \end{cases}$$

contains the subword 3^{l_n}. By Lemma 2.5, the subshift (X, σ) is not linearly recurrent. □

3 Lyapunov exponents

In this section we analyse the dynamics of the matrix products $\tilde{A}_1 \cdot \tilde{A}_2 \cdots \tilde{A}_n$, $n \to \infty$, and show that there is one positive Lyapunov exponent (as lim inf), one negative Lyapunov exponent (as lim sup), and the third Lyapunov is at least not negative (as lim inf). The usual Oseledets theory states that Lyapunov exponents exist (as limits) for typical parameters, where in our setting, typical is to be interpreted as w.r.t. a G-invariant measure on Ω_∞, in the line of [1] based on the construction of [16]. We need a statement for all $(\alpha, \beta) \in \Omega_\infty$, but lim inf / lim sup-results as in Proposition 3.1 suffice.

Because the matrices A_{k_i} have determinant -1, they can be viewed as automorphisms $M_k : \mathbb{T}^3 \to \mathbb{T}^3$ of the 3-torus $\mathbb{T}^3$, given by $M_k \vec{x} = \vec{x} A_k \bmod 1$. Condition (21) below can then be interpreted as that the line in $\mathbb{T}^3$ spanned by $(1, 1, 1)$ intersects the stable direction of $(0, 0, 0)$ for the infinite system $(M_{k_i} \circ \cdots \circ M_{k_2} \circ M_{k_1})_{i \geq 1}$ in $\mathbb{T}^3$. This explains the need of determining hyperbolicity and computing the dimension of the stable direction of this system in the first place. For conditions in Section 4 deciding whether the ITM is weak mixing or not, the important thing is that the stable space is one-dimensional. Therefore, if $\vec{u} := \vec{\xi} A_{k_1} \cdots A_{k_r} \bmod \mathbb{Z}^3 \in \mathit{span}(\vec{v}_2 A_{k_1} \cdots A_{k_r})$ for some $r \in \mathbb{N}$, then $\vec{\xi}$ doesn't contract and the arguments in Section 4 to get the estimates of (21) are all necessary and the same, regardless if $\vec{u} A_{k_{r+1}} \cdots A_{k_n}$ increases, increases exponentially or just stays bounded as $n \to \infty$.

Abbreviate $\widetilde{\mathbb{A}}^n = \tilde{A}_1 \cdots \tilde{A}_n$, where the $\tilde{A}_i$'s are the telescoped matrices from (6) in the previous section.

Proposition 3.1. *For every sequence $(k_i)_{i \in \mathbb{N}}$ satisfying* (3)*, the sequences of eigenvalues $(\lambda_{n,i})_{n \geq 1}$, $i = 1, 2, 3$, of $\widetilde{\mathbb{A}}^n$ satisfy*

$$\limsup_{n \to \infty} \frac{1}{n} \log \lambda_{n,3} < 0 \leq \liminf_{n \to \infty} \frac{1}{n} \log |\lambda_{n,2}| \leq \limsup_{n \to \infty} \frac{1}{n} \log |\lambda_{n,2}| \leq \liminf_{n \to \infty} \frac{1}{n} \log \lambda_{n,1}.$$

Proof. Let $\lambda_{n,i}$, $i = 1, 2, 3$, indicate the eigenvalues of $\widetilde{\mathbb{A}}^n$. Their product is ± 1, because $\det(A_k) = -1$ for all k. Each $\tilde{A}_j$ is strictly positive and maps the positive octant $Q^+ = \{x \in \mathbb{R}^3 : x_i \geq 0\}$ strictly into itself (except for the origin). As in (the proof of) the Perron-Frobenius Theorem, this implies that $\liminf_n \frac{1}{n} \log \lambda_{n,1} > 0$ and the corresponding unstable eigenspace E_1 goes through the interior of Q^+.

The octant $Q^- = \{x \in \mathbb{R}^3 : x_1, x_2 \geq 0 \geq x_3\}$ is preserved by each inverse matrix A_k^{-1}. A change of coordinates changes Q^- into Q^+. Indeed, take

$$U := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B_{k_i} = U A_{k_i}^{-1} U^{-1} = \begin{pmatrix} 0 & 1 & k_i - 1 \\ 1 & 0 & 0 \\ 0 & 1 & k_i \end{pmatrix}. \tag{7}$$

Let $\widetilde{\mathbb{B}}^n = \tilde{B}_n \cdots \tilde{B}_2 \cdot \tilde{B}_1$, where the $\tilde{B}_i$'s are blocks of matrices B_{k_j} telescoped in exactly the same way (but in the other direction) as the A_{k_i}'s. As will be shown in the proof of Lemma 4.3 all the $\tilde{B}_i$'s are strictly positive.

Then the Perron-Frobenius Theorem applies and we obtain $\liminf_n \frac{1}{n} \log(\lambda_{n,3})^{-1} > 0$ (because the eigenvalues of $\widetilde{\mathbb{A}}^n$ and $\widetilde{\mathbb{B}}^n$ are each other reciprocals). This gives a negative $\limsup_n \frac{1}{n} \log \lambda_{n,3} < 0$.

We will continue the proof using the original (i.e., non-telescoped) matrices A_k and B_k. For our choice of B_k, the corresponding transition graphs of A_k and B_k are the same except that the multiplicities of the 12-entry of 33-entry are swapped. (In particular, $A_k = B_k$ if $k = 1$.) The fact that B_k's have a loop $3 \to 3$ with multiplicity k (which has only multiplicity 1 in A_k) is the reason why $B_{k_n} \cdots B_{k_1}$ dominates $A_{k_1} \cdots A_{k_n}$ for sufficiently large n. Note that the matrix multiplication goes in opposite direction. This can be remedied by taking the transpose, amounting to a reversal of arrows in the transition graph, which has no influence on the number of paths.

The u, v-entries of the matrices $A_{k_1} \cdots A_{k_n}$ (or $B_{k_n} \cdots B_{k_1}$) represent the number of n-paths from u to v. Note, that in every iterate (i.e., multiplication with A_{k_i} or $B_{k_{N+1-i}}$, with N the total number of matrices involved in $\widetilde{\mathbb{B}}^n$) we use a different matrix, but their structure is the same, so that the

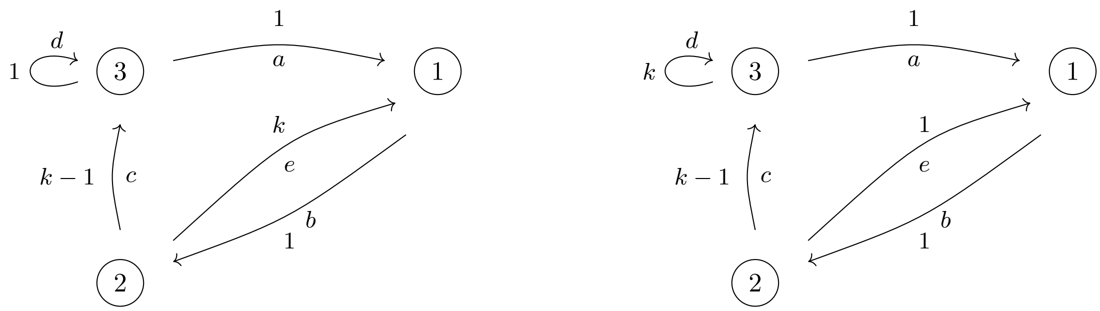

Figure 2: Transition graphs of A_k and B_k. The numbers at the edge-labels stand for weights (= multiplicities) of the edges.

corresponding transition graphs are the same, except for the labelling, see Figure 2. Let $x_1 \dots x_n$ be the edge-coding of an n-path. Then we call the product of the labels of the edges x_i the *weight* $w(x_1 \dots x_n)$. It represents the total number of paths corresponding to the single edge-code $x_1 \cdots x_n$.

We first look at the loops from vertex 1 back to 1, that is, edge-coded, from edge b to a. These are represented by words $bx_2 \dots x_{n-1}a$ where $x_j \neq a$.

Let $w_A(bx_2 \dots x_{n-1}a)$ and $w_B(bx_2 \dots x_{n-1}a)$ denote the weights of these paths in the transition graphs of A_{k_i} and $B_{k_{N+1-i}}$. If none of the symbols $x_j = a$, then there are only $\lfloor (n-1)/2 \rfloor$ possibilities, characterized by the symbol c at its unique even position. We call the collection of these words a *cluster of loops*.

We claim that for every $r \geq 1$, the sum of the weights of the $2r+1$-cluster satisfies

$$\sum_{j=0}^{r-1} w_A((be)^j bcd^{2(r-j-1)}a) = k_2 k_4 \cdots k_{2r} - 1 \leq \sum_{j=0}^{r-1} w_B((be)^j bcd^{2(r-j-1)}a), \tag{8}$$

and the inequality is strict if $k_{2j}, k_{2j'+1} > 1$ for some $1 \leq j \leq j' < r$.

Clusters of even length $2r+2$ are obtained from odd length clusters by adding an extra d at position $2r+1$. Therefore (8) implies that

$$\sum_{j=0}^{r-1} w_A((be)^j bcd^{2(r-j)-1}a) = k_2 k_4 \cdots k_{2r} - 1 \leq (k_2 k_4 \cdots k_{2r} - 1) k_{2r+1} \leq \sum_{j=0}^{r-1} w_B((be)^j bcd^{2(r-j)-1}a),$$

with strict inequality if $k_{2r-1} > 1$ or under the same condition as for (8).

Each edge b has weight $w_A(b) = w_B(b) = 1$; the edges c in position $2j$ have weight $w_A(c) = w_B(c) = k_{2j} - 1$ and the remaining edges in position i have weight $w_A(e) = w_B(d) = k_i$, $w_A(d) = w_B(e) = 1$.

We prove the equality regarding w_A in (8) by induction from 1 up to r. For $r = 1$ we have $w_A(bca) = k_2 - 1$. If the statement holds for $r - 1$, then

$$\begin{aligned}
\sum_{j=0}^{r-1} w_A((be)^j bcd^{2(r-j-1)}a) &= \sum_{j=0}^{r-2} w_A((be)^j bcd^{2(r-j-1)}a) + k_2 k_4 \dots (k_{2r} - 1) \\
&= k_2 k_4 \cdots k_{2r-2} - 1 + k_2 k_4 \dots (k_{2r} - 1) \\
&= k_2 k_4 \cdots k_{2r} - 1.
\end{aligned}$$

The inequality regarding w_B in (8) is also proved by induction, now from r down to 1, and assuming that all odd-indexed $k_{2j+1} = 1$. That is, we show that $\sum_{j=m}^{r-1} w_B((be)^j bcd^{2(r-j-1)}a) = k_{2m+2} \cdots k_{2r} - 1$ for all $r \in \mathbb{N}$ and $0 \leq m < r$. This is done by induction too, but now working downwards, first taking the For $m = r-1$, we get $w_B((be)^{r-1}bca) = k_{2r} - 1$. If the statement holds for m, then

$$\begin{aligned}\sum_{j=m-1}^{r-1} w_B((be)^j bcd^{2(r-j-1)}a) &= \sum_{j=m}^{r-1} w_B((be)^j bcd^{2(r-j-1)}a) + k_{2m}k_{2m+2}\cdots(k_{2r}-1) \\ &= k_{2m+2}\cdots k_{2r-2} - 1 + k_{2m}\cdots(k_{2r}-1) \\ &= k_{2m}\cdots k_{2r} - 1.\end{aligned}$$

This concludes the induction. Since each instance $w_B(d) > 1$ at an odd position (provided $w_B(c) > 1$ at the previous occurrence of symbol c) increases the terms in the sum of w_B-weights by a factor $\geq \frac{3}{2}$, the inequality follows. This proves (8).

For paths starting in vertices 2 and 3, the proof that the B-weight of clusters dominates the A-weight are similar.

Now decompose an n-path from u to $v \in \{1,2,3\}$ into loops, with a potential remaining piece that is shorter than a full loop[1]. Group together all n-paths with the same loop structure (i.e., the same endpoints of their loops) as they fall into the same cluster pateern. The above estimates show that the combined B-weight of the concatenation of clusters dominates the combined A-weight of the concatenation. Summing over all cluster patterns, we arrive at $(\widetilde{\mathbb{B}}^n)_{u,v} \geq (\widetilde{\mathbb{A}}^n)_{u,v}$ for all $u, v \in \{1,2,3\}$ and for sufficiently large n. Thus the leading eigenvalue of $\widetilde{\mathbb{B}}^n$ is larger than the leading eigenvalue of $\widetilde{\mathbb{A}}^n$. Since $\widetilde{\mathbb{A}}^n$ and $\widetilde{\mathbb{B}}^n$ have reciprocals eigenvalues, $\liminf_{n\to\infty} \frac{1}{n} \log|\lambda_{n,2}| \geq 0$. □

The next theorem establishes the Lyapunov exponents and invariant directions of $(\widetilde{\mathbb{A}}^n)_{n\geq 1}$.

Theorem 3.2. *Let $\lambda_{n,i}$, $i = 1,2,3$, be the eigenvalues of $\widetilde{\mathbb{A}}^n$. There exist a constant $C \geq 1$ and vectors $\vec{v}_i \in \mathbb{S}^2$, $i = 1,2,3$, such that*

$$\frac{1}{C}\lambda_{n,i} \leq \|\vec{v}_i \widetilde{\mathbb{A}}^n\| \leq C\lambda_{n,i},$$

for $i = 1,2,3$ and for all $n \geq 1$.

Proof. From the proof of Proposition 3.1 we have

$$\limsup_{n\to\infty} \frac{1}{n}\log\lambda_{n,3} < 0 \leq \liminf_{n\to\infty}\frac{1}{n}\log|\lambda_{n,2}| \leq \limsup_{n\to\infty}\frac{1}{n}\log|\lambda_{n,2}| \leq \liminf_{n\to\infty}\frac{1}{n}\log\lambda_{n,1},$$

and corresponding eigenvectors $\vec{v}_{n,1} \in Q^+ \cap \mathbb{S}^2$, $\vec{v}_{n,3} \in Q^- \cap \mathbb{S}^2$ and $\vec{v}_{n,2} \cap \mathbb{S}^2$ has a uniformly positive angle from the other eigenvectors.

Next define

$$Q_n^+ = \{\vec{v} \in \mathbb{S}^2 : \vec{v}(\widetilde{\mathbb{A}}^n)^{-1} \in Q^+\} \quad \text{and} \quad Q_n^- = \{\vec{v} \in \mathbb{S}^2 : \vec{v}(\widetilde{\mathbb{A}}^n) \in Q^-\}.$$

Since Q^+ is forward invariant and Q^- is backward invariant, these are both nested sequences of non-empty (because $\vec{v}_{n,1} \in Q_n^+$ and $\vec{v}_{n,3} \in Q_n^-$), compact sets, so we can choose $\vec{v}_1 \in \bigcap Q_n^+$ and $\vec{v}_3 \in \bigcap Q_n^-$. Recall that $\widetilde{\mathbb{A}}^n = \tilde{A}_1 \cdots \tilde{A}_n$. Since $\tilde{A}_i$ uniformly contracts Q^+ (in Hilbert semi-metric, see [12, Section 8.6]),

$$\frac{\vec{v}_1\widetilde{\mathbb{A}}^n}{\|\vec{v}_1\widetilde{\mathbb{A}}^n\|} - \frac{\vec{v}_{n,1}\widetilde{\mathbb{A}}^n}{\|\vec{v}_{n,1}\widetilde{\mathbb{A}}^n\|} \to 0 \quad \text{exponentially as } n \to \infty,$$

so there is $C_1 \geq 1$ such that

$$\frac{1}{C_1}\lambda_{n,1} \leq \|\vec{v}_1\widetilde{\mathbb{A}}^n\| \leq C_1\lambda_{n,1} \qquad \text{for all } n \geq 0. \tag{9}$$

[1]For these short pieces, similar cluster inequalities hold. The worst case is $w_A(be) + w_A(bc) = 2k - 1 > w_B(be) = w_B(bc) = k$ if k is the multiplicity of the e at the second position, but this single factor $k/(2k-1)$ is negligible for large n.

The same argument shows that there is $C_3 \geq 1$ such that

$$\frac{1}{C_3}\lambda_{n,3} \leq \|\vec{v}_3\widetilde{\mathbb{A}}^n\| \leq C_3\lambda_{n,3} \qquad \text{for all } n \geq 0. \tag{10}$$

Next define

$$P_n^+ = \{\vec{v} \in \mathbb{S}^2 : v(\widetilde{\mathbb{A}}^n) \notin int(Q^+)\} \quad \text{ and } \quad P_n^- = \{\vec{v} \in \mathbb{S}^2 : v(\widetilde{\mathbb{A}}^n)^{-1} \notin int(Q^-)\}.$$

Then $P_n^+ \cap P_n^-$ forms a nested sequences of non-empty (because $\vec{v}_{n,2} \in P_n^+ \cap P_n^-$) compact sets, so we can find $\vec{v}_2 \in \bigcap_n (P_n^+ \cap P_n^-)$. Since $\widetilde{\mathbb{A}}^1 = \tilde{A}_1$ is a strictly positive matrix, there is a neighborhood V^+ of $Q^+ \cap \mathbb{S}^2$ such that $\vec{v}\widetilde{\mathbb{A}}^1 \in Q^+$ for each $\vec{v} \in V^+$. Therefore $P^+ \cap V^+ = \emptyset$ for $P^+ := \bigcap_n P_n^+$. The same argument gives a neighborhood V^- of $Q^- \cap \mathbb{S}^2$ such that $P^- \cap V^- = \emptyset$ for $P^- := \bigcap_n P_n^-$. This shows that $\vec{v}_2\widetilde{\mathbb{A}}^n \in P^- \cap P^+$ has uniformly positive angles with $\vec{v}_3\widetilde{\mathbb{A}}^n$ and $\vec{v}_1\widetilde{\mathbb{A}}^n$, so there is a constant $C' \geq 1$ such that

$$|\lambda_{n,1}\lambda_{n,2}\lambda_{n,3}|\frac{4\pi}{3} = \mathrm{Vol}(\widetilde{\mathbb{A}}^n(B_1(\vec{0})) \leq C'\|\vec{v}_1\widetilde{\mathbb{A}}^n\|\|\vec{v}_2\widetilde{\mathbb{A}}^n\|\|\vec{v}_3\widetilde{\mathbb{A}}^n\| \leq C'C_1\lambda_{n,1}C_3\lambda_{n,3}\|\vec{v}_{n,2}\widetilde{\mathbb{A}}^n\|.$$

A similar argument gives a lower bound, so there is $C_2 \geq 1$ such that

$$\frac{1}{C_2}|\lambda_{n,2}| \leq \|\vec{v}_2\widetilde{\mathbb{A}}^n\| \leq C_2|\lambda_{n,2}| \qquad \text{for all } n \geq 0. \tag{11}$$

Combining (9), (10) and (11) finishes the proof the theorem for $C = \max\{C_1, C_2, C_3\}$. □

For further investigation into the number of concatenations of clusters, we use a lemma in terms of weighted trees. Consider a tree with root v_0 and vertex and edge sets V and E respectively. Let $V_0 = \{v_0\}$ be the root of the tree and $V_n = \{v \in V : v \text{ is } n \text{ edges away from } v_0\}$. Clearly $V = \sqcup_{n \geq 0} V_n$.

We assign to each $e \in E$ a weight $w(e) \geq 1$. Extend the weight function to V by setting $w(v_0) = 1$ and $w(v) = \prod\{w(e) : e \text{ lies between } v_0 \text{ and } v\}$ for $v \in V_n$, $n \geq 1$.

Lemma 3.3. *Assume that in the tree above each vertex $v \in V_{n-1}$ has $1 + b_n$ edges to V_n, one of these edges has weight $1 + \eta_n$ and the others have weight 1. Then*

$$\sum_{v \in V_n} w(v) = \#V_n \prod_{r=1}^{n} \left(1 + \frac{\eta_r}{1 + b_r}\right).$$

Proof. The proof of the claim goes by induction on n. For $n = 1$, this is trivially true. Assume now that the statement holds for $n - 1$. For $v \in V_{n-1}$, write $V_1(v)$ for the set of vertices $v' \in V_n$ such that there is an edge between v and v'. Then $\#V_1(v) = 1 + b_n$ for each $v \in V_{n-1}$ and

$$\begin{aligned}
\sum_{v \in V_n} w(v) &= \sum_{v \in V_{n-1}} \sum_{v' \in V_1(v)} w(v') = \sum_{v \in V_{n-1}} w(v) \left(\eta_n + \sum_{v' \in V_1(v)} 1\right) \\
&= \sum_{v \in V_{n-1}} w(v) \sum_{v' \in V_1(v)} \left(1 + \frac{\eta_n}{\#V_1(v)}\right) = \left(1 + \frac{\eta_n}{1 + b_n}\right)(1 + b_n) \sum_{v \in V_{n-1}} w(v) \\
&= (1 + b_n)\#V_{n-1} \prod_{r=1}^{n} \left(1 + \frac{\eta_r}{1 + b_r}\right) = \#V_n \prod_{r=1}^{n} \left(1 + \frac{\eta_r}{1 + b_r}\right).
\end{aligned}$$

The completes the induction and the proof. □

Assume that $\widetilde{\mathbb{A}}^n$ comprises s matrices A_{k_j}, $1 \le j \le s$. We call j a *marked position* if $k_j \ge 2$. Hence, each telescoped matrix $\tilde{A}_i$ contains at least two and $\widetilde{\mathbb{A}}^n$ contains at least $2n$ marked positions.

In (8) we computed the total weight of *clusters* of loops with the same start and end point. Each such cluster has a unique symbol m which must be at a marked position, and it must take an even position in the loop. If there were an earlier marked position at an even place in the loop, there is another loop in the same cluster. Hence, to identify a cluster, we need the *earliest* midpoint m (which we call the *mark* of the cluster) and an integer b such that $m - 2b + 1$ is the start of the cluster, and there are no marked position at an even place between $m - 2b + 1$ and m. The end of the cluster is one before the start of the next cluster, and hence determined by the next cluster.

A *cluster pattern* P is the concatenation of clusters that fits within the integer interval $[1, s]$, so its first cluster starts at 1 and its last cluster ends at s. According to (8), a cluster c has weight $w_B(c) \ge (\frac{3}{2})^t w_A(c)$ if it contains t marked positions m' after its mark m such that $m' - m$ is odd. If $t = 0$, then $w_B(c) = w_A(c)$. Let $w_B(P)$ (resp. $A(P)$) stand for the B-weight (resp. A-weight) of a cluster pattern, i.e., the product of the B-weights (resp. $A(P)$-weights) of its clusters.

Since $\sum_P w_B(P) / \sum_P w_A(P)$, where the sum runs over all cluster patterns, is an indication for the second eigenvalue $\lambda_{n,2}$, we need to identify all cluster patterns P and compare $w_B(P)$ and $w_A(P)$. To do that, we order the cluster patterns in a tree-structure and finally apply Lemma 3.3.

Among the marked positions, we choose a sequence as follows: j_0 is the first **even** marked position, and if j_{r-1} is chosen, then $j_r > j_{r-1}$ is the first marked position such that $j_r - j_{r-1}$ is **odd**. Let R be maximal such that $j_R < s$. We build a tree, thinking of its vertex sets V_r, $r = 0, \dots, R$, as being associated to the marked position j_r. $V_0 = \{v_0\}$ is the root. For $r \ge 1$, set $b_r = \lfloor (j_r - j_{r-1})/2 \rfloor$, and to each $v \in V_{r-1}$, attach $1 + b_r$ edges to vertices in V_r, namely

- one representing cluster patterns having no cluster that ends between j_{r-1} and j_r; this edge gets weight $1 + \eta_r \ge \frac{3}{2}$;
- one for each $1 \le b \le b_r$, representing cluster patterns having a cluster that ends at position $j_r - 2b$; such an edge gets weight 1.

Then every cluster pattern P is associated to a single R-path in the tree (i.e., a single $v \in V_R$), namely according whether P contains a cluster ending between j_r and j_{r-1}, and where that endpoint is. Moreover, $w_B(P) \ge w(v) w_A(P)$ where the weight $w(v)$ of $v \in V_R$ is the product of the weights of the edge between v_0 and v. If all these edge-weights are 1, then we have equality $w_B(P) = w_A(P)$.

Example 3.4. *This example shows that it is possible that the second Lyapunov exponent is zero, and a fortiori, there is no expansion in the direction $\vec{v}_2$.*

Take $k_j = 2$ for $j = 2^n - ((n+1) \bmod 2)$, that is, the marked positions are $j = 2, 3, 8, 15, 32, 63, \dots$ with $k_j = 2$, and $k_j = 1$ otherwise. Since these marked positions are alternatively even and odd, $j_r = 2, 3, 8, 15, 32, 63, \dots$, so $j_r = 2^{r+1} - (r \bmod 2)$ for $r \ge 0$. Also $\widetilde{\mathbb{A}}^n$ comprises 4^n matrices A_k and $2n$ marked positions. The corresponding numbers $b_r = 0, 2, 3, 8, 15, 32, \dots$, so $b_r = 2^{r-1} - (r \bmod 2)$ for $1 \le r \le R = 2n - 1$. From Lemma 3.3 with $1 + \eta_r \equiv 2$, we derive

$$\sum_P w_B(P) \le \sum_P w_A(P) \prod_{r=1}^{R} \left(1 + \frac{1}{1 + b_r}\right) \le 2e \sum_P w_A(P)$$

for the sums of weights of cluster patterns. Therefore, the sequence of second eigenvalues $(\lambda_{n,2})_{n \ge 1}$ is bounded.

Proposition 3.5. *Every linearly recurrent ITM satisfying* (3) *has two strictly positive (as* $\liminf$*) and one strictly negative (as* $\limsup$*) Lyapunov exponent.*

Proof. From Proposition 3.1 we have already that $\limsup_n \frac{1}{n} \log \lambda_{n,3} < 0 < \liminf_n \frac{1}{n} \log \lambda_{n,1}$. The second eigenvalue $\lambda_{n,2}$ is comparable to the quotient of the B-weight of all n-paths and the A-weight of all n-paths. For this, we construct the tree as described above. The characterization of linear recurrence in Theorem 2.6 implies now, that there is $K \in \mathbb{N}$ such that $1 + b_r \leq K$ for all $1 \leq r \leq R$ and also that $R \geq n/K$. Hence, Lemma 3.3 with $\eta_r \equiv \frac{1}{2}$ gives

$$\sum_P w_B(P) \geq \left(1 + \frac{1}{2K}\right)^R \sum_P w_A(P) \geq \left(1 + \frac{1}{2K}\right)^{n/K} \sum_P w_A(P).$$

This shows that $\liminf_{n\to\infty} \frac{1}{n} \log |\lambda_{n,2}| \geq \frac{1}{K} \log(1 + \frac{1}{2K}) > 0$. Theorem 3.2 then gives the two positive and one negative Lyapunov exponents. □

4 Weak mixing

4.1 Host's eigenvalue condition

Host [17] formulated a condition for primitive substitution shifts to have an eigenvalue: For $e^{2\pi i\xi}$ to be an eigenvalue the Koopman operator for some $\xi \in (0, 1)$ is:

$$\sum_{n=1}^{\infty} \|\vec{\xi} A^n\| < \infty, \qquad \vec{\xi} = (\xi, \xi, \xi), \tag{12}$$

where $\|x\|$ is the distance of a vector to the nearest integer lattice point and A is the associated matrix of the substitution. The condition is of the same gist as Veech criterion [21] for non-weak-mixing of IETs, although there A can be seen as the matrix representing the action of Rauzy induction on the homology $H^1(\mathbb{Z})$ of the translation surface obtain as suspension flow over the IET.

For primitive substitution shifts, (12) is both sufficient and necessary and every eigenfunction can be taken to be continuous. Later works, see e.g. [14, 15], showed that in the context of linearly recurrent S-adic shifts (i.e., subshifts based on a sequence of substitution rather than a single one), (12) is necessary and sufficient as well. For more general S-adic subshifts, more complicated conditions are needed, and we come back to them in Sections 4.4 and 4.5 where we distinguish between continuous and measurable eigenvalues.

4.2 Weak mixing for (pre-)periodic ITMs

Theorem 4.1. *For every (pre-)periodic sequence $(k_i)_{i\in\mathbb{N}}$ satisfying* (3)*, the corresponding system $(\Omega, T_{\alpha,\beta})$ is weakly mixing.*

Proof. Suppose that the sequence $(k_i)_{i\in\mathbb{N}}$ has period n and also satisfies (3), then the corresponding S-adic shift is linearly recurrent (and uniquely ergodic). If $A^n = A_{k_1} \cdots A_{k_n}$, then its eigenvalues are $\tilde{\lambda}_i$ where $\tilde{\lambda}_3 < 1 < |\tilde{\lambda}_2| < \tilde{\lambda}_1$. That is, there is only a one-dimensional stable subspace E_3 spanned by a vector $(u_1, u_2, -1) \in Q^-$. Also, since A^n is irreducible, the $\tilde{\lambda}_i$ are cubic numbers.

As (k_i) is periodic, telescoping gives a stationary sequence, so condition (12) decides on the eigenvalues of the the Koopman operator. To prove that there is a non-trivial eigenvalue, we need to show that $\vec{\xi}$ lies in some integer translation of E_3. That is

$$(\xi, \xi, \xi) + s(u_1, u_2, -1) = (p, q, r) \tag{13}$$

for some integers p, q, r and reals $u_1, u_2 > 0$. Eliminating s and ξ from this system, we obtain

$$(q - r)u_1 = (p - r)u_2 + (p - q). \tag{14}$$

For the matrix $A^n = (a_{ij})_{i,j=1}^3$, the third eigenvalue equation $\tilde{\lambda}_3(u_1, u_2, -1) = (u_1, u_2, -1)A$ gives

$$\begin{aligned} \tilde{\lambda}_3 u_1 &= a_{11}u_1 + a_{21}u_2 - a_{31}, \\ \tilde{\lambda}_3 u_2 &= a_{12}u_1 + a_{22}u_2 - a_{32}. \end{aligned}$$

Multiplying these equations and inserting (14) to eliminate u_1, we obtain

$$\begin{array}{rcl}\tilde{\lambda}_3((p-r)u_2+(p-q)) & = & a_{11}((p-r)u_2+(p-q))+a_{21}(q-r)u_2-(q-r)a_{31},\\ \tilde{\lambda}_3(q-r)u_2 & = & a_{12}((p-r)u_2+(p-q))+a_{22}(q-r)u_2-(q-r)a_{32}.\end{array}$$

Making u_2 subject of the second equation (so that the RHS is a fractilinear expression in $\tilde{\lambda}_3$) and then inserting it in the first equation gives an equation of two fractilinear expressions in $\tilde{\lambda}_3$. Therefore $\tilde{\lambda}_3$ is the solution of a quadratic equation contradicting that $\tilde{\lambda}_3$ is a cubic number.

For the preperiodic case, with preperiod m, we need to replace (13) by

$$(\xi,\xi,\xi)\cdot A_{k_1}\cdots A_{k_m}+s(u_1,u_2,-1)=(p,q,r).$$

This gives a more cumbersome version of (14) but the argument is essentially the same. $\square$

Remark 4.2. *Very recently*[2]*, Mercat [18] proved a general criterion for weak-mixing of S-adic shifts, which is directly computable if the involved sequence of substitutions is preperiodic. Mercat's criterion confirms our Theorem 4.1, see [18, Example 7.9].*

4.3 The stable direction.

Let $(\Omega,T_{\alpha,\beta})$ be an infinite type ITM based on a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (3). We call $W^s(\vec{0}) := \bigcap_n A_{k_1}^{-1}\circ\cdots\circ A_{k_n}^{-1}(Q^-)=\mathrm{span}(\vec{v}_3)$ the *stable space* of the sequence $(A_{k_i})_{i\in\mathbb{N}}$. In order to find $W^s(\vec{0})$, we iterate the matrices B_{k_i} from (7). This gives

$$(u,v,w)=\lim_{n\to\infty}\frac{\mathbb{B}^n(\vec{a})}{\|\mathbb{B}^n(\vec{a})\|_1}\qquad\text{for}\qquad\mathbb{B}^n=B_{k_n}\circ\cdots\circ B_{k_1}$$

and, apart from the assumption of unique ergodicity, the choice of the vector $\vec{a}\in Q^+$ can be arbitrary. Then $(v,u,-w)=(u,v,w)U$ (with U from (7)) is the stable direction of $A_{k_1}\cdot A_{k_2}\cdots$, normalised so that $u+v+w=1$.

Lemma 4.3. *The direction (u,v,w) is uniquely determined by the sequence $(k_i)_{i\in\mathbb{N}}$, provided it satisfies* (3).

Thus, even if $T_{\alpha,\beta}$ fails to be uniquely ergodic and there is no unique unstable direction in the first octant, the stable direction in Q^- is always well-defined.

Proof. Since $(k_i)_{i\in\mathbb{N}}$ satisfies (3), there are infinitely many i and integers $r\geq 0$ such that $k_i\geq 2$, $k_{i-1}=\cdots=k_{i-2r}=1$ and $k_{i-2r-1}\geq 2$. Abbreviate $a=k_i\geq 2$, $b=k_{i-2r-1}\geq 2$ and $c=k_{i-2r-2}\geq 1$. The telescoped block from k_i to k_{i+2r+2} gives

$$\begin{array}{rcl}\tilde{B} & = & B_a\cdot B_1^{2r}\cdot B_b\cdot B_c\\ & = & \begin{pmatrix}(a-1)(r+1) & (a-1)b(r+1)+1 & c((a-1)b(r+1)+1)-(r(a-1)+1)\\ 1 & b-1 & c(b-1)\\ a(r+1) & ab(r+1)+1 & c(ab(r+1)+1)-(ra+1)\end{pmatrix},\end{array}$$

which is a strictly positive matrix. Therefore it represents a strict contraction in the Hilbert metric on the first octant, see e.g. [12, Section 8.6]. The contraction factor is bounded by $\tanh(\frac{1}{2}\log(\rho))$, where[3]

$$\rho=\max_{1\leq j,j'\leq 3}\sqrt{\frac{\max\{\tilde{B}_{j,k}/\tilde{B}_{j',k}:1\leq k\leq 3\}}{\min\{\tilde{B}_{j,k}/\tilde{B}_{j',k}:1\leq k\leq 3\}}}.\qquad(15)$$

[2] Mercat's preprint was uploaded onto the arXiv four months after we first submitted our paper.

[3] We phrase this different from formula (8.29) in [12] because we are using left-multiplication with row vectors instead of right multiplication with column vectors as in [12].

The shape of $\tilde{B}$, where the rows R_i component-wise satisfy

$$\frac{a(r+1)}{3}R_2 < R_1 < 3a(r+1)R_2, \qquad \frac{1}{3}R_3 < R_1 < 3R_3, \qquad \frac{a(r+1)}{3}R_2 < R_3 < 3a(r+1)R_2.$$

Therefore $\rho \leq 3$, and hence the contraction factor is strictly smaller than 1, uniformly in a, b, c. As there are infinitely many such telescoped blocks $\tilde{B}$, the infinite matrix product contracts the positive octant into a single half-line ℓ. Thus $U\ell$ for the matrix U from (7) represents the unique stable direction $W^s(\vec{0})$. $\square$

To find an eigenvalue, we have to solve

$$(\xi, \xi, \xi) = (p, q, r) + s(v, u, u + v - 1) \text{ for some } p, q, r \in \mathbb{Z}. \tag{16}$$

Solving for ξ and s gives arcs $\ell_{p,q,r} = \{(u, v) \in \Delta : (1-u)(r-q) = (1-v)(r-p)\}$ in the simplex $\Delta = \{(u, v) : 0 \leq u \leq 1 - v\}$, or equivalently

$$\ell_{p,q,r} = \{(u, v) \in \Delta : u(q - r) = v(p - r) + q - p\}. \tag{17}$$

Expressed in terms of p, q, r, ξ, we find

$$u = \frac{\xi - q}{\xi + r - p - q}, \qquad v = \frac{\xi - p}{\xi + r - p - q}, \tag{18}$$

so that $\xi = \frac{r-p}{1-u} + p + q - r = \frac{r-q}{1-v} + p + q - r \in \mathbb{Q}$ if and only if both $u, v \in \mathbb{Q}$. We define $\ell_{p,q,r}(\xi)$ as those points $(u, v) \in \Delta$ such that (18) holds. Let $H_k(u, v)$ indicate the first two coordinates of $(u, v, 1 - (u + v))B_k$ normalised to unit length. Then $H_k : \Delta \to \Delta_k := H_k(\Delta)$ is a map from the unit simplex Δ, and it has the formula

$$H_k : (u, v) = \frac{1}{D_k}(v\ ,\ 1 - v) \quad \text{ for } \quad D_k = k(1 - v) + 1 - u, \tag{19}$$

$$H_k^{-1} : (x, y) \mapsto \left(\frac{x + (k+1)y - 1}{x + y}\ ,\ \frac{x}{x + y} \right) \quad \text{ for } \quad (x, y) \in \Delta_k. \tag{20}$$

The derivative and its determinant are

$$DH_k(u, v) = \frac{1}{D_k^2} \begin{pmatrix} v & k + 1 - u \\ 1 - v & u - 1 \end{pmatrix}, \qquad \det DH_k(u, v) = -\frac{1}{D_k}.$$

To find the set Δ_k, we compute the images of the three boundary lines of Δ:

$$\begin{aligned}
(a') \to (a) \qquad & H_k(\{0\} \times [0, 1]) = \left\{ \frac{1}{k(1 - v) + 1}(v, 1 - v) : v \in [0, 1] \right\}, \\
(b') \to (b) \qquad & H_k([0, 1] \times \{0\}) = \left\{ \frac{1}{k + 1 - u}(0, 1) : u \in [0, 1] \right\}, \\
(c') \to (c) \qquad & H_k(\{(1 - v, v) : v \in [0, 1]\}) = \left\{ \frac{1}{k(1 - v) + v}(v, 1 - v) : v \in [0, 1] \right\},
\end{aligned}$$

see Figure 3, left. Comparing (a) and (c) we can see that the triangles Δ_k and Δ_{k+1} are adjacent, and Δ_1 is adjacent to the upper boundary of Δ. That is, the Δ_k's have disjoint interiors and $\Delta = \overline{\bigcup_{k \geq 1} \Delta_k}$. Furthermore, unless $k = k' = 1$, $H_k \circ H_{k'}$ maps $\partial\Delta$ into the interior of Δ, and hence, for every $(u, v) \in \Delta$, there are no common boundaries of Δ_k's that are the image of any point $H_k \circ H_{k'} \circ H_{k''} \dots$. It follows that for each $(u, v) \in \Delta$, there is at most one sequence $(k_i)_{i \in \mathbb{N}}$ such that $\lim_{n \to \infty} H_{k_1} \circ H_{k_2} \circ \dots \circ H_{k_n}(\vec{a}) = (u, v)$.

The boundary points of the Δ_k cannot be reached as limit from an interior point $(u, v) \in \Delta^\circ$, and under later iterates of the H_k's, the iterated boundary points $E := \bigcup_{n \in \mathbb{N}} \bigcup_{k_1, \dots, k_n} H_{k_1} \circ \dots \circ H_{k_n}(\partial\Delta)$ cannot be equal to any point in $\bigcap_n \bigcup_{k_1, \dots, k_n} H_{k_1} \circ \dots \circ H_{k_n}(\Delta^\circ)$.

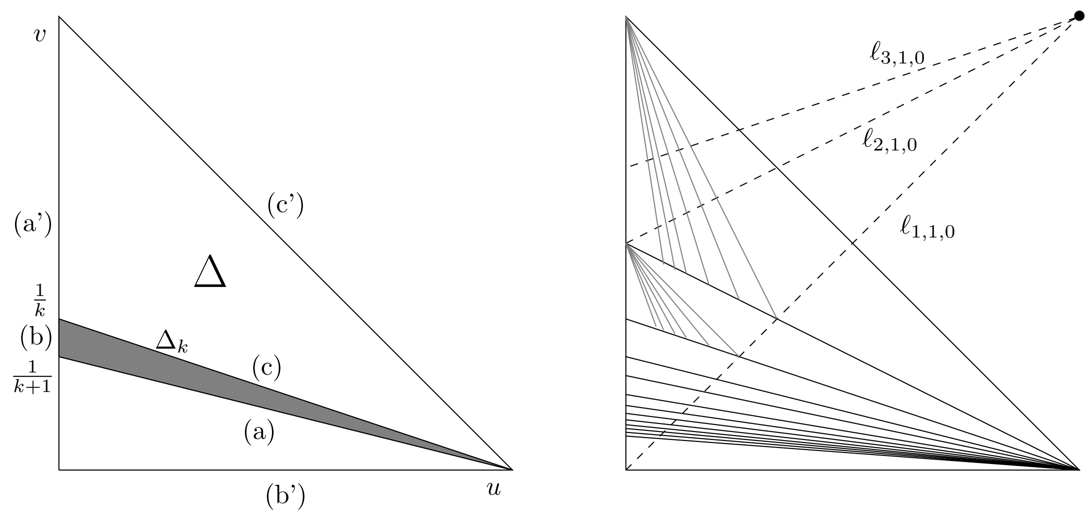

Figure 3: The simplex Δ and image $\Delta_k = H_k(\Delta)$ (left) and further images (right).

Remark 4.4. *The points $(x, y) \in \Delta$ which give rise to parameters $(k_n)_{n\in\mathbb{N}}$ such that the subshift associated is linearly recurrent, are bounded away from $\partial\Delta$ for all preimages $H_{k_n}^{-1} \circ \cdots \circ H_{k_1}^{-1}(x, y)$. For $(k_n)_{n\in\mathbb{N}}$ unbounded, a subsequence of preimages goes to the edges (a'), (b'); in case of unbounded blocks of 1's in $(k_n)_{n\in\mathbb{N}}$, a subsequence of preimages goes to (c'). Conjecturally, this set has full Hausdorff dimension, but for each fixed bound C, the set of parameters for which $\limsup_n k_n \le C$ and the lengths of blocks of 1's is eventually bounded by C has Hausdorff dimension strictly less than 1. Since the maps H_k are non-conformal, the current techniques of fractal geometry seem insufficient to prove this.*

The maps H_k send points with rational coordinates to points with rational coordinates, and hence the lines between such points (thus with rational slope) to lines of the same properties.

Proposition 4.5. *The sides of every triangle $H_{k_1} \circ \cdots \circ H_{k_n}(\partial\Delta)$ have negative slopes for all $n \ge 1$ and $k_1, \dots, k_n \in \mathbb{N}$.*

Proof. With some stretch of terminology, the statement holds for $n = 0$, so for Δ itself: here the "bottom" side $\partial_b\Delta$ has slope 0 and the left side $\partial_l\Delta$ needs to be interpreted as with slope $-\infty$. Only the "right" side $\partial_r\Delta$ has a proper negative slope -1.

Now $\bigcup_{k\in\mathbb{N}} \partial\Delta_k$ consists of $\partial_b\Delta$ and a fan F of straight lines stretching from $(1, 0)$ to $\partial_l\Delta$, see Figure 3, right. The slope of these lines are therefore all negative.

The maps H_k reverse orientation: they are reflections in lines of positive slope combined with a (non-linear) contraction. Since the vertex $V_k := (0, \frac{1}{k}) = H_k(1, 0)$ of Δ_k is the intersection of $\partial_l\Delta$ and $\partial_r\Delta_k$, $H_k(F)$ is a fan of lines stretching from V_k to the opposite boundary $\partial_b\Delta_k$. These lines have slopes between the slopes of $\partial_l\Delta_k$ and $\partial_r\Delta_k$, so they are negative.

The proof follows from repeating this argument inductively. □

4.4 Continuous eigenvalues for the general case.

Let $(E_n, V_n, \succ)$ be an ordered Bratteli diagram with Vershik transformation τ and associated matrices A_i. (See [15] or [12, Chapter 5.4] for the precise definitions.) Theorem 2 in [15] states for **positive** transition matrices M_j, $j \ge 1$, that $e^{2\pi i\xi}$ is a continuous eigenvalue if and only if

$$\sum_{n=1}^{\infty} \max_{x\in X} \| \langle s_n(x), \vec{\xi} M_1 \cdots M_n \rangle \| < \infty, \qquad \vec{\xi} = (\xi, \xi, \xi), \tag{21}$$

where $(s_n(x))_v = \#\{e \in E_{n+1} \ : \ e \succ x_{n+1}, \mathfrak{s}(e) = v\}$. Thus in the ordered Bratteli-Vershik diagram (with $\succ$ indicating the order of the incoming edges) the vector $s_n(x)$ counts the number of incoming edges that are higher in the order than edge x_{n+1} in the path x. This definition of s_n is phrased in terms of ordered Bratteli diagrams. In terms of S-adic shifts (X, σ) based on substitutions $\chi_n : \mathcal{A}_n \to \mathcal{A}_{n-1}$, every $x \in X$ can be written as

$$x = \lim_{n\to\infty} \sigma^{j_1} \circ \chi_1 \circ \sigma^{j_2} \circ \chi_2 \dots \sigma^{j_n} \circ \chi_n(a_n)$$

for some integer sequence (j_n) and symbols $a_n \in \mathcal{A}_n$ such that a_n is the first letter of $\sigma^{j_{n+1}} \circ \chi_{k_{n+1}}(a_{n+1})$. Then

$$s_{n,b}(x) = |\sigma^{j_{n+1}} \circ \chi_{k_{n+1}}(a_{n+1})|_b, \qquad b \in \mathcal{A}_n,$$

where $|w|_b$ indicates the number of symbols b in the word w.

We will use this for our situation, so with χ_{k_i} and A_{k_i} (or in fact, the telescoped version $\tilde{A}_i$ from (6)) replacing χ_i and A_i. The paper [15] is formulated for invertible S-adic shifts, but the Bratteli-Vershik system having a unique minimal path (which is true in our case because $\chi_{k_i} \circ \chi_{k_{i+1}}$ is left-proper) is sufficient.

Lemma 4.6. *Let $\{\tilde{A}_i\}_{i\geq 1}$ be the telescoped version of $(A_i)_{i\geq 1}$ given in* (6)*, and $\tilde{s}_{i+1}$ the associated s-value of $\tilde{A}_{i+1}$. Then*

$$\sum_{n=1}^{\infty} \max_{x\in X} \| \langle s_n(x), \vec{\xi} A_{k_1} \cdots A_{k_n} \rangle \| \geq \sum_{n=1}^{\infty} \max_{x\in X} \| \langle \tilde{s}_n(x), \vec{\xi} \tilde{A}_1 \cdots \tilde{A}_n \rangle \|. \tag{22}$$

Proof. Let $\tilde{h}_n = (1,1,1)\tilde{A}_1 \cdots \tilde{A}_n$. We can decompose the telescoped $\langle \tilde{s}_i(x), \tilde{h}_i \rangle$ back into levels n_i (the index of the first matrix in the product $\tilde{A}_i$) to $n_{i+1} - 1$ (the index of $A_{k_{i,m+2}}$). It follows that

$$\langle \tilde{s}_i(x), \tilde{h}_i \rangle = \sum_{j=n_i}^{n_{i+1}-1} \langle s_j(x), h_j \rangle.$$

Then

$$\begin{aligned}
\sum_{i=1}^{\infty} \max_{x\in X} \| \xi \langle \tilde{s}_i(x), \tilde{h}_i \rangle \| &= \sum_{i=1}^{\infty} \max_{x\in X} \| \xi \sum_{j=n_i}^{n_{i+1}-1} \langle s_j(x), h_j \rangle \| \\
&\leq \sum_{i=1}^{\infty} \max_{x\in X} \sum_{j=n_i}^{n_{i+1}-1} \| \xi \langle s_j(x), h_j \rangle \| \\
&\leq \sum_{j=1}^{\infty} \max_{x\in X} \| \xi \langle s_j(x), h_j \rangle \|,
\end{aligned}$$

as claimed. □

Remark 4.7. *For our specific case of infinite type ITMs $s_j(x)$ can be simplified to the vector $(r_j, 0, 0)^T$ with $r_j \in \{0, \dots k_j\}$ depending on the choice of x. Then the sum is*

$$\sum_{j=1}^{\infty} \max_{x\in X} \| \xi \langle s_j(x), h_j \rangle \| = \sum_{j=1}^{\infty} \max_{r_j \in \{0,\dots k_j\}} \| \xi r_j h_j(1) \|.$$

One would expect (in accordance with the Veech criterion [21]) the Koopman operator to have a non-trivial eigenvalue if and only if $(u, v) \in \ell_{p,q,r}$ for some $p, q, r \in \mathbb{Z}$ and otherwise the map is weak mixing. However, also because we have to deal with summability condition (21), especially the factor

$s_n(x)$, the story is not as clear-cut as that. In one direction, it is as expected. We will use the notation $W^s(\vec{0}) \bmod 1$ to indicate the stable manifold of $\vec{0}$ for the non-autonomous dynamical system on the 3-torus $\mathbb{T}^3$ induced by the sequence of matrices $(A_i)_{i\geq 1}$. It is an immersed line in the direction of $W^s(0)$ and wrapping densely through the torus.

Theorem 4.8. *If $\vec{\xi}$ does not belong to the stable space $W^s(\vec{0}) \bmod 1$ for $\xi \in (0,1)$, then the corresponding ITM has no continuous eigenvalue.*

Proof. Let $\vec{\xi}_0 = \vec{\xi} \bmod 1$ and $\vec{\xi}_{n+1} = \vec{\xi}_n \tilde{A}_{n+1} \bmod 1 \in (-\frac{1}{2}, \frac{1}{2}]^3$. Assume by contradiction that condition (21) holds, i.e., $\max_{x\in X} \|\langle \tilde{s}_n(x), \vec{\xi}_n\rangle\|$ is summable. Take $n_0 \in \mathbb{N}$ so that $\max_{x\in X} \|\langle \tilde{s}_n(x), \vec{\xi}_n\rangle\| < 1/4$ for all $n \geq n_0$.

The vectors $\tilde{s}_n(x)$ have non-negative entries, but are smaller than the v-th column of $\tilde{A}_{n+1}$ if the path $x \in X$ is such that $r(x_{n+1}) = v$. Using the language of BV-diagrams, if x_{n+1} is the smallest incoming edge to v, then $\tilde{s}_n(x)$ has the largest possible value, say $s_n^+(x)$, which is the v-th column of $\tilde{A}_{n+1}$ with one entry decreased by 1. Taking successive incoming edges for x_{n+1}, $\tilde{s}_n(x)$ decreases each time by one unit, depending on the source vertex $\mathfrak{s}(x_{n+1})$.

Now if $\vec{\xi}_n \in B_\varepsilon(\vec{0})$, but $\langle \tilde{s}_n^+(x), \vec{\xi}_n\rangle$ is close to a non-zero integer, then choosing paths x with x_{n+1} equal to successive incoming edges to v, $\langle \tilde{s}_n(x), \vec{\xi}_n\rangle$ decreases every step by an amount $< \varepsilon$. At some step, $\|\langle \tilde{s}_n(x), \vec{\xi}_n\rangle\| > \frac{1}{3}$, contradicting that $\max_{x\in X} \|\langle \tilde{s}_n(x), \vec{\xi}_n\rangle\| < 1/4$.

This implies that $\vec{\xi}_n \cdot \tilde{A}_{n+1} \cdots \tilde{A}_{n+m} \to \vec{0}$ as $m \to \infty$, contrary to the assumption that $\vec{\xi} \notin W^s \bmod 1$, and the proof is complete. □

The next proposition together with (18) shows that there cannot be an eigenvalue $e^{2\pi i\xi}$ for rational ξ, because $\frac{1}{\xi+r-p-q}(\xi - q, \xi - p) \in \Delta$ is contained in $H_{k_1} \circ \cdots \circ H_{k_n}(\partial\Delta)$ for n sufficiently large, and that means there is no sequence $(k_i)_{i\in\mathbb{N}}$ satisfying (3) such that $\ell_{p,q,r}(\xi) = \lim_{n\to\infty} H_{k_1} \circ \cdots \circ H_{k_n}(\Delta^\circ)$.

Proposition 4.9. *Every rational point in Δ belongs to $\bigcup_{n\geq 0} \bigcup_{k_1,\dots,k_n\in\mathbb{N}} H_{k_1} \circ \cdots \circ H_{k_n}(\partial\Delta)$.*

Proof. Suppose $(x,y) \in \Delta^\circ$ has rational coordinates. We can give them the same denominator, i.e., we write $(x,y) = (\frac{p}{q}, \frac{p'}{q})$ with $p + p' < q$ because $x + y < 1$. Take $k_1 \in \mathbb{N}$ such that $(x,y) \in H_{k_1}(\Delta)$. If $(x,y) \in H_{k_1}(\partial\Delta)$ we are done, so assume that $(x,y) \in H_{k_1}(\Delta^\circ)$. Then using (20) we get

$$H_{k_1}^{-1}(x,y) = \left(\frac{x + (k_1+1)y - 1}{x+y}\ ,\ \frac{x}{x+y}\right) = \left(\frac{p + (k_1+1)p' - q}{p+p'}\ ,\ \frac{p}{p+p'}\right),$$

which are fractions of common denominator $p + p' < q$, independently of k_1. Continuing this way, we find $n < q$ such that $H_{k_n}^{-1} \circ \cdots \circ H_{k_1}^{-1}(x,y)$ lies on the line $\{x + y = 1\} \subset \partial\Delta$. □

Theorem 4.10. *There exist parameters (α, β) such that $T_{\alpha,\beta}$ is of type ∞ and $\vec{\xi}$ does belong to the stable space $W^s(\vec{0}) \bmod 1$ for $\xi \in (0,1)$, but $e^{2\pi i\xi}$ is not a continuous eigenvalue of the Koopman operator.*

Proof. From (17) it follows that a parameter $(u,v) \in \Delta$ is such that $\vec{\xi}$ is in the stable subspace $W^s(\vec{0})$ only if it belongs to $\ell_{p,q,r}$ for some $p,q,r \in \mathbb{Z}$. The lines $\ell_{p,q,r}$ all pass through $(1,1)$ and have positive slope if they indeed intersect Δ. Therefore if $\ell_{p,q,r}$ intersections some subtriangle $H_{k_1} \circ \cdots \circ H_{k_n}(\Delta)$, it goes through its upper side.

Assume now that $\tilde{A}_1 \cdots \tilde{A}_i$ is a block of telescoped matrices, each of the form (6). Then $\varepsilon_i := \|\vec{\xi} A_{k_1} \cdots A_{k_n}\|$ is exponentially small in i (or even smaller). For the block $\tilde{A}_{i+1}$ as in (6), we choose $m = 1$, and let $r_{i+1,1}$ be some large integer and $k_{i+1,1} = k_{i+1,2} = 2$, then $\tilde{A}_{i+1} = A_1^r \cdot A_{k_{i+1,1}} \cdot A_{k_{i+1,2}} \cdot A_{k_{i+1,3}}$ is the next telescoped matrix, and the corresponding value $\tilde{s}_i \approx r$. Choosing r sufficiently large, we can assure that $\frac{1}{i} \leq |\langle \tilde{s}_i, \vec{\xi} \tilde{A}_1 \cdots \tilde{A}_i\rangle| \leq \frac{1}{2}$. Then the summability condition (21) for a continuous eigenvalue fails. □

Theorem 4.11. *The set of parameters $(\alpha, \beta) \in \Omega_\infty$ such that $T_{\alpha,\beta}$ does not have an eigenvalue $e^{2\pi i\xi}$ with $\xi \in (0, 1)$ contains a dense G_δ-subset of Ω_∞.*

Hence, for a dense G_δ-subset of parameters $(\alpha, \beta) \in \Omega_\infty$, the map $G_{\alpha,\beta}$ is weak mixing.

Proof. The map $G : U \to U \cup L$ from (1) is piecewise continuous. Let $U_k = \{(\alpha, \beta) \in \mathbb{R}^2 : \frac{1}{k+1} < \alpha \le \frac{1}{k}, 0 \le \beta \le \alpha\}$, $k \in \mathbb{N}$, be the domains on which G is continuous. The sets Ω_∞ and $\Delta_\infty = \bigcap_n \bigcup_{k_1,\dots,k_n \in \mathbb{N}^n} H_{k_1} \circ \dots \circ H_{k_n}(\Delta^\circ)$ are homeomorphic via the coding sequences $(k_i)_{i\in\mathbb{N}}$ satisfying (3). Indeed, every cylinder set $[k_1, \dots, k_n]$ corresponds to open triangles $\Omega_{k_1\dots k_n}$ in parameter space and $\Delta_{k_1,\dots,k_n}$ in Δ, and these triangles form a topological basis of Ω_∞ and Δ_∞ respectively. Common boundary points of such triangles don't belong to Ω_∞ or Δ_∞, so Ω_∞ and Δ_∞ are zero-dimensional sets without isolated points, but not compact. The itinerary maps

$$\begin{cases} (\alpha, \beta) \mapsto (k_i)_{i\in\mathbb{N}}, & G^{i-1}(\alpha, \beta) \in U^\circ_{k_i}, \\ (u, v) \mapsto (k_i)_{i\in\mathbb{N}}, & H^{1-i}(u, v) \in \Delta^\circ_{k_i}, \end{cases}$$

are homeomorphisms, and the composition $h : \Delta_\infty \to \Omega_\infty$ that assigns to $(u, v) \in \Delta_\infty$ the unique parameter pair $(\alpha, \beta) \in \Omega_\infty$ so that their itineraries coincide is the required homeomorphism between Δ_∞ and Ω_∞. Since $\Delta_\infty \setminus \bigcup_{p,q,r\in\mathbb{Z}} \ell_{p,q,r}$ is clearly a dense G_δ set and no (u, v) satisfies the summability condition (21), the h-image of this set is the required dense G_δ-subset of Ω_∞ of parameters failing (21). $\square$

4.5 Weak mixing and measurable eigenvalues for the general case.

For the Bratteli-Vershik representation $(V, E, \le)$ of the ITM let $h_n \in \mathbb{Z}^3$ be the vector with entries

$$h_n(v) = \#\{\text{paths from vertex } v_0 \text{ to } v \in V_n\} = (1, \dots, 1)M_1 \cdots M_n$$

and let μ be an ergodic probability measure. A necessary and sufficient condition for $e^{2\pi i\xi}$ to be a (measure-theoretic with respect to μ) eigenvalue is the following ([9] and [12, Prop 6.122]): There is a sequence of functions $\rho_n : \tilde{V}_{n+1} \to \mathbb{R}$ such that

$$g_n(x) := \left(\tilde{S}_n(x) + \rho_n(\mathfrak{t}(x_{n+1}))\right) \xi \bmod 1 \text{ converges for } \mu\text{-a.e. } x \in X_{BV} \text{ as } n \to \infty, \tag{23}$$

where $\tilde{S}_n(x) = \sum_{j=1}^n \langle \tilde{s}_j(x), h_j(\mathfrak{s}(x_{j+1}))\rangle$ is the minimal number $k \ge 1$ of iterates of the Vershik map such that $\tau^k(x)_{n+2} \ne x_{n+2}$. (Recall $\tilde{s}_j(x)$ from (21) with $\tilde{A}_k$ instead of M_k.) Furthermore, μ is a τ-invariant and ergodic probability measure. The condition $\liminf_j k_j < \infty$ made in this section implies that there is only one such measure, see Corollary 4.15.

Let Σ be the unit simplex in $\mathbb{R}^3$ and for $w \in \mathbb{R}^3_{\ge 0} \setminus \{0\}$, let $\pi(w) = \frac{w}{\|w\|_1}$ denote the projection of w onto Σ. In terms of properties of the matrices A_{k_j}, unique ergodicity is equivalent to

$$\pi\left(\bigcap_{j>m} A_{k_{m+1}} \cdots A_{k_j}(\mathbb{R}^3_{\ge 0})\right) = \ell_m$$

is a single point in Σ. This enables us, for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, to find a sequence $r_n \nearrow \infty$ and telescope the sequence of substitutions (and associated matrices) accordingly, such that for $\tilde{\chi}_{n+1} := \chi_{k_{r_{n-1}+1}} \circ \dots \circ \chi_{k_{r_{n+1}}}$ with associated matrices $\tilde{A}_{n+1} = A_{k_{r_n+1}} \cdots A_{k_{r_{n+1}}}$ we have for all $n \in \mathbb{N}$:

$$\sup_{a\in\{1,2,3\}} \sup_{b,b'\in\{1,2,3\}} \frac{f_a(b')}{f_a(b)} \le \sup_{a\in\{1,2,3\}} \sup_{x,y\in\mathbb{R}^3_{>0}} \frac{\pi(\tilde{A}_{n+1}x)_a}{\pi(\tilde{A}_{n+1}y)_a} \le 1 + \varepsilon_{n+1}, \tag{24}$$

for the frequencies $f_a(b) := \frac{|\tilde{\chi}_{n+1}(b)|_a}{|\tilde{\chi}_{n+1}(b)|}$.

Theorem 4.12. *If $\liminf_n k_n < \infty$ and $\vec{\xi}$ does not belong to the stable space $W^s(\vec{0}) \bmod 1$ for $\xi \in (0,1)$, then the corresponding ITM is weakly mixing.*

Contrary to Theorem 4.8 about the absence of continuous eigenvalues, the proof below does depend explicitly on the substitutions χ_{k_i} rather than only its abelianizations. It remains an open question whether our family of ITMs contains maps with a measurable non-continuous eigenvalue.

To prove Theorem 4.12, we need some lemmas, which feature the constant

$$C^* := 3 + \liminf_j k_j. \tag{25}$$

First we show the existence of a common prefix in words $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(v)$ for all $v \in \{1,2,3\}$. Then we show that the number of occurrences of any symbol in $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(v)$ are comparable for all $v \in \{1,2,3\}$ and thus, so are the heights $h_n(v)$. With these results we prove that ITMs with $\liminf_n k_n < \infty$ are uniquely ergodic and that such systems are typical in the parameter space. The last lemma shows that the common prefix holds significant mass, thus we can use the behaviour of points x in the prefix to show the non-existence of eigenvalues for the theorem.

Lemma 4.13. *For all values of $k_1, \dots, k_n$, the word $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(3)$ is a prefix of $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(2)$. If $k_n \geq 2$, then $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(1)$ is a prefix of $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(3)$, up to its last letter which is 1 or 2 when n is even or odd. If $k_n = 1$ and $n \geq 2$, then $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(3)$ is a prefix of $\chi_{k_1} \circ \dots \chi_{k_n}(1)$.*

Proof. Direct by induction on n. □

For the remainder of this section, set

Lemma 4.14. *If $2 \leq k_n \leq C^*$ or $k_n = 1, k_{n-1} \geq 2$, then for every j such that $A_{k_{n-j}} \cdots A_{k_n}$ is a full matrix,*

$$|\chi_{k_{n-j}} \circ \cdots \circ \chi_{k_n}(v)|_u \leq 4C^* |\chi_{k_{n-j}} \circ \cdots \circ \chi_{k_n}(v')|_u$$

for all $v, v' \in V_n$ and $u \in \{1,2,3\}$. In particular, $h_n(v) \leq 4C^* h_n(v')$ for all $v, v' \in V_n$.

The proof is deferred to Section 4.6.

Corollary 4.15. *If (3) holds and $\liminf_j k_j < \infty$, then the corresponding ITM is uniquely ergodic. In particular, for every invariant measure ν on Ω_∞, ν-a.e. $(\alpha, \beta) \in \Omega_\infty$ corresponds to a uniquely ergodic ITM.*

Proof. We telescope the sequence χ_{k_j} into blocks $\chi_{k_{n_{i-1}+1}} \circ \cdots \circ \chi_{k_{n_i}}$ such that $\tilde{A}_i = A_{k_{n_{i-1}+1}} \cdots A_{k_{n_i}}$ is a full matrix and $k_{n_i} \leq C^*$ with C^* as in (25) and if $k_{n_i} = 1$ then $k_{n_{i-1}} \geq 2$ for all i. Using Lemma 4.14 we estimate

$$\rho(L, L') := \sqrt{\frac{\max\{P_u/P'_u : u = 1,2,3\}}{\min\{P_u/P'_u : u = 1,2,3\}}} \quad \text{with } P, P' \text{ columns of } \tilde{A}_i$$

for the Hilbert metric[4]. For these associated matrices, we get $\rho \leq 4C^*$. This gives a contraction factor $\tanh(\frac{1}{2}\log(\rho)) < 1$ for the Hilbert metric, independently of i. Hence, unique ergodicity follows.

Now if ν is an ergodic G-invariant measure, then there is $k \in \mathbb{N}$ such that $\nu(L_k) > 0$ for the strip $L_k = \{(a,b) \in \Omega_\infty : \frac{1}{k+1} < \alpha \leq \frac{1}{k}\}$. By the Ergodic Theorem, $\#\{i \in \mathbb{N} : G^i(\alpha,\beta) \in L_k\} = \infty$ for ν-a.e. (α, β). Hence for such (α,β) $\liminf_i k_i \leq k < \infty$, and $T_{\alpha,\beta}$ is uniquely ergodic. □

In the sequel, write $\tilde{\chi}_n = \chi_{\tilde{k}_1} \circ \cdots \circ \chi_{\tilde{k}_m}$ where $m = m(n)$. We will use Bratteli-Vershik system arguments, but still prefer to keep a metric that agrees with the metric on the subshift. Hence if (X, τ) is the Bratteli-Vershik system containing distinct edge-labeled paths x and y, then we set $d(x,y) = 2^{-n}$ for $n = \min\{j \geq 0 : \tau^j(x)_1 \neq \tau^j(y)_1\}$.

[4]This time we multiply by column vector on the right, so the notation is exactly as in [12, Formula 8.29] and transposed compared to (15).

For $x \in X_{BV}$ and $n \in \mathbb{N}$ fixed, define recursively

$$y^0 = (y^0_m)_{m \geq 1} = x \quad \text{ and } \quad y^{\ell+1} = \tau^{h_n(\mathfrak{s}(y^\ell_{n+1}))}(y^\ell). \tag{26}$$

This is a speed-up of the τ-orbit of x which, if one observes only the levels $\geq n+1$ of the Bratteli diagram, acts as τ itself. The next lemma shows that given sufficiently large n, this speed-up doesn't change the path x on the first $n/(16C^*)$ levels of the Bratteli diagram.

Lemma 4.16. *Let μ be the uniquely ergodic measure. Then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ for which $\tilde{k}_{m(n)} \leq C^*$ from (25) and if $\tilde{k}_{m(n)} = 1$, then $\tilde{k}_{m(n)-1} \geq 2$, we have*

$$\mu(\{x \in [e] : d(x, y^\ell) < 2^{-n/16C^*}\}) \geq \frac{1}{16C^*}\mu([e]), \tag{27}$$

for all $\ell \geq 0$ and $e \in E_{n+1}$, where $[e] = \{x \in X_{BV} : x_{n+1} = e\}$.

Proof. The telescoping is such that $\tilde{\chi}_n(v) = \chi_{\tilde{k}_1} \circ \cdots \circ \chi_{\tilde{k}_m}(v)$ satisfies (24); in particular, for each $v \in V_n$, $\tilde{\chi}_n(v)$ contains every symbol in V_{n-1} multiple times, and $m = m(n)$ is so large that there are a minimal even $j' \geq 2$ and a minimal odd $j'' \geq 2$ both such that $\tilde{k}_{m-j'}, \tilde{k}_{m-j''} \geq 2$. Due to (3) such j' and j'' can always be found. Set $j = \max\{j', j''\}$. Then $\chi_{\tilde{k}_{m-j}} \circ \cdots \circ \chi_{\tilde{k}_m}(v)$, $v \in V_n$, have a maximal common prefix W' that contains every symbol. In the light of Lemma 4.13, W' equals $\chi_{\tilde{k}_{m-j}} \circ \cdots \circ \chi_{\tilde{k}_m}(1)$ minus its last letter if $\tilde{k}_m \geq 2$, and $W' = \chi_{\tilde{k}_{m-j}} \circ \cdots \circ \chi_{\tilde{k}_m}(3)$ if $\tilde{k}_{m-1} > \tilde{k}_m = 1$.

Clearly

$$W := \chi_{\tilde{k}_1} \circ \cdots \circ \chi_{\tilde{k}_{m-j-1}}(W')$$

is a common prefix of $\tilde{\chi}_n(v)$, $v \in V_n$, of length $|W| \geq \frac{1}{8C^*}|\tilde{\chi}_n(v)|$ according to Lemma 4.14, and W contains every symbol as well. Therefore

$$q := \left\lfloor \frac{1}{2}|\tilde{\chi}_1 \circ \cdots \circ \tilde{\chi}_{n-1}(W)| \right\rfloor \geq \frac{1}{16C^*} \max_{v \in V_n} |\tilde{\chi}_1 \circ \cdots \circ \tilde{\chi}_n(v)|.$$

Then as all paths from v_0 to $\mathfrak{t}(e)$ in the Bratteli diagram have the same mass, the union of the first q such paths in $[e]$ has mass $\geq \frac{1}{16C^*}\mu([e])$.

Take $v = \mathfrak{s}(e)$ and let $x_{\min}(v)$ be the minimal path from v_0 to v. For any $x \in [e] \cap \tau^j([x_{\min}(v)])$ with $0 \leq j \leq q$, if follows that $\tau^{h_n(v)}(x) \in \tau^j([x_{\min}(v')])$, where $v' \in V_n$ is the source of the successor edge to x_{n+1} (or if x_{n+1} is the maximal incoming edge, $v' \in V_n$ is the source of $\tau^{h_n(v)}(x)_{n+1}$). Because q is only half of the length of the common prefix W, $d(x, y^1) = d(x, \tau^{h_n(v)}(x)) \leq 2^{-n/16C^*}$. Since this is true for all $x \in \tau^j([x_{\min}(v)])$ and $0 \leq j \leq q$, and for every $\ell \geq 0$, $y^\ell \in \tau^j([x_{\min}(v'')])$ for some $v'' \in V_n$, the lemma follows. ☐

Proof of Theorem 4.12. Recall from (25) that $C^* = 3 + \liminf_j k_j$. We can assume that the telescoping of the BV-diagram can be done in such a way that (24) holds, and the last matrix in each telescoped block either has subscript $2 \leq k \leq C$ or is equal to 1, but then the subscript of the penultimate matrix is ≥ 2. Let $\vec{\xi}_0 = \vec{\xi}$ and $\vec{\zeta}_{n+1} = \vec{\xi}_n \cdot \tilde{A}_{n+1}$ and $\vec{\xi}_{n+1} = \vec{\zeta}_{n+1} - \vec{z}_{n+1} \in (-\frac{1}{2}, \frac{1}{2}]^3$, where $\vec{z}_{n+1} \in \mathbb{Z}^3$ is the closest integer vector to $\vec{\zeta}_{n+1}$. By assumption, $\vec{\xi} \notin W^s(\vec{0})$ mod 1, therefore there must be $\eta > 0$ such that $\|\vec{\zeta}_{n+1}\|_1 > \eta$ infinitely often.

Assume by contradiction that (23) holds. Let $g(x) = \lim_{n \to \infty} g_n(x)$ wherever it converges. By Lusin's Theorem, for every $\delta > 0$, there is a compact subset $X' \subset X_{BV}$ such that $\mu(X_{BV} \setminus X') < \delta$, g is uniformly continuous on X' and $\lim_{n \to \infty} g_n(x) = g(x)$ for all $x \in X'$. Pick $\delta < 1/(144C^*)$. Then, for $\varepsilon \in (0, \eta/(288C^*))$ arbitrary, there is $N \in \mathbb{N}$ such that for every $n > N$ and $x, y \in X'$ such that $d(x, y) < 2^{-N/16C^*}$ we have

$$|g(x) - g(y)| < \varepsilon \quad \text{ and } \quad |g_n(x) - g(x)| < \varepsilon. \tag{28}$$

It also follows that $|g_{n+1}(x) - g_n(x)| < 2\varepsilon$ for all $n \geq N$.

Let $X_n = \{x \in \tau^j([x_{\min}(v)]) : v \in V_n, 0 \le j < q\}$ with the notation q and $x_{\min}(v)$ from Lemma 4.16. In fact, X_n consists of all the paths $x \in X$ such that the finite path $(x_1, \dots, x_n)$ is among the first q paths from v_0 to $\mathfrak{s}(x_{n+1})$ and $\mu(X_n) \ge 1/(16C)$.

Recalling also the points y^ℓ from (26), we get from Lemma 4.16 that

$$d(x, y^\ell) < 2^{-n/16C^*} \text{ for all } x \in X_n \text{ and } \ell \ge 0. \tag{29}$$

Now take $x \in X_n$ and corresponding y^ℓ such that $x, y^\ell \in X'$ and $x_{n+2} = y^j_{n+2}$ for all $0 \le j \le \ell$. Then

$$\begin{aligned} \Big\| \sum_{j=0}^{\ell-1} \xi h_n(\mathfrak{s}(y^j_{n+1})) \Big\| &= \|\xi \cdot (\tilde{S}_n(x) - \tilde{S}_n(y^\ell))\| \\ &= \|\xi \cdot \tilde{S}_n(x) \bmod 1 - \xi \cdot \tilde{S}_n(y^\ell) \bmod 1\| = \|g_n(x) - g_n(y^\ell)\| \\ &\le \|g_n(x) - g(x)\| + \|g(x) - g(y^\ell)\| + \|g(y^\ell) - g_n(y^\ell)\| < 3\varepsilon. \end{aligned} \tag{30}$$

Now take $n \ge N$ such that $\|\vec{\zeta}_{n+1}\| \ge \eta$. Notice that the components of $\vec{\xi}_n$ are $\xi_n(v) = \xi h_n(v) \bmod 1 \in (-\frac{1}{2}, \frac{1}{2}]$.

$$\begin{aligned} \vec{\zeta}_{n+1} &= \vec{\xi}_n \cdot \tilde{A}_{n+1} = (\xi_n(1), \xi_n(2), \xi_n(3)) \cdot \tilde{A}_{n+1} \\ &= \left(\sum_{j=1}^{|\tilde{\chi}_{n+1}(1)|} \xi_n(\tilde{\chi}_{n+1}(1)_j), \sum_{j=1}^{|\tilde{\chi}_{n+1}(2)|} \xi_n(\tilde{\chi}_{n+1}(2)_j), \sum_{j=1}^{|\tilde{\chi}_{n+1}(3)|} \xi_n(\tilde{\chi}_{n+1}(3)_j) \right). \end{aligned}$$

By (24), for each $v \in V_n$, the frequencies $f_v(w)$ differ among the $w \in V_{n+1}$ by no more than a factor $1 + \varepsilon_{n+1}$, where we choose ε_{n+1} to be the smallest positive value among the sums $\left|\sum_{v \in V_n} \xi_n(v) f_v(w)\right|$ for $w \in V_{n+1}$. Lemma 4.14 applied to the block of substitution that produces $\tilde{\chi}_{n+1}$ gives $|\tilde{\chi}_{n+1}(w)| \le 4C^* |\tilde{\chi}_{n+1}(w')|$ for all $w, w' \in V_{n+1}$.

Since $\|\vec{\zeta}_{n+1}\|_1 \ge \eta$, we can pick $w \in V_{n+1}$ such that $\left| \sum_{j=1}^{|\tilde{\chi}_{n+1}(w)|} \xi_n(\tilde{\chi}_{n+1}(w)_j) \right| \ge \eta/3$. First assume that $\sum_{v \in V_n} \xi_n(v) f_v(w) > 0$. Then, recalling that $\sum_{v \in V_n} |\xi_n(v)| f_v(w) \le \frac{1}{2}$, we have

$$\begin{aligned} \frac{\eta}{24C^*} &\le \frac{1}{8C^*} \sum_{j=1}^{|\tilde{\chi}_{n+1}(w)|} \xi_n(\tilde{\chi}_{n+1}(w)_j) \le \frac{1}{4C^*} |\tilde{\chi}_{n+1}(w)| \cdot \frac{1}{2} \sum_{v \in V_n} \xi_n(v) f_v(w) \\ &\le \frac{1}{4C^*} |\tilde{\chi}_{n+1}(w)| \left(\sum_{v \in V_n} \xi_n(v) f_v(w) - \varepsilon_{n+1} \sum_{v \in V_n} |\xi_n(v)| f_v(w) \right) \\ &\le \frac{1}{4C^*} |\tilde{\chi}_{n+1}(w)| \left(\sum_{\xi_n(v) > 0} \xi_n(v) \frac{f_v(w)}{1 + \varepsilon_{n+1}} + \sum_{\xi_n(v) < 0} \xi_n(v) f_v(w)(1 + \varepsilon_{n+1}) \right) \\ &\le |\tilde{\chi}_{n+1}(w')| \left(\sum_{\xi_n(v) > 0} \xi_n(v) f_v(w') + \sum_{\xi_n(v) < 0} \xi_n(v) f_v(w') \right) \\ &= |\tilde{\chi}_{n+1}(w')| \sum_{v \in V_n} \xi_n(v) f_v(w') = \sum_{j=1}^{|\tilde{\chi}_{n+1}(w')|} \xi_n(\tilde{\chi}_{n+1}(w')_j). \end{aligned}$$

If on the other hand $\sum_{v\in V_n} \xi_n(v) f_v(w) < 0$, we change signs:

$$\begin{aligned}
\frac{\eta}{24C^*} &\leq \frac{1}{8C^*}\left|\sum_{j=1}^{|\tilde\chi_{n+1}(w)|} \xi_n(\tilde\chi_{n+1}(w)_j)\right| \leq \left|\frac{1}{4C^*}|\tilde\chi_{n+1}(w)|\cdot\frac{1}{2}\sum_{v\in V_n}\xi_n(v)f_v(w)\right| \\
&\leq -\frac{1}{4C^*}|\tilde\chi_{n+1}(w)|\left(\sum_{v\in V_n}\xi_n(v)f_v(w) - \varepsilon_{n+1}\sum_{v\in V_n}|\xi_n(v)|f_v(w)\right) \\
&\leq -\frac{1}{4C^*}|\tilde\chi_{n+1}(w)|\left(\sum_{\xi_n(v)>0}\xi_n(v)f_v(w)(1+\varepsilon_{n+1}) + \sum_{\xi_n(v)<0}\xi_n(v)\frac{f_v(w)}{1+\varepsilon_{n+1}}\right) \\
&\leq -|\tilde\chi_{n+1}(w')|\left(\sum_{\xi_n(v)>0}\xi_n(v)f_v(w') + \sum_{\xi_n(v)<0}\xi_n(v)f_v(w')\right) \\
&= -|\tilde\chi_{n+1}(w')|\sum_{v\in V_n}\xi_n(v)f_v(w') = \left|\sum_{j=1}^{|\tilde\chi_{n+1}(w')|}\xi_n(\tilde\chi_{n+1}(w')_j)\right|.
\end{aligned}$$

Recall that the $\tilde\chi_{n+1}(w)$ gives the order of incoming edges to $w \in V_{n+1}$. Without loss of generality, we can assume that (24) applies to any subword of $\tilde\chi_{n+1}(w)$ of length at least $|\tilde\chi_{n+1}(w)|/3$. That is, if $1 \leq a < a + |\tilde\chi_{n+1}(w)|/3 < b$, then

$$\frac{\eta}{72C^*} \leq \left|\sum_{j=a}^{b-1}\xi_n(\tilde\chi_{n+1}(w)_j)\right|. \tag{31}$$

Indeed, since the frequencies of letters in all of these subwords are almost the same, these sums (consisting of $b-a$ terms of three types $\xi_n(1), \xi_n(2)$ and $\xi_n(3)$) indicate almost collinear vectors of lengths almost proportional to $|b-a|/|\tilde\chi_{n+1}(w)|$.

Now choose the most frequent (measure-wise) $w \in V_{n+1}$ and $v \in V_n$; that is

$$\mu(\{x \in X_n' : \mathfrak{s}(x_{n+1}) = v, \mathfrak{t}(x_{n+1}) = w\}) \geq \frac{1}{144C^*} - \delta > 0.$$

For a path $u := u_1 \dots u_n \in \tau^j([x_{min}(v)])$ from v_0 to $v \in V_n$ and $0 \leq j \leq q$ and for each edge $a \in E_{n+1}$ with $\mathfrak{s}(a) = v$ and $\mathfrak{t}(a) = w$, let $x^{(a)}(u)$ be a path from v_0 to $w \in V_{n+1}$ such that

(i) $x^{(a)}_{n+1}(u) = a$, and

(ii) $x^{(a)}_k(u) = u_k$ for all $1 \leq k \leq n$.

Hence for edges $a, b \in E_{n+1}$ connecting $v \in V_n$ and $w \in V_{n+1}$ such that (31) holds. Then

$$\mu(X' \cap [x^{(a)}(u)]) > \frac{1}{2}\mu([x^{(a)}(u)]) \quad \text{ and } \quad \mu(X' \cap [x^{(b)}(u)]) > \frac{1}{2}\mu([x^{(b)}(u)]).$$

This implies that there exists $x \in [x^{(a)}(u)] \cap X'$ and $y \in [x^{(b)}(u)] \cap X'$ such that $y = y^{\ell}$ for some $\ell \geq 1$ in the sense of (26) (because $y^{(j)} \mapsto y^{(j+1)}$ is measure-preserving).

Then $\tilde S_{n-1}(x^{(a)}(u)) = \tilde S_{n-1}(x^{(b)}(u))$ and hence $g_{n-1}(x^{(a)}(u)) = g_{n-1}(x^{(b)}(u))$. Because $\mathfrak{s}(a) = \mathfrak{s}(b)$ and $b - a > |\tilde\chi_{n+1}(w)|/3$, it follows from (31) that

$$\left|g_n(x^{(b)}(u)) - g_n(x^{(a)}(u))\right| = \left|\xi\cdot\left(\tilde S_n(x^{(b)}(u)) - \tilde S_n(x^{(a)}(u))\right)\right| = \left|\sum_{j=a}^{b-1}\xi_n(\tilde\chi_{n+1}(w)_j)\right| \geq \frac{\eta}{72C^*}.$$

Therefore

$$\begin{aligned}
4\varepsilon &\geq \Big|(g_n(x^{(b)}(u)) - g_{n-1}(x^{(b)}(u))) + (g_{n-1}(x^{(b)}(u)) - g_{n-1}(x^{(a)}(u))) \\
&\qquad + (g_{n-1}(x^{(a)}(u)) - g_n(x^{(a)}(u)))\Big| \\
&= |g_n(x^{(b)}(u)) - g_n(x^{(a)}(u))| \geq \frac{\eta}{72C^*},
\end{aligned}$$

which contradicts the choice of ε. This finishes the proof. □

Corollary 4.17. *A linearly recurrent ITM of infinite type is weakly mixing if and only if $\vec{\xi}$ does not belong to the stable space $W^s(\vec{0}) \bmod 1$ for $\xi \in (0,1)$. Furthermore, any measurable eigenvalue is continuous.*

Proof. It follows from Theorem 4.12 that if $\vec{\xi} \notin W^s(\vec{0}) \bmod 1$, then the system is weakly mixing. If on the other hand $\vec{\xi} \in W^s(\vec{0}) \bmod 1$, then the convergence of $\|\vec{\xi} A_{k_1} \cdots A_{k_n}\|$ to zero is exponential. By [9, Theorem 1], a measurable eigenvalue $e^{2\pi i\xi}$ for linearly recurrent system exists if and only if

$$\sum_{n\geq 1} |||\vec{\xi}\tilde{A}_1 \cdots \tilde{A}_n|||^2 < \infty$$

and additionally the eigenvalue is continuous if and only if

$$\sum_{n\geq 1} |||\vec{\xi}\tilde{A}_1 \cdots \tilde{A}_n||| < \infty.$$

Thus if $\vec{\xi} \in W^s(\vec{0}) \bmod 1$, then the sum converges in both cases and $e^{2\pi i\xi}$ is a continuous eigenvalue of the ITM. □

4.6 The proof of Lemma 4.14

Proof of Lemma 4.14. Let us write $\mathbb{A}^m = A_{k_m} \cdots A_{k_n}$ for the matrix associated to the substitution $\chi_{k_m} \circ \cdots \circ \chi_{k_n}$. Then $\mathbb{A}^n = A_{k_n}$ and the columns $\mathbb{A}^m_2 \geq \mathbb{A}^m_3$ element-wise for every $m \leq n$.

Let

$$\mathbb{D}_m := A_{k_{m-2}} \cdot A_{k_{m-1}} = \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix}.$$

be the matrix associated to the next pair of substitutions $\chi_{k_{m-2}} \circ \chi_{k_{m-1}}$. Our proof is of algorithmic nature, illustrated by the following scheme:

The properties of the matrices in each state are:

State 1: $\mathbb{A}^m = \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n-1)+1 \end{pmatrix}$ for some integer $r \geq 0$.

If $k_n \geq 2$, then $\mathbb{A}^n$ is in this state, with $r = 0$.

State 2:

$$\mathbb{A}^m = \begin{pmatrix} 0 & k_n & k_n-1 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix} \qquad \begin{array}{rl} \text{for integers} & 1 \leq \min_i Q_i \leq \max_j Q_j < 3C^* \min_i Q_i - C^* \\ \text{satisfying} & 1 \leq \min_i R_i \leq \max_j R_j < 3C \min_i R_i - C^*. \end{array}$$

State 3:

$$\mathbb{A}^m = \begin{pmatrix} P_1 & P_2 & P_3 \\ q & 1-q & 0 \\ R_1 & R_2 & R_3 \end{pmatrix} \qquad \begin{array}{rl} \text{for integers} & 1 \leq \min_i P_i \leq \max_j P_j < 3C^* \min_i P_i \\ \text{satisfying} & 1 \leq \min_i R_i \leq \max_j R_j < 3C^* \min_i R_i \\ & q \in \{0,1\}. \end{array}$$

If $1 = k_n < k_{n-1}$, then $\mathbb{A}^n$ is in this state, with $q = 0$.

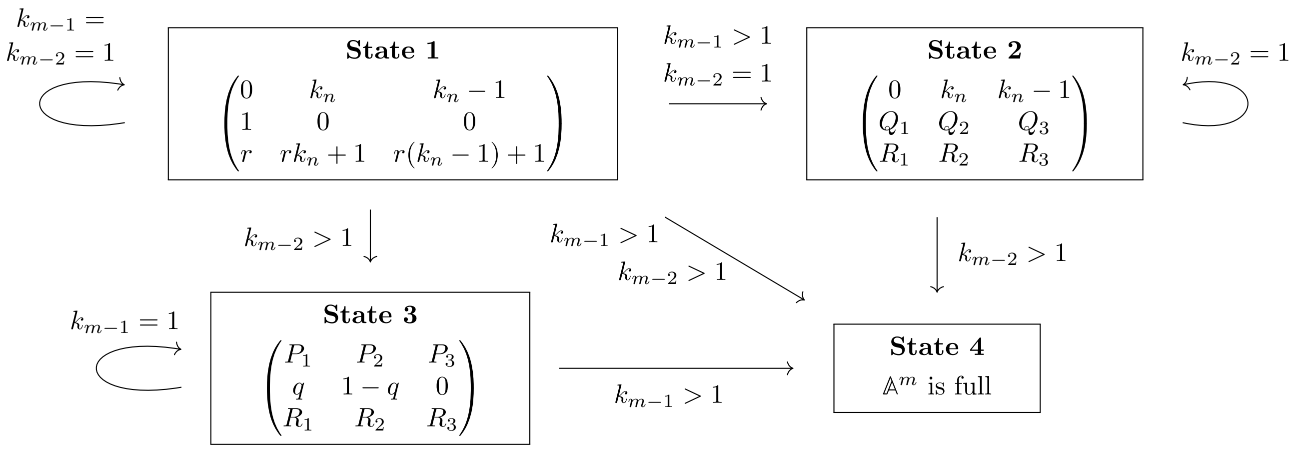

Figure 4: Every arrow stands for left multiplication with $\mathbb{D}_m$

State 4: $\mathbb{A}^m$ is full and $(\max_i \mathbb{A}_i^m)_u \leq 4C^*(\min_j \mathbb{A}_j^m)_u$ for all $u \in \{1, 2, 3\}$. The lemma follows from these inequalities.

Now we verify the transitions between the states.

- From State 1 to State 1: $k_{m-1} = k_{m-2} = 1$. In this case

$$\mathbb{D}_m = \mathbb{J} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

  and we compute:

$$\mathbb{J}^r \cdot \mathbb{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & r & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n - 1) + 1 \end{pmatrix}, \tag{32}$$

  as required.

- From State 1 to State 2: $k_{m-1} > 1 = k_{m-2}$. From (32) we get

$$\begin{aligned} \mathbb{A}^{m-2} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{m-1} & k_{m-1} - 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n - 1) + 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & k_n & k_n - 1 \\ X + k_{m-1} & Xk_n + k_{m-1} - 1 & X(k_n - 1) + k_{m-1} - 1 \\ r + 1 & (r+1)k_n + 1 & (r+1)(k_n - 1) + 1 \end{pmatrix} \end{aligned}$$

  for $X = r(k_{m-1} - 1) \geq 0$, and the properties of State 2 hold.

- From State 1 to State 3: $k_{m-2} > 1 = k_{m-1}$. From (32) we get

$$\begin{aligned} \mathbb{A}^{m-2} &= \begin{pmatrix} k_{m-2} & k_{m-2} - 1 & k_{m-2} - 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n - 1) + 1 \end{pmatrix} \\ &= \begin{pmatrix} Y - 1 & Yk_n + k_{m-2} - 1 & Y(k_n - 1) + k_{m-2} - 1 \\ 1 & 0 & 0 \\ r + 1 & (r+1)k_n + 1 & (r+1)(k_n - 1) + 1 \end{pmatrix} \end{aligned}$$

for $Y = r(k_{m-2}-1)+k_{m-2} \geq 2$, and the properties of State 3 hold with $q = 1$.

- From State 1 to State 4: $k_{m-2}, k_{m-1} > 1$. From (32) we get

$$\begin{aligned}\mathbb{A}^{m-2} &= \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n-1 \\ 1 & 0 & 0 \\ r & rk_n+1 & r(k_n-1)+1 \end{pmatrix} \\ &= \begin{pmatrix} Y-1 & Yk_n+k_{m-2}-1 & Y(k_n-1)+k_{m-2}-1 \\ X+k_{m-1} & Xk_n+k_{m-1}-1 & X(k_n-1)+k_{m-1}-1 \\ r+1 & (r+1)k_n+1 & (r+1)(k_n-1)+1 \end{pmatrix}\end{aligned}$$

for $X = r(k_{m-1}-1) \geq 0$, $Y = r(k_{m-2}-1)+k_{m-2} \geq 2$, and the properties of State 4 hold.

- From State 2 to State 2: $k_{m-1} \geq 1 = k_{m-2}$. Left multiplication with

$$\mathbb{D}_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix}$$

leaves the first row of $\mathbb{A}^m$ unchanged. For the second row the inequalities for the new $\tilde{Q}_i$ follow from

$$\begin{aligned}\tilde{Q}_i + C^* &= k_{m-1}Q_i + (k_{m-1}-1)R_i + C^* \\ &< 3C^*k_{m-1}Q_j + 3C^*(k_{m-1}-1)R_j = 3C^*\tilde{Q}_j\end{aligned}$$

and for the last row

$$\tilde{R}_i + C^* \leq Q_i + R_i + 2C^* < 3C^*Q_j + 3C^*R_j = 3C^*\tilde{R}_j.$$

So the conditions of State 2 remain valid.

- From State 3 to State 3: $k_{m-2} \geq 1 = k_{m-1}$. Left multiplication with

$$\mathbb{D}_m = \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

keeps the conditions of State 3 valid. By $R_i + 1 \leq 3C^*R_j$ it holds that for the new $\tilde{P}_1$

$$\tilde{P}_1 = P_1 + R_1 + q < 3C^*P_i + 3C^*R_i \leq 3C^*\tilde{P}_i$$

for all i and so on for the other entries.

- From State 2 to State 4: $k_{m-2} > 1$. Left multiplication with $\mathbb{D}_m$ gives

$$\begin{aligned}\mathbb{A}^{m-2} = \mathbb{D}_m \cdot \mathbb{A}^m &= \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n-1 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix} \\ &= \begin{pmatrix} (k_{m-2}-1)(Q_1+R_1) & \begin{array}{c} k_nk_{m-2}+ \\ (k_{m-2}-1)(Q_2+R_2) \end{array} & \begin{array}{c} (k_n-1)k_{m-2}+ \\ (k_{m-2}-1)(Q_3+R_3) \end{array} \\ k_{m-1}Q_1+(k_{m-1}-1)R_1 & k_{m-1}Q_2+(k_{m-1}-1)R_2 & k_{m-1}Q_3+(k_{m-1}-1)R_3 \\ Q_1+R_1 & k_n+Q_2+R_2 & k_n-1+Q_3+R_3 \end{pmatrix}\end{aligned}$$

keeps the inequality in the second and third row as before. For the first row

$$\begin{aligned}\tilde{P}_2 &= k_n k_{m-2} + (k_{m-2}-1)(Q_2+R_2)\\ &\leq (k_{m-2}-1)(Q_2+R_2+C^*) + C^*\\ &< 4C^*(k_{m-2}-1)(Q_j+R_j) \leq 4C^*\tilde{P}_j\end{aligned}$$

and analogously the inequalities hold for the other entries. Thus the conditions of State 4 hold.

- From State 3 to State 4: $k_{m-2} > 1$. Left multiplication with $\mathbb{D}_m$ gives

$$\mathbb{A}^{m-2} = \mathbb{D}_m \cdot \mathbb{A}^m = \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1\\ 0 & k_{m-1} & k_{m-1}-1\\ 1 & 1 & 1\end{pmatrix} \cdot \begin{pmatrix} P_1 & P_2 & P_3\\ q & 1-q & 0\\ R_1 & R_2 & R_3\end{pmatrix} =$$

$$= \begin{pmatrix} \begin{matrix} k_{m-2}P_1 + \\ (k_{m-2}-1)(q+R_1)\end{matrix} & \begin{matrix} k_{m-2}P_2 + \\ (k_{m-2}-1)(R_2+1-q)\end{matrix} & \begin{matrix} k_{m-2}P_3 + \\ (k_{m-2}-1)R_3\end{matrix}\\ qk_{m-1} + (k_{m-1}-1)R_1 & (1-q)k_{m-1} + (k_{m-1}-1)R_2 & (k_{m-1}-1)R_3\\ P_1+q+R_1 & P_2+1-q+R_2 & P_3+R_3\end{pmatrix}$$

keeps the required inequalities of the first and third row as before. For the second row we see that

$$\begin{aligned}\tilde{Q}_1 &= qk_{m-1} + (k_{m-1}-1)R_1 = (k_{m-1}-1)(R_1+q)+q\\ &\leq 3C^*(k_{m-1}-1)R_j + 1 < 4C^*(k_{m-1}-1)R_j \leq 4C^*\tilde{Q}_j\end{aligned}$$

for all j and analogously for the other entries. Hence the conditions of State 4 hold.

Since for a full matrix in State 4, any further left multiplications with $\mathbb{D}_{m-2}$ etc., preserves the conditions of State 4, this proves the lemma for $k_n \geq 2$. □

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