NEW EXAMPLES OF TOPOLOGICALLY EQUIVALENT S-UNIMODAL MAPS WITH DIFFERENT METRIC PROPERTIES

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with an Appendix

QUASICONFORMAL DEFORMATION OF MULTIPLIERS

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Abstract. We construct examples of topologically conjugate unimodal maps, such that both of them have an absolutely continuous invariant measure, but for one of them that measure is finite, and for another one it is σ-finite and infinite on every interval. The work is based on the results of Al-Khal, Bruin and Jakobson [2].

1. Introduction

The existence of infinite σ-finite absolutely continuous measures (σacim) for smooth interval maps has been discussed in several papers, see [14, 10, 4, 3, 22]. If the map is only C^1, then the existence of a σacim is not guaranteed, and even rare in the appropriate topology, see [21, 6] (although these papers focus on expanding circle maps rather that interval maps). In [2] we showed the existence of quadratic maps whose σacim is infinite on every nondegenerate interval, a phenomenon previously encountered only in invertible dynamics (circle diffeomorphisms) by Katznelson [15, Part II, Section 2]. In this paper, we will extend [2] by showing that the above property is not topological; it is not preserved under topological conjugation even within the class of S-unimodal maps of the interval with quadratic critical points. We prove

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Theorem 1. *Let $Q_t : x \mapsto 1 - tx^2$ be the quadratic family. Let $\mathcal{U}$ be any C^3 neighborhood of Q_2. There exist uncountably many parameter values t_* such that $Q_{t_*} \in \mathcal{U}$ has a finite acim, and for each Q_{t_*} there is an analytic unimodal map $f \in \mathcal{U}$ conjugate to Q_{t_*} such that f has an infinite σacim. Moreover, the σacim of f gives infinite mass to every non-degenerate interval.*

In this proof we are exploiting a possible difference in derivative $A_* := Q_t'(q_t)^2$ for $t = t_*$ at the fixed point $q_t = \frac{1}{2t}(\sqrt{1+4t}-1)$ and the corresponding squared derivative A_f for f. One can get f with $A_* > A_f$ by a local surgery, and in the appendix to this paper G. Levin proves using quasiconformal deformations that one can get such f by an arbitrary small analytic perturbation of Q_{t_*}.

The construction of a σacim μ such that $\mu(J) = \infty$ for every non-degenerate interval J was demonstrated in [2]. To this end Johnson boxes were used; a *Johnson box* $\mathcal{B}$ is a closed neighbourhood of the critical point such that $f^n(\mathcal{B}) \supset \mathcal{B}$ and $f^n(\mathcal{B})$ is contained in a small neighborhood of itself. More precisely, $\mathcal{B}$ has an n-periodic boundary point p and a smaller neighbourhood $\mathcal{H}$ of the critical point, called the *hat* maps to an interval of size $h \asymp |\mathcal{H}|^2$ adjacent to $\mathcal{B}$. For very small hats, points in $\mathcal{B}$ will stay a long time in $\mathcal{B}$ under iteration of f^n, and then linger a long time near $\mathcal{B}$ before leaving a neighbourhood of $\mathcal{B}$. Johnson [14] was the first to use a sequence of such boxes $\mathcal{B}_i$, with increasing periods p_i, to show that there are non-trivial unimodal maps without finite acim.

In [2] this idea was combined with constructing a dense critical orbit which avoids a particular Cantor set of positive Lebesgue measure C constructed inductively in such a way that $\mu(C) < \infty$.

The new ingredient is long runs of the critical orbit near the fixed point q of the map. As the derivatives at q are different for Q_{t_*} and f, the impact on the sizes of the Johnson boxes is different for both maps, and this is the crucial difference for the final estimates.

This latter method was used in [5] (for “almost saddles nodes” rather than Johnson boxes) to prove that the existence of a finite acim is not a topological property, which however requires less subtle estimates.

To begin, choose t_0 close to 2 and let $G : [-q, q] \to [-q, q]$ be a *power map* (or induced map) over Q_{t_0} having monotone branches except for a central parabolic branch defined on a small neighborhood δ_0 of the critical point. The remaining branches will all be monotone onto, and extendible to cover a fixed neighborhood of $[-q, q]$. Since $t_0 \approx 2$, the first return map to $[-q, q]$ will have this property.

The proof continues by constructing inductively a *power map* over G with countably many monotone branches onto $[-q,q]$ with uniformly bounded distortion. At the r-th step of induction, a map G_r is constructed, along with a partition ξ_r of $[-q,q]$ into "good branches" (on which $G_r : \Delta \to [-q,q]$ is onto, and which remain unchanged at later steps in the induction) and "holes" (which are filled in at later steps by new good branches or holes).

For a good branch $G_r : \Delta \to [-q,q]$, there is a number N, the "power" or "induce time" such that $G_r|\Delta = G^N|\Delta$. The aim of the construction is to estimate the *expectation* for the final partition ξ: $\sum_{\Delta\in\xi} N(\Delta)|\Delta|$. This expectation is finite if and only if there is a finite acim. The proof of this statement, the details of the inductive construction of the G_rs, and the construction of a Cantor set of finite mass can be found in [2]. (Obtaining convergence of $\sum_{\Delta\in\xi} N(\Delta)|\Delta|$ is an issue for basically all power map constructions that are by now around in the literature, but we will use the construction from e.g. [13] because we need very precise estimates both for $\sum_{\Delta\in\xi} N(\Delta)|\Delta|$ and the Cantor set C.)

In this paper we give the additional arguments concerning the long runs of the critical orbit near the fixed point q, and the impact of different derivatives at q on the increments of each inductive step of the construction on the expectation $\sum_{\Delta\in\xi} N(\Delta)|\Delta|$.

Remark: For the property that $\mu(J) = \infty$ for every non-degenerate interval J it is essential that the critical orbit is dense. Any unimodal map with a non-dense critical orbit has a σacim μ such that $\mu(J) < \infty$ for any closed set J in $[-1,1] \setminus \omega(0)$, see e.g. [6]. Our main theorem without the requirement that the σacim gives infinite mass to intervals follows from [5], which showed that (even within the class of analytic unimodal maps with quadratic critical points) the existence of a finite acim is not a property that is preserved under conjugacy.

The structure of the paper is as follows. In Section 2 we give some estimates on how long orbits are expected to linger in Johnson boxes and in the parabolic branch of a power map. In Section 3 we describe how precisely to combine Johnson boxes with close visits of the critical orbit to the fixed point. Section 4 then estimates the growth of the expected value of the "induce time" N of every step in the inductive procedure creating the final power map. In the final section the main theorem is proved.

2. Estimates on parabolic branches

Let us briefly outline the construction and terminology of [2]. The inductive construction of [2] starts from a partition of $[-q, q]$ into domains of the first return map

$$\xi_0' \colon I = (\cup_i \Delta_i) \cup \delta_0 \tag{1}$$

where Δ_i denote domains of uniformly expanding and uniformly extendible monotone branches and δ_0 is the domain of the central parabolic branch φ_0, see Figure 1. Next some of Δ_i are refined by using the monotone pullback of ξ_0' and one gets what is called the *initial partition*

$$\xi_0 \colon I = (\cup_i \Delta_i) \cup (\cup_k \delta_0^{-k}) \cup \delta_0 \tag{2}$$

where Δ_i and δ_0 are as above, and δ_0^{-k} are preimages of δ_0 by uniformly extendible diffeomorphisms.
If parameter values are close enough to 2, then for any $\varepsilon > 0$ one can construct a finite partition ξ_0 with the following properties:

(i) Each monotone domain has length less than ε.

(ii) The sum of lengths of the "holes" $\delta_0^{-k}, k \geq 0$ is less than ε.

Remark. The existence of ξ_0 with uniform estimates for all parameters within a certain parameter interval is sufficient for our results. Such partition exists for maps close to Chebyshev (which are discussed in this paper) and in many other cases.
In general we conjecture that maps for which Theorem 1 is true are dense in the set of finitely renormalizable maps.

In the course of induction we construct the power map by using several standard operations (see [2] for details).

(1) *Monotone Pullback:* Suppose
$$f_0 \colon \Delta_0 \to I$$
is a monotone branch and let ξ denote a partition of I. Then we refer to $f_0^{-1}(\xi)$ as the monotone pullback of the partition ξ onto Δ_0.
(2) Let $\xi_m,\ 0 \leq m \leq n-1$, be a partition constructed at the previous steps of induction. Assume the critical value $\varphi_{n-1}(0)$ belongs to a domain of a certain monotone branch $\Delta_m^* \in \xi_m$. We refer to this as a *Basic step* and we proceed with the construction using the following procedures.
(3) *Critical Pullback*: We induce on δ_{n-1} the partition $\varphi_{n-1}^{-1}(\xi_m)$ thus creating preimages of all the elements of ξ_m that are contained in the image of h_{n-1}.

(4) *Boundary Refinement Procedure*: Suppose $F\colon \Delta \to I$ is an extendible monotone branch, where $\Delta \in \xi_m$, $\Delta \subset \varphi_{n-1}(\delta_{n-1})$, and $\varphi_{n-1}(0) \notin \Delta$. If Δ is too close to $\varphi_{n-1}(0)$ then when we do critical pullback onto δ_{n-1}, the monotone domain $\varphi_{n-1}^{-1}(\Delta)$ may be not extendible. In this case, we perform the *boundary refinement procedure* as follows.
The initial partition (2) contains the boundary branch $F_0\colon \Delta_0 \to I$ which has a repelling fixed point q. We refine Δ_0 by monotone pullback, thus creating the partition $F_0^{-1}(\xi_0)$ which has a boundary domain Δ_{00} adjacent to q. Then we refine Δ_{00} by *monotone pullback* of ξ_0 by F_0^{-2} and so on. The domain $\Delta_{\underbrace{00\ldots0}_{i}}$ is called *the i-th step of the staicase*. As the sizes of extensions of $\Delta_{\underbrace{00\ldots0}_{i}}$ decrease exponentially in i, after several refinements we get a partition such that the critical pullbacks of all its elements are extendible. Let η_j be the partition of a central parabolic domain δ_j obtained by the above critical pullback.

(5) *Filling-in* : We fill each preimage
$$\delta_j^{-k} = \chi^{-1}(\delta_j), \quad j = 0, 1, \ldots, n-1$$
with the pullback $\chi^{-1}(\eta_j)$. In this way we get a copy of the elements of η_j inside each δ_j^{-k}.

(6) According to the construction of [2] we alternate *basic* steps described above and *Johnson* steps described below, see [2] for details. At a Johnson step the critical value of a parabolic branch is contained inside its domain, but outside the respective box. After a Johnson step we get a partition η_j inside some central domain, and then as above we use filling-in to pullback η_j inside each δ_j^{-k} .

At step n of the induction we apply the above operations and get a new partition ξ_n which has the form

$$\xi_n = (\bigcup \Delta) \cup (\bigcup_{j\le n} \bigcup_{p>0} \delta_j^{-p}) \cup \delta_n. \tag{3}$$

The rest of ths section is devoted to estimate for parabolic branches creating Johnson boxes. Throughout this section we let $g : [-1, 1] \to \mathbb{R}$ be an S-unimodal map $g = f \circ Q$, where f is a diffeomorphism with a uniformly bounded distortion, and $Q[x] = x^2$. If A, B are two quantities depending on a parameter, say t, then we write $A \approx B$ if $\lim_{t\to\infty} A(t)/B(t) = 1$, and $A \asymp B$ if there is $c > 0$ such that $c < A(t)/B(t) < 1/c$ for all t.

Lemma 1. *Assume $g(-1) = g(1) = -1$ and g has a hat $\mathcal{H} \ni 0$ such that $g(x) \ge 1$ for $x \in \mathcal{H}$. Let $\tau_0(x) = \min\{n \ : \ g^n(x) \in \mathcal{H}\}$ be the entrance time* *into the hat. Then the expectation*

$$\mathbf{E}(\tau) := \int_{-1}^{1} \tau_0(x)\ dx$$

satisfies

$$\frac{c_1}{\sqrt{h}} < \mathbf{E}(\tau) < \frac{c_2}{\sqrt{h}}$$

where h is the height of the hat, and c_1, c_2 are uniform constants.

Proof. Replace g by a map $\tilde{g} : [-1, 1] \to [-1, 1]$ such that $\tilde{g} : H \to [-1, 1]$ is a single linear onto branch, and $\tilde{g}(x) = g(x)$ if $x \notin \mathcal{H}$. Then $\tilde{g}$ is eventually hyperbolic, see [18], and it has a finite acim ν such that $\frac{d\nu}{dx}$ is bounded and bounded away from 0 by uniform constants. The upper bound depends on the size of the hat; it will have large peaks near 1 and -1 growing to square-root singularity as $|\mathcal{H}| \to 0$. However, in a neighborhood of 0, the density is bounded and bounded away from 0 uniformly in the hat-size, cf. [16].

Now Kac's Lemma implies that

$$\int_{\mathcal{H}} \tilde{\tau}_0(x)\ dx \asymp \int_{\mathcal{H}} \tilde{\tau}_0(x)\ d\nu = 1$$

where $\tilde{\tau}_0$ is the first return map to $\mathcal{H}$. The set $\{x \in \mathcal{H} \mid \tilde{\tau}_0(x) = n\}$ maps linearly (with slope $2/|\mathcal{H}|$) onto $\{x \in [-1, 1] \mid \tau_0(x) = n - 1\}$ and hence

$$\int_{-1}^{1} \tau_0(x)\ dx \asymp \frac{2}{|\mathcal{H}|} \int_{\mathcal{H}} (\tilde{\tau}_0(x) + 1)\ dx \asymp \frac{2}{|\mathcal{H}|}.$$

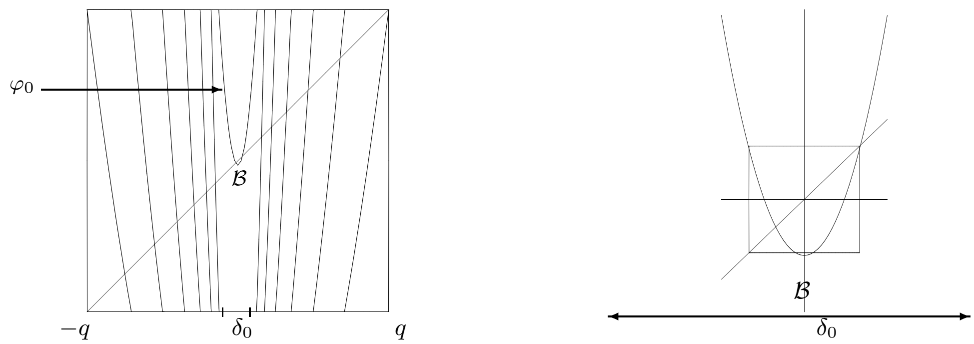
The lemma now follows from the fact that $h \asymp |\mathcal{H}|^2$. □

At every Johnson step the respective box $\mathcal{B}$ is created by a certain parabolic branch φ of the power map; Figure 1 depicts the situation at the first Johnson box.

Let p be the number of iterates of the initial map in φ. We call p the period of the box. Let δ be the domain of φ. In our construction, the domains δ are small and converging to zero with the step of induction. As φ is a composition of a diffeomorphism with uniformly bounded distortion and Q, we get

$$|\mathcal{B}| \asymp |\delta|^2 \tag{4}$$

A typical point $x \in \mathcal{B}$ is mapped $j_1(x)$ times by φ within the box, then escapes out of the box through the hat, then is mapped $j_2(x)$ times by h within the domain of φ and after that is mapped onto one of the elements of the previously constructed partitions.



$G : [-q, q] \to [-q, q]$ $\qquad$ $\varphi_0 : \delta_0 \to [-q, q]$

FIGURE 1. The map G with central parabolic branch $\varphi_0 : \delta_0 \to [-q, q]$ and a zoom-in at the first Johnson box $\mathcal{B}$.

The above maps consist respectively of $pj_1(x)$ and $pj_2(x)$ iterates of the initial transformation.

Let $[-B, B] = \mathcal{B}$ be a box in our construction, and let $[-H, H]$ be the base of the respective hat $\mathcal{H}$. Then $2B$, $2H$ are the widths of the box and of the hat, and

$$h \asymp \frac{2}{B} H^2 \tag{5}$$

is the height of the hat. Let n be the period of the box.

A linear map $x \to \frac{1}{B}x$ conjugates $\varphi|[-B, B]$ to a map g satisfying conditions of Lemma 1. Let $2H_0$ be the width of the respective hat for g . Then $H = BH_0$. Lemma 1 implies that the expectation of the exit time from the box $\mathcal{B}$ into the hat $\mathcal{H}$

$$E_1 := \int_{\mathcal{B}} \tau \; dx \asymp p\frac{B}{H_0} \asymp p\frac{B^2}{H} \asymp nB\sqrt{\frac{B}{h}}. \tag{6}$$

The next lemma is used to estimate the number of times that the critical value of the parabolic branch is mapped by the parabolic branch before it leaves its domain.

Lemma 2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a quadratic map with Johnson box $[-B, B]$ where $g(-B) = g(B) = B, \quad B > 0,\ g(0) < -B$. Let $h = -B - g(0)$ be the height of the hat, and a the minimal number of iterates such that $g^a(0) \geq 1$. Then*

$$a \approx \frac{\log B/2h}{\log(4 + 2h/B)} + \frac{\log\log 2/B}{\log 2}. \tag{7}$$

Note that the second term shows that, for fixed a, the ratio h/B depends on B. If the box size decreases, the hat is relatively taller to enable it to reach large scale in the same number of iterates.

Proof. The quadratic map described in this lemma has the form

$$g(x) = \frac{2x^2}{B}\ (1 + \frac{h}{2B}) - B - h.$$

The width of the hat is $B\sqrt{\frac{h}{2B+h}}$. Assume $h \ll B$. Let $x_n \in (0, \infty)$ be such that $g^n(x_n) = 1$. Then, as long as $x_n \gg B$, we can approximate $x_{n+1}^2 = (B/2)x_n$, and we find $x_n \asymp (B/2)^{1-2^{-n}}$.
Let l be minimal such that $x_l \leq 5B$. Then we get $10 \asymp (B/2^{-2^{-l}}$, so $l \asymp \frac{\log\log 2/B}{\log 2}$. At the same time $-g'(-B) = g'(B) = 4 + \frac{2h}{B}$, so $|g^k(0) - B| \asymp (4 + \frac{2h}{B})^{k-1}h$. Finding k such that $5B \geq g^k(0) > 4B$ gives $k = \frac{\log(B/2h)}{\log(4+2h/B)}$.

Therefore $a = k + l \approx \frac{\log(B/2h)}{\log(4+2h/B)} + \frac{\log\log 2/B}{\log 2}$. □

From the estimates of the proof, we also obtain

$$\frac{|\mathcal{B}|}{h} \asymp (4+\frac{2h}{B})^k = (4+\frac{2h}{B})^{a-l} \asymp (4+\frac{2h}{B})^a(2^{-l})^2 \asymp (4+\frac{2h}{B})^a \left(\frac{1}{\log|\mathcal{B}|}\right)^2. \tag{8}$$

Neglecting small terms in the formulation of the previous lemma, we obtain

$$a \asymp \left(\log\frac{B}{h} + \log\log\frac{1}{B}\right).$$

This implies that the expected value of the time spent by $x \in \mathcal{B}$ outside of the box, but inside the domain of φ satisfies

$$E_2 \asymp Bp\left(\log\frac{B}{h} + \log\log\frac{1}{B}\right), \tag{9}$$

where p is the period of the box.

3. Combining Johnson steps with long runs near the fixed point

We will construct the partition of the power map inductively, and use n to count the induction steps. In some of these step, Johnson box are created, and we will use r to count the Johnson boxes. Thus n_r is the induction step at which the r-th Johnson box is created. Let $\varphi_{r-1} : \delta_{r-1} \to I$ be the corresponding parabolic branch, which exhibits Johnson box $\mathcal{B}_{r-1}$ of period p_{r-1} and has hat $\mathcal{H}_{r-1}$. Then φ_r is created in a following way:

(1) The point $\varphi_{r-1}^2(0)$ is close to the fixed boundary point of box $\mathcal{B}_{r-1}$. It is then mapped $a_r = k_r + l_r$ times by φ_{r-1}, to a point y say, belonging to the domain of some good branch constructed at a previous step of induction. This constitutes one Johnson step. The remainder of this description amounts to basic induction steps in between the Johnston steps.

(2) Then y is mapped a prescribed number of times by previously constructed branches of the power map in order to visit neighborhoods of certain points in I. This is done in order to make the orbit of the critical point dense. Let m_r be the number of iterates of the initial map in this part of the trajectory. We can choose parameter in such a way that y is mapped inside a domain constructed much earlier in the induction. This makes m_r much smaller than p_{r-1} and therefore m_r is negligible in the following estimates.

(3) Let w be the m_r-th iterate of y. In our construction we choose parameter in such a way that w which is the critical value of the respective parabolic branch is located very close to the fixed point q. By continuity of the kneading invariant there is a parameter value such that w coincides with q. Also there is a parameter interval τ_{d_r} such that when $t \in \tau_{d_r}$, w moves through the d_r-th step of the staircase $\Delta_{\underbrace{00\ldots0}_{d_r}}$

(4) Let ξ_0 be the initial partition of the interval $[-q, q]$ and let Δ_0 be the element of ξ_0 adjacent to q. Let F_0 be the branch of the initial map, which maps Δ_0 onto I. The point q is the repelling fixed point of F_0. Let t_r^* be an arbitrary parameter value in τ_{d_r}. Let

$$A_r = A_r(t_r^*) = \left| \frac{\partial F_0(q, t_r^*)}{\partial x} \right|. \tag{10}$$

In our construction d_r is much larger than the total number of all previous iterates. We shall specify the requirements on d_r later.

(5) Let $F_0^{d_r}$ be the map from the respective step onto I. As q is the repelling fixed point of F_0 and distortions of F_0^l for all l are uniformly bounded, there exists a uniform constant C such that for any point x inside step d_r we have

$$C^{-1} A_r^{d_r} < \left| \frac{\partial F_0^{d_r}(x, t_r^*)}{\partial x} \right| < C A_r^{d_r}.$$

Up to a uniform constant the length of d_r-th step equals $A_r^{-d_r}$.

(6) Let $b_r = m_r + d_r$. Then in our construction the critical point of the parabolic branch φ_{r-1} is iterated $(2+a_r)p_{r-1}$ times within domain δ_{r-1}, then it is iterated b_r times as described above, and after that we get

the new parabolic branch φ_r with the new box $\mathcal{B}_r$. The period p_r of $\mathcal{B}_r$ satisfies $p_r = (a_r + 2)p_{r-1} + b_r$.

Any parabolic branch in our construction is a composition $\varphi_r = F_r \circ Q$, where F_r is a diffeomorphism with uniformly bounded distortion and Q is a quadratic map. For any ε_r we can choose d_r so large, that for any t_r^* such that the critical value belongs to the d_r-th step, the derivative of F_r at any point of its domain of definition satisfies

$$C^{-1} A_r^{b_r(1-\varepsilon_r)} < \left| \frac{\partial F_r(x, t_r^*)}{\partial x} \right| < C A_r^{b_r(1+\varepsilon_r)},$$

where C is a uniform constant.

Remark: The above estimate is uniform in the following sense. If $A_r \geq A_0 > 1$ for all r, then for any $\varepsilon > 0$ one can make all $\varepsilon_r < \varepsilon$ by choosing the same sequence d_r.

Then the length of the domain δ_r of the parabolic branch φ_r satisfies

$$C^{-1} A_r^{-\frac{1}{2} b_r(1+\varepsilon_r)} < |\delta_r| < C A_r^{-\frac{1}{2} b_r(1-\varepsilon_r)}.$$

We use $|\mathcal{B}_r| \asymp |\delta_r|^2$ to obtain

$$|\mathcal{B}_r| \asymp A_r^{-b_r(1+\varepsilon_r)}. \tag{11}$$

Suppose that we have now constructed the next generation of the box map. Let $N(x)$ be the total number of iterates of x until its domain is mapped onto I by the power map. Let us estimate the expected value

$$E_r = \int_{\mathcal{B}_{r-1} \setminus \delta_r} N(x) dx.$$

The contribution to the expectation after the points leave the domain δ_{r-1} and before they are mapped near to q is small because the number of such iterates is small comparatively to the other terms.
Then combining (6) and (9), we get

$$E_r \asymp p_{r-1} \left[|\mathcal{B}_{r-1}| \sqrt{\frac{|\mathcal{B}_{r-1}|}{h_{r-1}}} + |\mathcal{B}_{r-1}| \left(\log \frac{|\mathcal{B}_{r-1}|}{h_{r-1}} + \log\log \frac{1}{|\mathcal{B}_{r-1}|} \right) \right]. \tag{12}$$

From (8) and (11) we get an estimate

$$\frac{|\mathcal{B}_{r-1}|}{|h_{r-1}|} \asymp \frac{4^{a_r}(1 + C_r)^{a_r}}{(b_{r-1})^2}, \tag{13}$$

where $|C_r| < C|\mathcal{B}_{r-1}|$.

We use that for large $\frac{|\mathcal{B}_{r-1}|}{|h_{r-1}|}$

$$\sqrt{\frac{|\mathcal{B}_{r-1}|}{|h_{r-1}|}} \gg \log \frac{|\mathcal{B}_{r-1}|}{|h_{r-1}|}.$$

Then (12) is equivalent to

$$E_r \asymp p_{r-1} A_r^{-(1+\varepsilon_r)b_{r-1}} \left(\frac{2^{a_r}(1+C_r)^{\frac{a_r}{2}}}{b_{r-1}} + \log b_{r-1} \right) \tag{14}$$

where $|C_r| < C|\mathcal{B}_{r-1}|$.

4. Growth of the expected value

At every inductive step in the creation of the final power map, good old branches remain unchanged, while old holes are filled in by new good branches or new holes. In this section we estimate how much this procedure adds to the total expectation $\sum_\Delta N(\Delta)|\Delta|$.

(1) Let Δ be one of the domains appearing in our construction, and suppose Δ is mapped by an iterate $N(\Delta)$ of the initial map diffeomorphically with uniformly bounded distortion onto the initial interval I. Then we call

$$\mathcal{C}(\Delta) := N(\Delta)|\Delta|$$

the *contribution* of Δ. Let δ_i be a domain of some parabolic branch, let δ_i^{-k} be one of its preimages, and let $g : \delta_i^{-k} \to \delta_i$ be the respective diffeomorphism. Domains δ_i^{-k} are called *holes*. Let $N(\delta_i^{-k})$ be the number of iterates of the initial map in g . Then the contribution of the hole is

$$\mathcal{C}(\delta_i^{-k}) := N(\delta_i^{-k})|\delta_i^{-k}|.$$

Good domains Δ mapped onto I are not changed anymore, but central domains δ_i are substituted by elements of the new partition ξ_i constructed at step i, when we are doing critical pull-back. After that, δ_i^{-k} are substituted by the elements of $g^{-1}\xi_i$.

(2) *Contribution of the critical pull-back.*
Let ξ_{n-1} be the partition constructed after the $n-1$ step of induction. Then the contribution after step $n-1$ is

$$\mathcal{C}(\cup\Delta) := \sum_\Delta N(\Delta)|\Delta|,$$

where Δ are all elements of the partition ξ_{n-1} (both holes and good domains). In particular if φ_{n-1} is the parabolic branch with the domain

δ_{n-1}, then the contribution of δ_{n-1} taken into account after step $n-1$ equals

$$\mathcal{C}(\delta_{n-1}) := N(\varphi_{n-1})|\delta_{n-1}|.$$

Here we estimate the total contribution from the new domains constructed by the critical pull-back operation, assuming these new domains do not belong to the Johnson box.

Remark. In our estimates we use repeatedly the following way of counting contributions: Let Δ be an element of one of the partitions constructed before step n. Assume at step n we pull back that partition by g^{-1} onto the domain D of the map g, where D is also an element constructed before step n. Here g can be a parabolic branch φ_{n-1} or it can be a map from a preimage δ_i^{-k} onto a central domain δ_i. Let $\Delta^{-1} = g^{-1}\Delta$. The contribution of Δ^{-1} along its trajectory under g until it is mapped onto Δ has been already taken into account, because on that part of its orbit every iterate of Δ^{-1} is just a piece of the respective iterate of D. The contribution from Δ^{-1} *added* at step n is

$$\mathcal{C}(\Delta^{-1}) := N(\Delta)|\Delta^{-1}|.$$

Let us estimate the sizes of critical preimages $\Delta^{-1} = \varphi_{n-1}^{-1}\Delta$ which are needed in (2). As φ_{n-1} is a composition of the quadratic map with a diffeomorphism of uniformly bounded distortion, we get the following estimate for the size of Δ^{-1}: If d is the distance from Δ to the critical value of φ_{n-1}, then

$$|\Delta^{-1}| < c\frac{|\Delta|}{\sqrt{d}}|\delta_{n-1}|, \tag{15}$$

where c is a uniform constant.

(a) *Contribution of a basic step.*
At a basic step we can pull-back any of the partitions ξ_k constructed at the previous steps of induction by the parabolic branch. Typically, i.e., when the iterates of the critical point are near to the fixed point q, we pull-back the initial partition ξ_0. Since one of our goals is to construct a dense critical orbit, we should also pull-back ξ_i with growing i, but we can always choose at step n one of the partitions ξ_i constructed much earlier in the induction. In this way, we can put the critical value inside a sufficiently large domain Δ, such that the distance between the critical value of the parabolic branch and the boundary of Δ is greater than $|\delta_{n-1}|$. Then d in (15) satisfies $d > |\delta_{n-1}|$ and the new contribution of

Δ^{-1} is less than

$$cN(\Delta)|\Delta|\sqrt{|\delta_{n-1}|}. \tag{16}$$

Summation over all Δ gives the new contribution less than

$$c\sum_{\Delta} N(\Delta)|\Delta|\sqrt{|\delta_{n-1}|}.$$

As $|\delta_{n-1}|$ decreases at least exponentially with the step of induction, we get that the contribution of the critical pull-back at basic steps are exponentially small compared to the sum $\sum_{\Delta} N(\Delta)|\Delta|$ accumulated at all previous steps.

(b) *Contribution outside of the box at a Johnson step.*
We have already estimated contribution of the box, so let us estimate the contribution of all elements Δ^{-1} located inside δ_{n-1} but outside the box. Let $j_2(x)$ be the number of iterates of φ_{n-1} it takes $x \in \delta_{n-1} \setminus \mathcal{B}_{n-1}$ to leave δ_{n-1}. Notice that the first steps (i.e., the intervals with low values of j_2, decrease double exponentially and after that subsequent steps decrease as $\asymp 4^{-k}$. This implies that added expected value of this "staircase" is comparable to the contribution of the step with $j_2(x) \equiv 1$. On the other hand by the definition of the first step its distance from the critical value is close to $\frac{1}{2}|\delta_{n-1}|$. This implies the same estimate (16) which proves that contributions from the critical pull-backs are exponentially small compared to the contribution of all previous steps.

(3) *Filling-in.*
Here we estimate the contribution of the new domains constructed when we are filling-in preimages of central domains $\delta_i, i = 0, 1, \ldots$. Let $\mathcal{C}(D)$ denote the contribution of D. Let δ_i be a central domain, and δ_i^{-k} the preimages of δ_i. In our construction each preimage δ_i^{-k} is mapped onto δ_i with a small distortion.

At step $i+1$ of the induction we construct a new partition inside δ_i. Note that the contribution of the orbit of δ_i^{-k} mapped onto δ_i has been already taken into account at the previous step. The additional contribution of δ_i^{-k} is less than

$$(1+\varepsilon_i)\frac{|\delta_i^{-k}|}{|\delta_i|}\mathcal{C}(\delta_i),$$

where $1+\varepsilon_i$ is an upper bound of the distortion of all maps $\Delta_i^{-k} \to \Delta_i$ used in the filling in and $\prod_i(1+\varepsilon_i) < \infty$. To get contribution of all preimages δ_i^{-k} we take the sum over all of them and get

$$\mathcal{C}_{new}\left(\bigcup \delta_i^{-k}\right) < (1+\varepsilon_i)\frac{\sum_{\delta_i^{-k}} |\delta_i^{-k}|}{|\delta_i|}\mathcal{C}(\delta_i). \tag{17}$$

In order to estimate (17) we use the same approach as [2]. In this paper, however, we do not delete orbits of small intervals containing the boxes $\mathcal{B}_i$, but need more precise estimates of the size of Johnson boxes and their orbits. Then it turns out that the filling-in inside the boxes provides the main contribution, and the measure of the domains inside the boxes is important.

Let r and $r+1$ be two consecutive Johnson steps. Between r and $r+1$ there are b_r basic steps, most of them corresponding to the iterates near the fixed point q. Note that up to a set of measure zero the box $\mathcal{B}_r$ is partitioned into preimages $\mathcal{H}_r^{-k}$ of the hat $\mathcal{H}_r$. Much smaller domains δ_{r+1}^{-k}, which are preimages of δ_{r+1} containing the next Johnson box, are located in the middle of $\mathcal{H}_r^{-k}$.

We construct hats $\mathcal{H}_r$ small compared to the boxes $\mathcal{B}_r$. Therefore the distortion of the maps

$$\varphi_r^k : \mathcal{H}_r^{-k} \to \mathcal{H}_r$$

are less than $1+\varepsilon_r$, where $\prod_{r=1}^{\infty}(1+\varepsilon_r)$ converges. Small distortion implies that the measure of the union of preimages δ_{r+1}^{-k} inside the box $\mathcal{B}_r$ satisfy

$$\frac{\sum |\delta_{r+1}^{-k}|}{|\delta_{r+1}|} \asymp \frac{|\mathcal{B}_r|}{|\mathcal{H}_r|}. \tag{18}$$

Thus after the filling-in at step $r+1$, the additional contribution of the preimages δ_{r+1}^{-k} located inside the box $\mathcal{B}_r$ can be estimated, using (18), as

$$\mathcal{C}(\cup \delta_{r+1}^{-k}) \leq (1+\varepsilon_r)\ \mathcal{C}(\delta_{r+1})\frac{|\mathcal{B}_r|}{|\mathcal{H}_r|}. \tag{19}$$

(4) As the combined measure of preimages δ_{r+1}^{-k} located inside the box $\mathcal{B}_r$ is much bigger than the measure of the central domain δ_{r+1}, we introduce new "combined" objects.

Let D_0 be the finite union of δ_0 and all preimages δ_0^{-k} constructed at the preliminary (zero) step of the induction. Assume the first step is a Johnson step. Let

$$D_1' = \bigcup_{k=0}^{\infty} \delta_1^{-k}$$

be the union of δ_1 and preimages of δ_1 located inside the box $\mathcal{B}_1$, and let D_1 be the union of D'_1 and preimages $(D'_1)^{-k}$ located inside δ_0^{-k}, which constitute D_0. Assume next that the steps $2, \ldots, N-1$ are basic. Then $D_i,\ i = 1, \ldots, N-1$ consist of δ_i^{-k} located at the middle of respective δ_{i-1}^{-k}, which constitute D_{i-1}. Each δ_i^{-k} is mapped onto δ_i by some g which is a restriction of the respective $g : \delta_1^{-k} \to \delta_1$.
Let the next step N be a Johnson step again, and we construct inside $\mathcal{B}_{N-1}$ a set

$$D'_N = \bigcup_{k=0}^{\infty} \delta_N^{-k}.$$

Then D_N is the union of D'_N and preimages $(D'_N)^{-k}$ located inside δ_{N-1}^{-k}, which constitute D_{N-1}.
Similarly at any basic step $n+1$ we define D'_{n+1} as the domain δ_{n+1} of the parabolic branch φ_{n+1}. At Johnson step $n+1$ we define D'_{n+1} as the union of δ_{n+1} and all preimages δ_{n+1}^{-k} located inside the box B_n. Then we define D_{n+1} as the union of D'_{n+1} and preimages $(D'_{n+1})^{-k}$ located inside δ_n^{-k} , which constitute D_n.

(5) The next proposition shows that the total measure of preimages δ_i^{-k} constructed at all steps of induction is comparable up to a uniform constant with the measure of such preimages constructed at step i of the induction.

The estimates below are similar to the estimates from Proposition 5.11.7. of [2]. Equation (18) motivates the following expression

$$\Sigma_{n-1} = \frac{|\mathcal{B}_{m-1}|}{|\mathcal{H}_{m-1}|} + \frac{|\mathcal{B}_{m-1}|}{|\mathcal{H}_{m-1}|}\frac{|\mathcal{B}_{m-2}|}{|\mathcal{H}_{m-2}|} + \ldots + \prod_{i=1}^{m-1} \frac{|\mathcal{B}_i|}{|\mathcal{H}_i|}$$

where $m-1$ is the number of Johnson steps between 1 and $n-1$ including $n-1$.

Proposition 1. • *After induction step $n-1$ the measure of all preimages δ_{n-1}^{-k} is less than*

$$M_{n-1} := c_0|\delta_{n-1}|\Sigma_{n-1} \prod_{j=0}^{n-1}(1+\varepsilon_j), \tag{20}$$

where $\prod_{j=0}^{\infty}(1+\varepsilon_j)$ converges and

$$c_0|\delta_0| := |\bigcup_{k=0}^{p_0} \delta_0^{-k}|$$

is the measure of the union of preimages of δ_0 in the initial partition ξ_0.

- *After step $n-1$ the measure of all preimages*

(21) $$|\cup_k \delta_i^{-k}| \leq \gamma_{n-1-i} M_i$$

where γ_l decreases exponentially in l.

Remark: Recall that after step $n-1$ we construct the parabolic branch φ_{n-1}, but do not partition its domain δ_{n-1}. Then we call step n Johnson, if φ_{n-1} has a box $\mathcal{B}_{n-1}$, and basic otherwise. At step n we are doing critical pull-back by φ_{n-1}^{-1} and after that we are doing filling-in for all preimages δ_i^{-k} located outside δ_{n-1}.

Proof of Proposition 1. (a) Suppose step n is basic. Then no preimages of δ_n are created by the critical pull-back. Preimages δ_n^{-k} are created by the filling-in of δ_{n-1}^{-k}. Taking into account that at step n of the induction we can make the distortion of the filling-in smaller than $1+\varepsilon_n$ where $\prod_{n=0}^{\infty} 1+\varepsilon_n$ converges, we get, after filling-in at step n, that the measure of $\cup_k \delta_n^{-k}$ is less than

(22) $$c_0 |\delta_{n-1}| \Sigma_{n-1} \left(\prod_{j=0}^{n-1} (1+\varepsilon_j) \right) \frac{|\delta_n|}{|\delta_{n-1}|} (1+\varepsilon_n) = M_n.$$

(b) Suppose step $n = n_r$ is the r-th Johnson step. Taking into account that distortions $1+\varepsilon_n$ of the maps from $\mathcal{H}_{r-1}^{-k}$ onto $\mathcal{H}_{r-1}$ are exponentially close to 1, we get from (18) that the measure of new preimages δ_n^{-k} constructed inside δ_{n-1} is bounded by

$$|\delta_n| \frac{|\mathcal{B}_r|}{|\mathcal{H}_r|} (1+\varepsilon_n).$$

After the filling-in we get as above that the total measure of preimages δ_n^{-k} does not exceed

(23) $$c_0 |\delta_{n-1}| \Sigma_{n-1} \left(\prod_{j=1}^{n-1} 1+\varepsilon_j \right) \frac{|\delta_n| \frac{|\mathcal{B}_r|}{|\mathcal{H}_r|} (1+\varepsilon_n)}{|\delta_{n-1}|} \leq M_n.$$

(c) Next we turn to the proof of (21) at step n. Let us first estimate the quantity M_n. Assume $n = n_r$ is the r-th Johnson step. Since the critical orbit lingers a long time near q, we can assume that $|\delta_r| \ll |\mathcal{H}_{r-1}|$. Therefore

(24) $$|\delta_r| \frac{|\mathcal{B}_{r-1}|}{|\mathcal{H}_{r-1}|} \ll |\mathcal{B}_{r-1}| \asymp |\delta_{r-1}|^2.$$

There are many basic steps between Johnson steps $r-1$ and r and at each basic step the central domains shrink by a small factor α.

Note that α can be made arbitrary small by an initial refining of the elements of ξ_0 , see[2]. Applying (24) repeatedly we get that

$$M_n < c_1 \alpha^{b_1+b_2+\ldots+b_{r-1}} \prod_{j=1}^{r-1} |\delta_j|, \tag{25}$$

where c_1 is a uniform constant. Therefore the numbers M_n are decreasing fast between consecutive Johnson steps. In particular $M_n \ll |\delta_{r-1}|$.

(d) Let $\alpha_j^{(k)}$ denote the measure of all preimages of δ_j at step k. Similarly to the formula (45) from [2]

$$\alpha_i^{(n+1)} < \beta\alpha_{i-1}^{(n)} + c_1 \sum_{j=i}^{n} \alpha_j^{(n)} \left(\sqrt{\alpha_i^{(j)}} + |\delta_j|^{1/2} \right), \tag{26}$$

we get if i is a basic step

$$\alpha_i^{(n+1)} < \frac{|\delta_i|(1+\varepsilon_n)}{|\delta_{i-1}|} \alpha_{i-1}^{(n)} + c_1 \sum_{j=i}^{n} \alpha_j^{(n)} \left(\sqrt{\alpha_i^{(j)}} + |\delta_j|^{1/2} \right). \tag{27}$$

If i is a Johnson step, we get

$$\alpha_i^{(n+1)} < \frac{|\delta_i| \frac{|\mathcal{B}_i|}{|\mathcal{H}_i|}(1+\varepsilon_n)}{|\delta_{i-1}|} \alpha_{i-1}^{(n)} + c_1 \sum_{j=i}^{n} \alpha_j^{(n)} \left(\sqrt{\alpha_i^{(j)}} + |\delta_j|^{1/2} \right). \tag{28}$$

As in [2] we choose $\gamma_l = \gamma_0^l$, with the constant $\gamma_0 < 1$ satisfying

$$\gamma_0 \gg \alpha \tag{29}$$

where α is the constant from (25).

We substitute $\alpha_i^{(j)}$ in (27) and (28) by their inductive estimates (21), use (25) and (29) and get that the sum of all terms except the first one in (27) and (28) is small comparatively to $\gamma_{n+1-i}M_i$ and the first terms give the required estimate.

This concludes the proof of (21) and hence of Proposition 1. □

(6) Note that (14) estimates contributions of elements located in the annulus between δ_{n-1} and δ_n. If at each step n of induction we count the contribution only from the preimages δ_n^{-k} which belong to the set D_n described above, then we get that (up to a uniform distortion factor) that their contribution is greater than

$$\sum_{m=1}^{\infty} \mathcal{C}(\mathcal{B}_m \setminus \delta_{m+1}) \left(\frac{|\mathcal{B}_{m-1}|}{|\mathcal{H}_{m-1}|} + \frac{|\mathcal{B}_{m-1}|}{|\mathcal{H}_{m-1}|}\frac{|\mathcal{B}_{m-2}|}{|\mathcal{H}_{m-2}|} + \ldots + \prod_{i=1}^{m-1} \frac{|\mathcal{B}_i|}{|\mathcal{H}_i|} \right), \tag{30}$$

where $\mathcal{C}(\mathcal{B}_m \backslash \delta_{m+1})$ is the quantity E_m from (14) (with r replaced by m).

On the other hand by taking the sum of preimages δ_m^{-k} at all steps of induction, we get from (21) that it is less than

$$C_0 \left(\frac{|\mathcal{B}_{m-1}|}{|\mathcal{H}_{m-1}|} + \frac{|\mathcal{B}_{m-1}|}{|\mathcal{H}_{m-1}|} \frac{|\mathcal{B}_{m-2}|}{|\mathcal{H}_{m-2}|} + \dots + \prod_{i=1}^{m-1} \frac{|\mathcal{B}_i|}{|\mathcal{H}_i|} \right),$$

where C_0 is a uniform constant. This implies the following

Corollary 1. *If ξ is the partition into branches of the final power map, then the sum $\sum_{\Delta \in \xi} N(\Delta)|\Delta|$ converges or diverges simultaneously with*

$$\sum_{m=1}^{\infty} \mathcal{C}(\mathcal{B}_m \setminus \delta_{m+1}) \left(\frac{|\mathcal{B}_{m-1}|}{|\mathcal{H}_{m-1}|} + \frac{|\mathcal{B}_{m-1}|}{|\mathcal{H}_{m-1}|} \frac{|\mathcal{B}_{m-2}|}{|\mathcal{H}_{m-2}|} + \dots + \prod_{i=1}^{m-1} \frac{|\mathcal{B}_i|}{|\mathcal{H}_i|} \right).$$

(7) As φ_i restricted to $\mathcal{B}_i$ are compositions of quadratic map with diffeomorphisms of small distortion we get

$$\frac{|\mathcal{B}_i|}{|\mathcal{H}_i|} = \sqrt{\frac{|\mathcal{B}_i|}{|h_i|}} (1 + \varepsilon(\mathcal{B}_i)), \tag{31}$$

where $\varepsilon(\mathcal{B}_i) < \varepsilon_i$ and $\prod_{i=1}^{\infty}(1+\varepsilon_i)$ is close to 1.
We also note that $|\mathcal{B}_i|/|\mathcal{H}_i|$ increases rapidly with i. Therefore the convergence in (30) is equivalent to the convergence of

$$\sum_{m=1}^{\infty} \mathcal{C}(\mathcal{B}_m \setminus \delta_{m+1}) \prod_{i=1}^{m-1} \sqrt{\frac{|\mathcal{B}_i|}{|\mathcal{H}_i|}}. \tag{32}$$

5. Proof of the main theorem

Proof of Theorem 1. Consider the quadratic family Q_t and let Q_2 be the Chebyshev polynomial. Let $q_t = q(Q_t)$ be the fixed point of $Q(t)$, in particular $q(Q_2) = 1/2$ the fixed point of the Chebyshev polynomial.
For any large n there is a parameter interval $\tau_n(Q)$ such that for $t \in \tau_n(Q)$ the first return map to the interval $I_t = [-q_t, q_t]$ consists of $2n$ monotone branches and the central parabolic branch φ_0. When t varies in $\tau_n(Q)$ the critical value of φ_0 moves from the top to the bottom of I_t . For any t_a in the interior of $\tau_{n+1}(Q)$ and any $t_b \in \tau_n(Q)$ the kneading invariant of Q_{t_a} is greater than the kneading invariant of Q_{t_b}.
Let f_1 be an analytic perturbation of Q_2 topologically conjugate to Q_2. Let $q(f_1)$ be the fixed point of f_1 corresponding to $q(Q_2) = 1/2$. By construction

of the Appendix we can find f_1 in an arbitrary C^3 neighborhood $\mathcal{U}$ of Q_2 such that

$$(Q_2'(q(Q_2)))^2 - (f_1'(q(f_1)))^2 = \varepsilon > 0 \tag{33}$$

One can include f_1 in a one parameter family of maps $f_t \in \mathcal{U}$, such that for $t < 1$ the kneading invariant of f_t is less than the maximal kneading invariant of f_1. For example if $f_1(0) = f_1(1) = 0$ and the critical value is 1 then we consider $f_t = tf_1$ for $t \leq 1$ and close to 1.

Although the kneading invariant for f_t may be not monotone, it depends continuously on the parameter. This implies that for all sufficiently large n there is an interval of parameter $\tau_n(f)$ similar to $\tau_n(Q)$. When $t \in \tau_n(f)$, the first return map to the interval $I(f_t)$ consists of $2n$ monotone branches and a central parabolic branch φ_0, see Figure 1. When t runs through $\tau_n(f)$ the critical value of φ_0 moves from the top to the bottom of $I(f_t)$.
Moreover for large n geometric properties of the monotone branches and φ_0 (scaling of the domains, extendibility) are determined by the geometry of f_1, see for example [12]. As f_1 belongs to a small neighborhood $\mathcal{U}$ of Chebyshev polynomial we get uniform geometric properties independent on n. That implies that the maps that we construct following [2] have σ-finite acim.

We start our inductive construction by choosing a small parameter interval $J_1 \subset \tau_n(Q)$ close to the middle of $\tau_n(Q)$ such that for $t \in J_1$ the parabolic branch exhibits the first Johnson box. After that we construct a sequence of nested parameter intervals J_r near t_r^* (with $t_{r+1}^* \in J_r$ for each r) such that all maps Q_t with $t \in J_r$ have r Johnson boxes and satisfy all the estimates of the r-th step in the construction of the previous sections.
At each step of the construction one can find many disjoint intervals $J_{r+1} \subset J_r$ satisfying the required estimates. That gives a Cantor set of possible limit values $t_* = \cap_r J_r$, and we fix one of them.

For sufficiently large n the maps f_t with $t \in \tau_n(f)$ are arbitrary close to f_1 , and similarly Q_t with $t \in \tau_n(Q)$ are arbitrary close to Q_2. From 33 we get that the derivatives at the fixed points for maps f_t with $t \in \tau_n(f)$ and $Q_{t'}$ with $t' \in \tau_n(Q)$ are uniformly separated, say by $\varepsilon/2$.

Recall that the estimates of Section 3 are valid for any $t \in J_r$. Thus they are valid for t_* for all r. So for all r we get

$$A_r = A_r(t^*) = A(Q_{t_*}) = \frac{\partial F_0(q, t_*)}{\partial x}. \tag{34}$$

In the course of induction of the previous sections, we can construct numbers a_r, b_r increasing very fast to satisfy the below conditions. Suppose we have chosen $a_i, b_i,\quad i \le r-1$. Choose a_r such that

$$\frac{2^{a_r}}{b_{r-1}} \gg \log b_{r-1}$$

Then we rewrite (14) as

$$E_r \asymp p_{r-1} A_r^{-(1+\varepsilon_r)b_{r-1}} \frac{2^{a_r}(1+C_r)^{\frac{a_r}{2}}}{b_{r-1}},$$

On the other hand we can choose a_r such that

$$a_r A_r^{-b_{r-1}(1+\varepsilon_r)} < \alpha_r, \tag{35}$$

where $\prod_{r=1}^{\infty}(1+\alpha_r) \asymp 1$. Also we choose b_r so large that

$$\frac{p_r}{b_r} = 1 + \beta_r$$

where $\prod_{r=1}^{\infty}(1+\beta_r) \asymp 1$. Then we get

$$E_r \asymp A_r^{-(1+\varepsilon_r)b_{r-1}} 2^{a_r}.$$

With the above choice of a_r, b_r the convergence in (32) is equivalent to the convergence of the series

$$\sum_r A_r^{-(1+\varepsilon_r)b_{r-1}} 2^{a_r} \frac{\prod_{j=1}^{r-1} 2^{a_j}}{\prod_{j=1}^{r-2} b_j}. \tag{36}$$

We choose b_{r-1} much larger than the total number of iterates preceding the string of b_{r-1} iterates, and such that

$$A_r^{-(1+\varepsilon_r)b_{r-1}} \frac{\prod_{j=1}^{r-1} 2^{a_j}}{\prod_{j=1}^{r-2} b_j}$$

is small. Next we choose a_r so large that

$$A_r^{-(1+\varepsilon_r)b_{r-1}} 2^{a_r} \frac{\prod_{j=1}^{r-1} 2^{a_j}}{\prod_{j=1}^{r-2} b_j} \asymp \frac{1}{r^2}$$

but at the same time $a_r A_{1,r}^{-(1+\varepsilon_r)b_{r-1}}$ is small enough, so that (35) holds. By Corollary 1, this implies that $\sum_{\Delta \in \xi} N(\Delta)|\Delta|$ is finite and hence Q_{t_*} has a finite acim.

As for $t \in \tau_n(f)$ the kneading invariants of f_t take all possible values, there is a map $f = f_{\tilde{t}_*}$ with $\tilde{t}_* \in \tau_n(f)$ topologically conjugate to Q_{t_*}.

By construction the respective derivatives at the fixed points satisfy

$$A(Q_{t_*}) - A(f) > \varepsilon/2 \tag{37}$$

As Q_{t_*} and f are topologically conjugate, a_r, b_r, d_r are the same for Q_{t_*} and f. Notice that our conditions allow arbitrary fast growth of d_r. Then we can choose $\varepsilon_{1r} = \varepsilon_r(Q_{t_*})$ and $\varepsilon_{2r} = \varepsilon_r(f)$ arbitrary small. From 37 it follows that one can take d_r so large that there exists $\lambda > 1$ such that

$$A(f)^{-(1+\varepsilon_{2,r})} > \lambda A(Q_{t_*})^{-(1+\varepsilon_{1,r})}$$

for all r. As a_r, b_r are the same for Q_{t_*} and f, we get that the terms in (36) for f are greater than

$$\frac{c\ \lambda^{b_r}}{r^2} \to \infty. \tag{38}$$

Therefore f does not have finite acim. At the same time the arguments [2] apply and f has a σacim. As the length of the orbit of the r-th box is comparable to b_r up to a uniform factor, we get from (38) that the measure of each iterate of the r-th box under f is greater than

$$\frac{c\ \lambda^{b_r}}{b_r r^2} \to \infty. \tag{39}$$

By construction the orbit of the critical point is dense in I. Then each interval contains iterates of infinitely many boxes, and we get that the invariant measure of each interval is infinite. □

Remark: Similarly one can get the situation that Q_{t_*} has the infinite σacim and f has the finite acim.

Appendix.

Quasiconformal deformation of multipliers.

Genadi Levin

We show that the quadratic polynomial Q_t admits an arbitrary small analytic perturbation in the space of topologically conjugated maps, such that the multiplier of the fixed point q changes. In fact, our statement is much more general, see Theorems 2- 3 and Remark below.

Recall that a polynomial-like map [8] is a finite holomorphic branched d-covering map $f : U \to U'$, where U, U' are topological disks in the plane, and $\overline{U} \subset U'$. The set of non-escaping points $\{z : f^n(z) \in U \ for\ all\ n = 0, 1, ...\}$ is called the filled-in Julia set K_f of f; its boundary is the Julia set. Every polynomial is a polynomial-like map if one takes U' to be a large enough geometric disk around the origin, and U the full preimage of U'. In the opposite direction, the Straigthening Theorem of Douady and Hubbard [8] states that every polynomial-like map is hybrid equivalent to a polynomial of the same degree d, that is, there exists a quasiconformal homeomorphism of the plane, which conjugates the map and the polynomial near their filled-in Julia sets and is conformal almost everywhere on these sets.

In Theorem 2 we assume (mainly, for simplicity) that the Julia set is connected while in Theorem 3 we drop this assumption.

Theorem 2. *Let P be an arbitrary non-linear polynomial with connected Julia set, and a its repelling fixed point. Denote $\lambda = P'(a)$ the multiplier of a. Then P can be included in a family of polynomial-like maps $f_t : U_t \to U'_t$, $t \in \mathbb{D} = \{|t| < 1\}$, such that the following conditions hold.*

(a) *$f_0 = P$ and f_t depends analytically in $t \in \mathbb{D}$, moreover, the modulus of the annulus $U'_t \setminus U_t$ is bounded away from zero uniformly in $t \in \mathbb{D}$.*
(b) *each f_t is hybrid equivalent to P by a quasiconformal homeomorphism h_t, that is, $f_t = h_t \circ P \circ h_t^{-1}$,*
(c) *the multiplier $\lambda(t)$ of the fixed point $a_t = h_t(a)$ of f_t is a non-constant analytic function in t,*
(d) *if P and a are real (on the real line), then h_t, f_t and a_t are real too, for t real.*

Proof. Making a linear change of variable, one can normalize P so that $a = 0$ and $P(1) = 1$. The proof consists of the following steps.

(1) For a polynomial-like map $f : U \to U'$ of degree d with connected filled-in Julia set K_f, its external map g_f is defined by $g_f = B_f \circ f \circ B_f^{-1}$, where B_f is an analytic isomorphism of $U' \setminus K_f$ onto a geometric annulus $\{1 < |z| < R\}$, see [8]. Then g_f extends through the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ by symmetry to an expanding analytic d-covering map of $\mathbb{S}^1$. For the polynomial P, the map B_P is a restriction of the Bottcher coordinate defined in the basin of infinity $\mathbb{C} \setminus K_P$, and the external map g_P is $\sigma : z \mapsto z^d$.

There exists a cycle R of external rays (gradient lines of Green's function $\log |B_P|$) that land at the fixed point 0, see [7, 9]. Let $p \geq 1$ be its period. Then the set of limit values of B_P along these rays consists of a cycle C_0 of period p for the map $\sigma : \mathbb{S}^1 \to \mathbb{S}^1$.

(2) Let g be an expanding real-analytic d-covering map of $\mathbb{S}^1$ preserving orientation. It is conjugated to σ. Denote by C a cycle of g corresponding to C_0 by this conjugacy. Approximating smooth maps by polynomials, one can choose g in such a way, that modulus of the multiplier ρ_g of the cycle C of $g : \mathbb{S}^1 \to \mathbb{S}^1$ is as close to 1 as we wish. Fix g such that $|\rho_g| < |\lambda|^{1/2}$.

(3) By Proposition 5 of [8], given the map $g : \mathbb{S}^1 \to \mathbb{S}^1$, there exists a polynomial-like map $f : U \to U'$, which is hybrid equivalent to P, and such that the external map of f is g. Denote the quasiconformal conjugacy between f and P by h, so that $f \circ h = h \circ P$ near K_P. One can assume $h(0) = 0$, $h(1) = 1$. Consider the fixed point $0 = h(0)$ of f, and its multiplier λ_f. By an inequality for the multipliers [17] (see also [20] and [11]), $|\lambda_f| \leq |\rho_g|^{2/p}$. Together with the inequality of Step (2), we have: $|\lambda_f| < |\lambda|$.

(4) Let $\mu = h'_{\overline{z}}/h'_z$ be the Beltrami coefficient of the quasiconformal map h. Since f and P are holomorphic, μ is invariant under P. Hence, $t\mu$ is invariant under P, too, for every $|t| < 1/||\mu||$. By the Measurable Riemann Theorem, see [1], there exists a unique quasiconformal homeomorphism h_t of the plane with the Beltrani coefficient $t\mu = (h_t)'_{\overline{z}}/(h_t)'_z$, which fixes 0 and 1. Clearly, $h_1 = h$, $h_0 = id$. Let us show that $f_t = h_t \circ P \circ h_t^{-1}$ is the required family. Indeed, for every fixed t, f_t is a holomorphic (hence, polynomial-like) map because $t\mu$ is invariant under P. Besides, $f_0 = P$, and $f_1 = f$. Furthermore, using the relation $f_t \circ h_t = h_t \circ P$ and the analytic dependence of h_t on t [1], we see that f_t is analytic in t, too. By the Cauchy formula, the multiplier $\lambda(t)$ of its fixed point $0 = h_t(0)$ is also analytic in t. Moreover, $\lambda(t)$ is not a constant analytic function because $|\lambda(1)| = |\lambda_f| < |\lambda(0)| = |\lambda|$.

(5) If P and a are real, one can arrange all maps and Beltrami coefficients to be symmetric with respect to the real line for real t. □

Let us prove a stronger result. It gives also a different proof of Theorem 2. In the course of the proof, we establish a formula for the derivative of the multiplier, which is interesting by itself.

Let P be an arbitrary non-linear polynomial, and a its repelling fixed point.

Theorem 3. *The polynomial P can be included in an analytic family of polynomial-like maps f_t as in Theorem 2 in such a way that $\lambda'(0) \neq 0$.*

The proof is based on the formula for the derivative of the multiplier $\lambda(t)$ at $t = 0$. We keep the normalization that $a = 0$ and 1 are fixed points of P. Let μ with $||\mu||_\infty \leq 1$ be any invariant Beltrami coefficient of the polynomial P, let the quasiconformal map h_t have the complex dilatation $t\mu$ and fix $0, 1, \infty$, and $f_t = h_t \circ P \circ h_t^{-1}$, $|t| < 1$, be the corresponding family of polynomial-like maps. As above, denote $\lambda(t) = f'_t(0)$, so that $\lambda(0) = \lambda = P'(0)$. Then the formula says that

$$\frac{\lambda'(0)}{\lambda} = \frac{1}{\pi} \lim_{A \to \{0\}} \int_A \frac{\mu(z)}{z^2} dxdy, \tag{40}$$

where A is a fundamental region of the dynamics near the fixed point 0.

To obtain (40), let us linearize P near 0 by fixing a disk $D = \{|z| < r_0\}$ and a univalent map $K : D \to \mathbb{C}$, such that $P \circ K = K \circ \lambda$ in D. Let $\hat{\mu} = |K'|^2/(K')^2 \mu \circ K$ be the pullback of μ to D. Then $\hat{\mu}$ is invariant by the linear map $\lambda : w \mapsto \lambda w$, i.e. $\hat{\mu} = |\lambda|^2/\lambda^2 \hat{\mu} \circ \lambda$. Extend $\hat{\mu}$ to $\mathbb{C}$ by the latter equation. For $|t| < 1/||\hat{\mu}||$, denote by φ_t a unique quasiconformal map with the complex dilatation $t\hat{\mu}$, that fixes $0, 1$, and ∞. Then the map $\varphi_t \circ \lambda \circ \varphi_t^{-1}$ is again linear $w \mapsto \rho(t)w$, for some $|\rho(t)| > 1$. By the construction, it is easy to see that the linear map $w \mapsto \rho(t)w$ is analytically conjugate to f_t near 0. Therefore, $\rho(t) = \lambda(t)$. By change of coordinates $z = K(w)$, the formula (40) reads:

$$\frac{\lambda'(0)}{\lambda} = \frac{1}{\pi} \int_{\hat{A}} \frac{\hat{\mu}(w)}{w^2} d\sigma_w, \tag{41}$$

where $\hat{A}$ is a fundamental region of $w \mapsto \lambda w$. We prove the latter formula. By the invariance equation, one can assume that $\hat{A} = \{w : 1 < |w| < |\lambda|\}$. Differentiating the equation

$$\lambda(t)\psi_t(w) = \psi_t(\lambda w)$$

by t at $t = 0$, we get, for $w \neq 0$:

$$\lambda'(0) = \frac{1}{w}(\frac{d}{dt}|_{t=0}\psi_t(\lambda w) - \lambda \frac{d}{dt}_{t=0} \psi_t(w)).$$

By [1],

$$\frac{d}{dt}|_{t=0}\psi_t(w) = -\frac{1}{\pi} \int_{\mathbb{C}} \hat{\mu}(u) R(u, w) d\sigma_u,$$

where

$$R(u, w) = \frac{w(w-1)}{u(u-1)(u-w)}.$$

Then, after elementary transformations and using the invariance of $\hat{\mu}$, we get

$$\lambda'(0) = -\frac{\lambda(\lambda-1)}{\pi} w \sum_{n \in \mathbf{Z}} \int_{\hat{A}} \frac{\hat{\mu}(\lambda^n z)|\lambda|^{2n}}{\lambda^n z(\lambda^n z - \lambda w)(\lambda^n z - w)} d\sigma_z =$$

$$= -\frac{\lambda}{\pi} \int_{\hat{A}} \frac{\hat{\mu}(z)}{z} \lim_{N \to +\infty} \sum_{n=-N}^{N} (\frac{\lambda^{n-1}}{\lambda^{n-1} z - w} - \frac{\lambda^n}{\lambda^n z - w}) d\sigma_z =$$

$$= -\frac{\lambda}{\pi} \int_{\hat{A}} \frac{\hat{\mu}(z)}{z} \lim_{N \to +\infty} (\frac{\lambda^{-N-1}}{\lambda^{-N-1} z - w} - \frac{\lambda^N}{\lambda^N z - w}) d\sigma_z = \frac{\lambda}{\pi} \int_{\hat{A}} \frac{\hat{\mu}(z)}{z^2} d\sigma_z,$$

because $|\lambda| > 1$.

Proof of Theorem 3. By (40), it is left to choose μ in such a way that the right-hand side does not vanish (if the map is real, μ should be chosen symmetrically w.r.t. $\mathbb{R}$). Note that μ can be chosen arbitrary between any equipotential

$\{z : \log|B_P(z)| = c\}$ of P and its P-preimage $\{z : \log|B_P(z)| = c/d\}$, and then extended up to the filled-in Julia set K_P by the invariance property. On K_P and outside of the domain $\{z : \log|B_P(z)| < c\}$, μ is defined to be zero. The fundamental region A will be of the form $K(A(r))$, where $A(r) = \{|\lambda|^{-1}r < |z| < r\}$, and r is small. We consider preimages in D by K of equipotentials of P, and call them local equipotentials. Let us fix a local equipotential $\Gamma = K^{-1}(\gamma)$, where γ is an equipotential of P, such that Γ intersects the annulus $A(r_0)$. We fix a subset of points Ω in $A(r_0)$, which are between the local equipotentials Γ and $K^{-1}(P^{-1}(\gamma))$. Given a positive integer m, denote $A_m = \lambda^{-m}A(r_0)$, and $\Omega_m = \lambda^{-m}\Omega$. Let us fix m and define a Beltrami coefficient μ_m as follows. First, set $\hat{\mu}(z) = w^2/|w|^2$ in Ω_m. Now we define μ_m in $K(\Omega_m)$ by pushing forward $\hat{\mu}$ by the map K. The set $K(\Omega_m)$ is a subset of the domain bounded by the equipotentials $P^{-m}(\gamma)$ and $P^{-m-1}(\gamma)$. Then we define μ_m to be zero in the rest of this domain, and extend μ_m up to K_P by the invariance. We want to show that, for m large enough, the corresponding family $\{f_t\}$ of polynomial-like mappings satisfies the conditions of Theorem 3. Consider the set of points z in $A_m \setminus \Omega_m$, such that $\mu_m(K(z)) \neq 0$. We claim that the area of this set in the metric $|dz|/|z|$ tends to zero, as $m \to \infty$. Indeed, if z is such a point, then the corresponding pont $Z = P^m(K(z)) = K(\lambda^m z)$ must lie in $K(A(r_0))$ as well as under the equipotential $P^{-m}(\gamma)$ of P and outside of the filled-in Julia set of P. The area of such points Z tends to zero as $m \to \infty$, and the claim follows. It implies that the integral $\int_{A_m} \mu_m(z)/z^2 dxdy$ in the formula for $\lambda'(0)$ is asymptotically, as $m \to \infty$, equal to $\int_\Omega |z|^{-2} dxdy$, which is away from zero. It remains to fix m big enough. □

Remark: The statements and the proofs (with straightforward changes) hold if one replaces the polynomial by a polynomial-like map and the fixed point by a repelling periodic orbit.

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