COMBINATORICS OF THE KNEADING MAP

H. Bruin

Delft University of Technology
Department of Pure Mathematics
P.O.Box 5031
2600 GA Delft, The Netherlands

July 1994

Abstract. The kneading map and Hofbauer tower are tools, developed by F. Hofbauer and G. Keller, to study unimodal maps and the kneading theory. In this paper we survey on the geometric properties of these tools. Results concerning the topological structure of the critical omega-limit set are obtained.

1. Introduction.

In the seventies and eighties, F. Hofbauer and G. Keller wrote several papers in which kneading theory plays an important role, e.g. [H1,H2,HK1,HK2,K1,K2]. Independently of the well-known work of Milnor and Thurston [MT], they developed tools to describe the kneading invariant and certain geometric aspects of the corresponding unimodal map. These tools are the kneading map and the Hofbauer tower. Hofbauer and Keller used the word Markov graph and also canonical Markov extension, because this object ties the dynamics of unimodal maps to the theory of (countable) Markov chains. Hofbauer and Keller proved some beautiful results, mainly concerning invariant measures, for unimodal maps.

In this paper we want to overview some of the properties of kneading map and Hofbauer tower. Hofbauer and Keller [H1,HK1] take a rather combinatorial viewpoint. We will argue geometrically as much as possible, hoping to make things more accessible. Also we will give extensions of the kneading map and Hofbauer tower: the co-kneading map and extended tower. We use these tools to clarify a part of the recurrence behaviour of the critical orbit that is less visible in the ordinary kneading map and tower.

Let $f : I \to I$, $I = [0, 1]$, be a unimodal map. f has a unique critical point c, and we assume that $f(c)$ is a maximum. The forward images of c will be denoted by $c_1, c_2, c_3, \ldots.$ We assume that $f(\partial I) \subset \partial I$, but the interesting dynamics take place on the *dynamical core* $[c_2, c_1]$.

The kneading map is a tool to describe a unimodal map f symbolically, with emphasis on the recurrence behaviour of the critical point. Therefore kneading

Typeset by $\mathcal{AMS}$-TEX

maps can be used to define, or classify, the combinatorial types of unimodal maps. The kneading map also describes the combinatorics of the Hofbauer tower. Using this tower, one can picture the map f as a Markov chain with countably many states. From a geometric viewpoint, the tower is important because it sheds light on the image of the central branch of f^n. (If H is a maximal interval on which f^n is monotone, then the restriction $f^n_{|H}$ is called a *branch* of f^n. A branch is *central* if H contains the critical point in its boundary.) In many proofs concerning the growth rate of the derivatives along the critical orbit, cf. [B1,NS,S], this is important information. Let us present the outline of this paper, and give some more motivation along the way.

Section 2: We define the Hofbauer tower, the extended tower and give a geometric interpretation of the kneading map and co-kneading map. We extend the Hofbauer tower to show the maximal branch images of the branches of f^n at the critical value $f(c)$. If the reader is acquainted with the Koebe Principle (see e.g. [MMS], he will see the importance of these branch images.

Section 3: We will discuss the splitting of the kneading invariant, indicating the (original) combinatorial definition of the kneading map. Apart from the proof that the combinatorial agrees with the geometric definition, we need this part to explain admissibility conditions better.

Section 4: It is not true that every sequence in $\{0,1\}^{\mathbb{N}}$ can be the kneading invariant of some unimodal map. In this section we give admissibility conditions for the kneading and co-kneading map. A good acquaintance with admissibility conditions is necessary to construct, explicitly, unimodal maps with certain intricate combinatorial properties. For instance, a unimodal map for which the critical orbit is dense in the dynamical core.

Section 5: In this section we describe some basic notions as periodic attractor, renormalization and restrictive interval in terms of the kneading map.

Section 6: The topology of the critical omega-limit set $\omega(c)$ is very important for the metric behaviour of unimodal maps. For example, one can prove that absorbing Cantor sets [BL2,GJ] can only occur if $\omega(c)$ satisfies a very rigid minimality property. In this section we will present conditions in terms of the kneading map, that guarantee that $\omega(c)$ is nowhere dense, minimal, etc.

Many results in this paper are due to Hofbauer and Keller. However, many proofs are new, and have a geometric flavour. In this paper we tried to be concise; more details and results can be found in [B2,S,T], and in Hofbauer's and Keller's papers.

We are indebted to Duncan Sands for sharing his idea on extended towers, and also to Sebastian van Strien for the many encouraging discussions. We like the thank the referee for his careful reading of the manuscript.

2. Definitions.

Throughout this paper (a,b) denotes an interval with end-points a and b, also if $b < a$. If $x \neq c$ then $\hat{x}$ denotes the *symmetric point*, i.e. the unique point different from x such that $f(\hat{x}) = f(x)$. The *Hofbauer tower*, or simply tower, is the disjoint union $\check{I} = \bigsqcup D_n$ of intervals $D_n \subset I$. $D_1 = (c, c_1)$ and inductively

$$D_{n+1} = \begin{cases} f(D_n) & \text{if } c \notin cl\, D_n, \\ (c_{n+1}, c_1) & \text{if } c \in cl\, D_n. \end{cases} \tag{1}$$

(If $c_n = c$, then $D_{n+1} = \emptyset$; the tower consists of a finite number of levels in this case.) Notice that c_n is always an end-point of D_n. If the second case of (1) applies then n is called a *cutting time*. The cutting times are denoted by S_k, where $S_0 = 1$. So $D_{S_k+1} = (c_{S_k+1}, c_1)$ for all k. Moreover, for every $S_k < n \leq S_{k+1}$,

$$D_n = (c_n, c_{n-S_k}). \tag{2}$$

It follows inductively that $D_n \subset D_{n-S_k}$, and that the end-point c_{n-S_k} of D_{n-S_k} is also an end-point of D_n. For $n = S_{k+1}$, $c \in D_{S_{k+1}} \subset D_{S_{k+1}-S_k}$, hence $S_{k+1} - S_k$ is again a cutting time. This leads to the definition of the *kneading map* Q:

$$Q : \mathbb{N} \to \mathbb{N} \cup \infty, \quad S_{Q(k)} = S_k - S_{k-1}. \tag{3}$$

For the sake of completeness, let $Q(0) = 0$. If S_k does not exists, we set $S_k = \infty$ and $Q(k) = \infty$, and leave S_{k+j}, $Q(k+j)$ undefined for $j > 1$.

The tower is endowed with an action $\check{f}$. If $x \in D_n$, $x \neq c$, then

$$\check{f}(x) = f(x) \in \begin{cases} D_{n+1} & \text{if } c \notin (x, c_n), \\ D_{S_{Q(k)}+1} & \text{if } c \in (x, c_n), \end{cases} \tag{4}$$

where the second case can only occur when $n = S_k$ is a cutting time. Let $\pi : \check{I} \to I$ be the natural projection. Clearly $\pi \circ \check{f} = f \circ \pi$. Moreover, the construction is such that for every interval $J \subset \check{I}$,

$$f^n|\pi(J) \text{ is monotone if and only if } \check{f}^n|J \text{ is continuous.} \tag{5}$$

Indeed, suppose that $J = (x, y) \subset D_k$ for some k. Then $\check{f}^n|J$ is continuous precisely if $\pi^{-1}(c) \cap (\check{f}^j(x), \check{f}^j(y)) = \emptyset$ for all $0 \leq j < n$. But then also $c \notin (f^j(\pi(x)), f^j(\pi(y)))$ for $0 \leq j < n$, so $f^n_{|\pi(J)}$ is monotone.

Example 1. Figure 1 shows the tower for a map with kneading map $Q(k) = \max(0, k-2)$. In this case $S_0 = 1$, $S_1 = 2$ and $S_k = S_{k-1} + S_{k-2}$, so the cutting times are the Fibonacci numbers. This map, the Fibonacci map, received much attention in the literature, e.g. [BKNS,HK2,LM,KN]. It has very extreme recurrence properties, which leads under certain additional circumstances to the existence of an absorbing Cantor set.

In the Fibonacci case, there are natural finite covers of $\omega(c)$ with disjoint intervals from the tower. For example, the union $cl(D_6 \cup D_7 \cup D_8 \cup D_12 \cup D_{13})$ covers $\omega(c)$. In general, the levels between two subsequent cutting levels cover $\omega(c)$:

$$\omega(c) \subset cl \left(\bigcup_{i=1}^{S_{k-1}} D_{S_k+i} \cup \bigcup_{i=S_{k-1}+1}^{S_k} D_{S_{k+1}+i} \right).$$

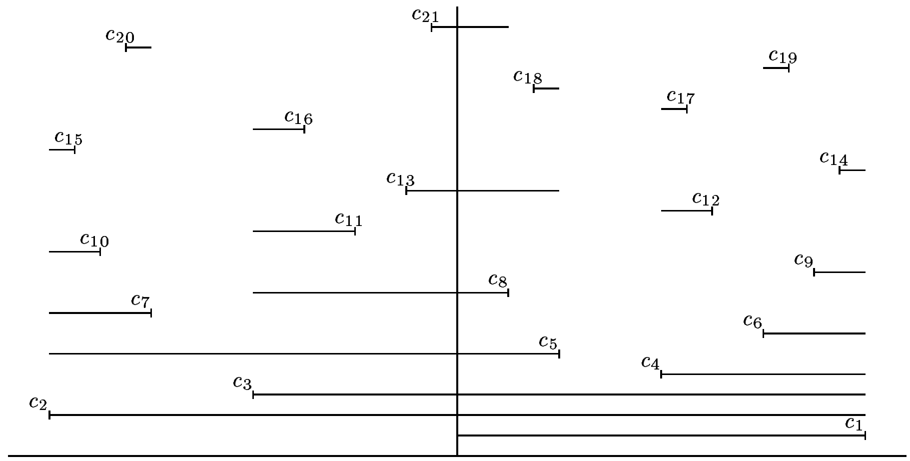

Figure 1: The tower for the Fibonacci map.

Except for $D_{S_{k+1}}$ and $D_{S_{k+2}}$, the intervals of this covering are pairwise disjoint. Details can be found in [LM].

The point $c_{-n} \in f^{-n}(c)$ is a *closest preimage* of c if $f^m((c_{-n}, c)) \not\ni c$ for $m < n$. Clearly c_{-n} and $\hat{c}_{-n}$ are closest preimages simultaneously. An equivalent way to define the closest preimages is the following: (c_{-S_k}, c) and $(c, \hat{c}_{-S_k})$ are the result of a monotone pull-back of the cutting level $D_{S_{k+1}}$ along the critical orbit $c, c_1, \dots c_{S_{k+1}}$. This definition makes clear that $f^{-n}(c)$ contains a closest preimage if and only if n is a cutting time. Let us write

$$A_k = (c_{-S_{k-1}}, c_{-S_k}) \cup (\hat{c}_{-S_k}, \hat{c}_{-S_{k-1}}).$$

By definition, $(c_{-S_{k-1}}, c)$ and $(c, \hat{c}_{-S_{k-1}})$ are maximal open intervals on which f^{S_k} is monotone. Because also $S_{Q(k)}$ is the largest integer such that $f^{S_{Q(k)}}_{|(c, c_{S_{k-1}})}$ is monotone,

$$c_{S_{k-1}} \in cl\, A_{Q(k)}, \tag{6}$$

for every k. Using closest preimages the next lemma is easy to prove.

Lemma 1. *If f has no periodic attractor, then $Q(k) < k$ for every $k \geq 1$.*

Proof. Suppose $Q(k) \geq k$ for some k. By (6), $c_{S_{k-1}} \in (c_{-S_{k-1}}, \hat{c}_{-S_{k-1}})$. Moreover $f^{S_{k-1}}$ maps both $(c_{-S_{k-1}}, c)$ and $(c, \hat{c}_{-S_{k-1}})$ monotonically onto $(c, c_{S_{k-1}})$. Hence $f^{S_{k-1}}$ maps either $(c_{-S_{k-1}}, c)$ or $(c, \hat{c}_{-S_{k-1}})$ monotonically onto itself, yielding a periodic attractor. □

Some important combinatorial concepts are not well incorporated in the Hofbauer tower. The branch of f^n at c_1 and the size of its image is of interest. In many proofs concerning the growth-rate of $|Df^n(c_1)|$ this is essential information, cf. [B1,LM,NS,S]. Also, not every the closest returns of c is clearly visible in the Hofbauer tower. We call c_n a *closest return* if $c_n \in (c_m, \hat{c}_m)$ for every $m < n$. So let us extend the usual Hofbauer tower a little. The *extended tower* is the disjoint union of intervals $\tilde{D}_n \subset I$. $\tilde{D}_1 = (c, 1)$, and

$$\tilde{D}_{n+1} = \begin{cases} f(\tilde{D}_n) & \text{if } c \notin cl\, \tilde{D}_n, \\ f(\tilde{E}_n) & \text{if } c \in cl\, \tilde{D}_n, \end{cases} \tag{7}$$

where $\tilde{E}_n$ is the component of $\tilde{D}_n \setminus \{c\}$ containing c_n. (If $c_n = c$, we take $\tilde{E}_n = \tilde{D}_n \setminus cl\, D_n$.) It follows that $D_n \subset \tilde{D}_n$ for all n. The usual cutting times $\{S_k\}$ are also cutting times for this tower, but there are other cutting times. If $\tilde{D}_n \ni c$, but $D_n \not\ni c$, then n is *co-cutting time*, denoted as $\tilde{S}_l$. In Figure 2 the extended tower of the Fibonacci map is shown. The intervals $\tilde{D}_i \setminus D_i$ are indicated by thick lines.

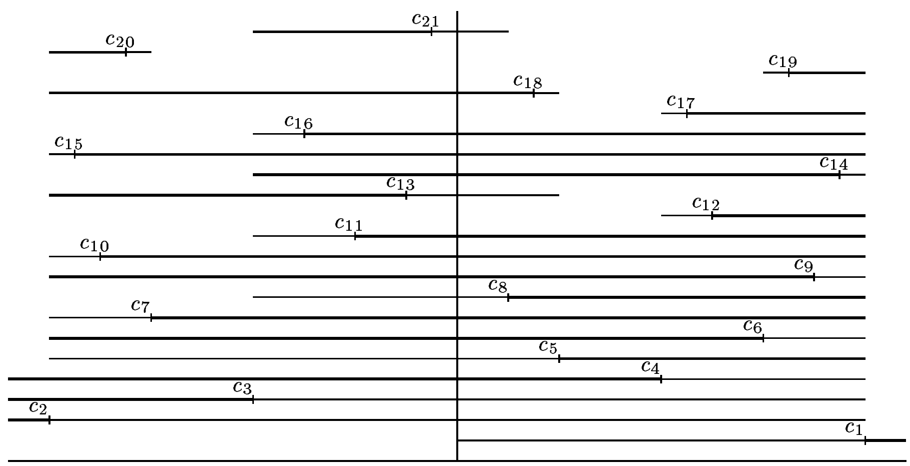

Figure 2: The extended tower for the Fibonacci map.

Let $S\langle n\rangle = \max\{S_k \mid S_k < n\}$ and $\tilde{S}\langle n\rangle = \max\{\tilde{S}_l \mid \tilde{S}_l < n\}$.

Lemma 2. *The following properties are true:*

i) A co-cutting time is never a cutting time.

ii) $\tilde{S}_0 = \kappa$ where $\kappa > 1$ is the smallest integer such that $c_\kappa > c$.

iii) For $n \geq \tilde{S}_0$, $\tilde{D}_n = (c_{n-\tilde{S}\langle n\rangle}, c_{n-S\langle n\rangle})$.

iv) The difference between two subsequent co-cutting times is a cutting time. So one can define the co-kneading map $\tilde{Q} : \mathbb{N} \to \mathbb{N}$ as

$$\tilde{Q} : \mathbb{N} \to \mathbb{N} \cup \infty, \quad \tilde{S}_l - \tilde{S}_{l-1} = S_{\tilde{Q}(l)}, \tag{8}$$

with the same convention $\tilde{S}_l = \infty$ and $\tilde{Q}(l) = \infty$ if $\tilde{S}_l$ as such does not exist.

v) Assume that c is not periodic and let $H_n \ni c_1$ be the maximal open interval such that $f^{n-1}_{|H_n}$ is monotone. Then $f^{n-1}(H_n) = \tilde{D}_n$.

vi) In particular $\tilde{D}_{S_k} = (c_{S_k - \tilde{S}\langle S_k \rangle}, c_{S_{Q(k)}})$, and $\tilde{D}_{\tilde{S}_l} = (c_{S_l - S\langle \tilde{S}_l \rangle}, c_{S_{\tilde{Q}(l)}})$ for every k and $l \in \mathbb{N}$. The integers $S_k - \tilde{S}\langle S_k \rangle$ and $\tilde{S}_l - S\langle \tilde{S}_l \rangle$ are also cutting times.

vii) Statement (5) is also true for the extended tower.

viii) c_1 is approximated from the left by closest preimages c_{1-S_k} and from the right (i.e. from the outside of $[c_2, c_1]$) by the preimages $c_{1-\tilde{S}_l}$ of c. Indeed, $H_n = (c_{1-S\langle n \rangle}, c_{1-\tilde{S}\langle n \rangle})$.

ix) Closest return appear either at cutting or at co-cutting times.

Proof. The proof of these statements is more or less the same as for the Hofbauer tower. As an example, let us prove that $S_k - \tilde{S}\langle S_k \rangle$ is a cutting time. Consider the interval H_{S_k} and its image $f^{S_k - 1}(H_{S_k}) = \tilde{D}_{S_k}$. $c \in f^{S_k-1}(H_{S_k})$, so $c_{-(S_k - \tilde{S}\langle S_k \rangle)} \in f^{\tilde{S}\langle S_k \rangle - 1}(H_{S_k})$. As $\tilde{S}\langle S_k \rangle$ is the largest co-cutting time less than S_k, also $c \in \partial f^{\tilde{S}\langle S_k \rangle - 1}(H_{S_k})$. Hence $f^{\tilde{S}\langle S_k \rangle - 1}(H_{S_k}) \supset (c, c_{-(S_k - \tilde{S}\langle S_k \rangle)})$. Because $f^{S_k - \tilde{S}\langle S_k \rangle}_{|f^{\tilde{S}\langle S_k \rangle - 1}(H_{S_k})}$ is monotone, $c_{-(S_k - \tilde{S}\langle S_k \rangle)}$ must be a closest preimage, and consequently $S_k - \tilde{S}\langle S_k \rangle$ is a cutting time. $\square$

3. The splitting of kneading invariants.

First we will recall some definitions from the kneading theory, and introduce some notation. The *itinerary* of a point $x \in I$ is the sequence $\nu(x) = e_1(x)e_2(x)e_3...$, where

$$e_i(x) = \begin{cases} 0 & \text{if } f^i(x) \in [0, c), \\ C & \text{if } f^i(x) = c, \\ 1 & \text{if } f^i(x) \in (c, 1]. \end{cases}$$

As we neglect the position of x itself, x and $\hat{x}$ have the same itinerary. The left shift σ commutes with the map f. The itinerary of the critical point, denoted as ν, is called the *kneading invariant.* We assume that ν starts with 10, otherwise the dynamics are rather uninteresting. Let

$$\vartheta(n) = \#\{1 \le i \le n \mid e_i = 1\}$$

and

$$\#(n) = \min\{j \ge 1 \mid e_j \ne e_{n+j}\}.$$

It is easy to see that $A_k = \{x \mid \nu(x) = \nu \text{ up to exactly entry } S_k\}$, cf. formula (6).

There is a unique way to split ν into *basic* blocks:

$$\nu = 1\Delta_1\Delta_2\Delta_3...,$$

where each block $\Delta_j = e_{i+1}e_{i+2}...e_{i+k-1}e_{i+k} = e_1e_2...e_{k-1}e'_k$. Here $e'_k = 0$ if $e_k = 1$ and vice versa. (If c happens to be $i+k$ periodic, so $e_{i+k} = C$, then we must take

$e'_k = C$.) The block Δ_j coincides with ν up to the $k-1$th entry. Let $|\Delta_j|$ be the number of entries of Δ_j. We claim that

$$S_k = 1 + \sum_{i=1}^{k} |\Delta_i|,$$

or equivalently $|\Delta_i| = S_{Q(i)} = \#(S_{i-1})$. Indeed, $j = S_{Q(i)}$ is the smallest positive integer such that $f^j(D_{S_{i-1}}) \ni c$. So $j = S_{Q(i)}$ is also the smallest positive integer such that $e_j \neq e_{S_{i-1}+j}$.

As ν is determined completely by its splitting into basic blocks, f is, up to homtervals (i.e. intervals on which f^n is homeomorphic for all n), determined completely by its kneading map.

By splitting the kneading invariant ν, we regain the cutting times. The co-cutting times can also be regained, by a different splitting of ν, called *co-splitting.*

$$\nu = e_1..e_\kappa \tilde{\Delta}_1 \tilde{\Delta}_2....$$

Here $\kappa > 1$ is the smallest integer such that $e_\kappa = 1$. (This is equivalent to the definition of κ in Lemma 2.) $\tilde{\Delta}_i$ are basic blocks. The co-splitting is nothing but an ordinary splitting starting at entry κ. In the same way as above we obtain $\tilde{S}_l = \kappa + \sum_{i=1}^{l} |\tilde{\Delta}_i| = \tilde{S}_0 + \sum_{i=1}^{l} |\tilde{\Delta}_i|$.

Lemma 3. *Suppose that c is not periodic. Then $\vartheta(S_k)$ is odd for all k.*

Proof. At cutting times, either $c_{S_k} > c$ and f^{S_k} has a local maximum at c, or $c_{S_k} < c$ and f^{S_k} has a local minimum at c. In the first case, f^{S_k-1} is increasing in a neighbourhood of c_1, so c_1 visits $(c, c_1]$ an even number of times in the first $S_k - 1$ iterates. Hence $\vartheta(S_k - 1)$ is even and, as $e_{S_k} = 1$, $\vartheta(S_k)$ is odd. The second case goes likewise. □

Remark. In fact, this shows that $\vartheta(n)$ is odd if and only if $c_n > c$ and f^n has a local maximum at c, or $c_n < c$ and f^n has a local minimum at c. It follows that $\vartheta(\tilde{S}_l)$ is even for all l and that the number of ones in each basic block is even. Furthermore, if $f^n_{|T}$ is a branch of f^n such that $f^n(T) = (c_a, c_b) \not\ni c$, then $\vartheta(a)$ is odd and $\vartheta(b)$ is even or vice versa. In this sense one can speak about the *odd* and *even* end-point of $f^n(T)$, being the end-points farthest from and closest to c.

4. Admissibility conditions.

In this section we confine ourselves to maps for which the critical point is not periodic. It is not true that every sequence in $\{0,1\}^{\mathbb{N}}$ corresponds to a kneading invariant. Neither is it true that every map on the natural numbers is a kneading map corresponding to a unimodal map. A map Q (sequence ν) is an *admissible* kneading map (invariant) if there exists a unimodal map having Q (ν) as kneading map (invariant). In this section we will discuss two equivalent admissibility conditions on kneading maps. But let us first recall the admissibility condition for kneading invariants.

There is a standard order relation $\prec$ for itineraries: Let n be the smallest entry for which $\nu(x)$ and $\nu(y)$ differ, then $\nu(x) \prec \nu(y)$ if either $e_n(x) < e_n(y)$ and $\vartheta(n-1)$ is even, or $e_n(x) > e_n(y)$ and $\vartheta(n-1)$ is odd. As $f(x) \mapsto \nu(x)$ is increasing, $\nu = \nu(c) \succeq \nu(x)$ for every $x \in I$. This is the key observation in the admissibility condition A1

Admissibility condition A1. *ν is an admissable kneading invariant if and only if $\sigma^n(\nu) \preceq \nu$ for every $n \in \mathbb{N}$.*

The proof can be found in [CE,MT]. More generally, let $x \in I$ be *nice* if $f^n(x) \notin (x, \hat{x})$ for every $n \geq 1$. A point x is nice if and only if $\sigma^n(\nu(x)) \preceq \nu(x)$ for every $n \geq 1$.

A family of unimodal maps is called *full*, if it exhibits all admissable kneading invariants. The quadratic family $x \mapsto ax(1-x)$ is a full family, as was proved in [MT]. The family of tent maps $T_s : x \mapsto s(\frac{1}{2} - |x - \frac{1}{2}|)$ is not full, since it does not admit every possible renormalization, see Section 5.

In the language of kneading maps, Hofbauer [H1] gave the following condition:

Admissibility condition A2. *Q is an admissable kneading map if and only if*

$$\{Q(k+j)\}_{j\geq 1} \succeq \{Q(Q^2(k)+j)\}_{j\geq 1} \tag{9}$$

for all $k \geq 1$. Here $\succeq$ denotes lexicographical order.

In [H1] one can find a combinatorial proof that A1 and A2 are equivalent. We will give a geometric proof.

Proof of A2. First we will show that (9) is necessary. Choose k arbitrary. By (6), $c_{S_{k-1}} \in A_{Q(k)}$ and as $f^{S_{Q(k)}}$ is monotone on $A_{Q(k)}$, $c_{S_k} \in [c_{S_{Q^2(k)}}, c]$, see Figure 3. This observation is the geometric interpretation of (9).

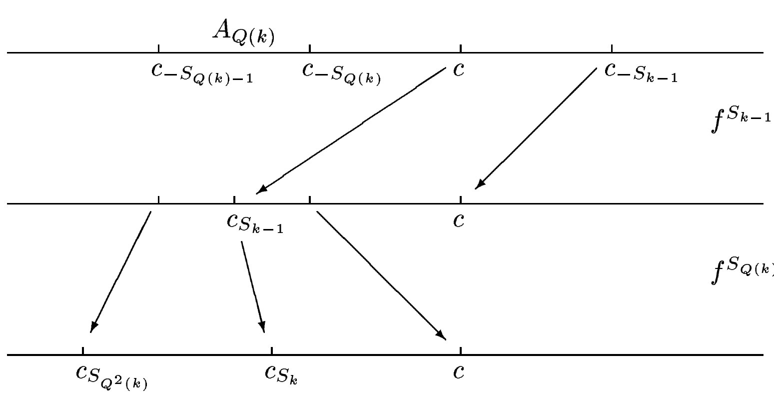

Figure 3.

We will prove it by induction, using the induction hypothesis

$$c_{S_{k+j}} \in [c_{S_{Q^2(k)+j}}, c].$$

We just showed it for $j = 0$. Suppose it is true for j, then we obtain from (6) that $Q(k+j+1) \geq Q(Q^2(k)+j+1)$. If the inequality holds, (9) is true for this value of k. If $Q(k+j+1) = Q(Q^2(k)+j+1)$, then $c_{S_{k+j}}$ and $c_{S_{Q^2(k)+j}}$ are both contained in $A_{Q(k+j+1)}$. If we substitute $k+j+1$ for k in Figure 3, then we obtain $c_{S_{k+j+1}} \in [f^{S_{Q(k+j+1)}}(c_{S_{Q^2(k)+j}}), c] = [f^{S_{Q(Q^2(k)+j+1)}}(c_{S_{Q^2(k)+j}}), c] = [c_{S_{Q^2(k)+j+1}}, c]$, the hypothesis for $j+1$. Hence (9) is necessary.

Now let us prove that (9) is sufficient. Let Q be a kneading map satisfying (9), and let $\nu \in \{0,1\}^{\mathbb{N}}$ and $\{S_k\}_k$ be the corresponding kneading invariant and cutting times. (If c is periodic, then argue as if c is attracted to a periodic attractor.) Define

$$R(k) = Q^2(l) + k - l,$$

where l is the smallest integer such that $Q(Q^2(l)+j) = Q(l+j)$ for all $1 \leq j \leq k-l$. Notice that l exists; possibly $l = k$, so that $R(k) = Q^2(k)$.

$f_a(x) = 4ax(1-x)$ is known to be a full family of unimodal maps. In the rest of the proof we will write ν_a and $c_n(a)$ to denote parameter dependence. Let

$$M_k = \{a \mid \nu_a = \nu \text{ at least up to entry } S_k\}.$$

Clearly $M_0 \supset M_1 \supset ....$ We will prove that M_k is an interval, say $cl\, M_k = [a_k, b_k]$, and $M_\infty = \bigcap_k cl\, M_k \neq \emptyset$. We use the induction hypothesis:

$$\begin{aligned} &a \mapsto c_{S_k}(a) \text{ is monotone on } M_k, \\ &c_{S_k}(a_k) = c \text{ and } c_{S_k}(b_k) = c_{S_{R(k)}}(b_k). \end{aligned} \tag{10}$$

Clearly $\nu_a = \nu$ for every $a \in M_\infty$.

We assume that ν starts with 10. The other cases are trivial. It is easy to check that $M_0 = (\frac{1}{2}, 1]$ and $M_1 = (\frac{1+\sqrt{5}}{4}, 1]$, and that (10) is fulfilled. So assume that (10) is true for k and M_k is found. As $\nu_a = \nu$ up to S_k for all $a \in M_k$, the closest preimages c_{-S_j} can be found for $j \leq k$. By hypothesis $cl\, c_{S_k}(M_k) = [c_{S_{R(k)}}(b_k), c]$, and $c_{S_{R(k)}} \in A_{Q(R(k)+1)}$.

i) If $Q(k+1) = \infty$, so $\nu = e_1...e_{S_k}e_1...e_{S_k}e_1...$, then M_{k+1} is such that $a_{k+1} = a_k$ and b_{k+1} satisfies $c_{S_k}(b_{k+1}) = c_{-S_k}(b_{k+1})$. The induction terminates here, and that suffices. Indeed, if $a \in M_{k+1} = M_\infty$, then $c_{S_k}(a) \in f_a^{S_k}([c, c_{-S_k}(a)]) \subset [c, c_{-S_k}(a)]$. As $f_a^{S_k}|[c, c_{-S_k}(a)]$ is monotone, there exists a periodic attractor with itinerary $e_1...e_{S_k}e_1...e_{S_k}e_1...$, which attracts c.

ii) If $Q(k+1) > Q(R(k)+1)$, then $R(k+1) = Q^2(k+1)$. M_{k+1} is such that $c_{S_k}(a_{k+1}) = c_{-S_{Q(k+1)}}(a_{k+1})$ and $c_{S_k}(b_{k+1}) = c_{-S_{Q(k+1)-1}}(b_{k+1})$. As $f_a^{S_{Q(k)}}$ is monotone on $A_{Q(k+1)}(a)$ for all $a \in M_k$, $a \mapsto c_{S_{k+1}}(a)$ is monotone on M_{k+1}. Moreover $c_{S_{k+1}}(a_{k+1}) = c$ and $c_{S_{k+1}}(b_{k+1}) = f^{S_{Q(k+1)}}(c_{-S_{Q(k+1)-1}}(b_{k+1})) = c_{S_{Q^2(k+1)}}(b_{k+1}) = c_{S_{R(k+1)}}(b_{k+1})$. This proves (10) for $k+1$.

iii) If $Q(k+1) = Q(R(k)+1)$, then $R(k+1) = R(k)+1$. M_{k+1} is such that $c_{S_k}(a_{k+1}) = c_{-S_{R(k)+1}}(a_{k+1})$ and $c_{S_k}(b_{k+1}) = c_{S_{R(k)}}(b_{k+1})$. So $b_{k+1} = b_k$. As before $a \mapsto c_{S_{k+1}}(a)$ is monotone on M_{k+1}, $c_{S_{k+1}}(a_{k+1}) = c$ and $c_{S_{k+1}}(b_{k+1}) = f^{S_{Q(R(k)+1)}}(c_{S_{R(k)}}(b_{k+1})) = c_{S_{R(k)+1}}(b_{k+1}) = c_{S_{R(k+1)}}(b_{k+1})$. Again we obtain (10) for $k+1$. □

Having defined both the kneading and co-kneading map, we can formulate another admissibility condition.

Admissibility condition A3. *Suppose* $\nu = e_1 e_2..$ *such that* $e_i \neq C$ *for all* i, *then* ν *is admissible if and only if* Q *satisfies* (3) *and* $\tilde{Q}$ *satisfies* (8).

A similar admissibility condition is discussed in [T].

Sketch of proof of A3. Since always $c_n \in A_m$ for some m, $\#(n) = S_m$. This is true in particular for $n \in \{S_k\}$ or $\{\tilde{S}_l\}$, proving the only if part.

For the if part we use the same argument as in the proof of A2. However, we need an (equivalent) definition of $R(k)$ in terms of kneading and co-kneading maps. Recall that $\tilde{S}\langle n\rangle = \max\{\tilde{S}_l \mid \tilde{S}_l < n\}$. Let $R(k)$ be such that

$$S_{R(k)} = S_k - \tilde{S}\langle S_k\rangle.$$

This implies that we also have to prove that $S_k - \tilde{S}\langle S_k\rangle$ is indeed a cutting time. We keep the definitions of M_k, a_k and b_k, and prove (10) inductively with this new definition of $R(k)$. Any kneading invariant starts with

$$1\underbrace{00...00}_{m\ 0\text{'s}}1\underbrace{00...00}_{n\ 0\text{'s}}1$$

for some $m, n \in \mathbb{N}$. $m = S_{m-1}$ is a cutting time. Moreover, $n \leq m$. Indeed, if $n > m$, then $S_m = 2m+1$ and $S_m - S_{m-1} = m+1$ is not a cutting time, violating (3). If $n = m$ then ν must be periodic: S_m does not exist, because otherwise $S_m - S_{m-1}$ is not a cutting time. Finally, if $n < m$, then $S_m = n+m+3$ and $\tilde{S}\langle S_m\rangle = S_m - 1$. So $R(m) = Q^2(m) = 0$ and $S_m - \tilde{S}\langle S_m\rangle$ is indeed a cutting time.

Now for the induction step, let us only give the necessary adjustments concerning the values of $R(k)$. Assume (10) is true up to k.

i) $Q(k+1) = \infty$. See A2.

ii) $Q(k+1) > Q(R(k)+1)$. Then $(c, c_{S_k}) \subset (c, c_{-S_{Q(k+1)-1}}) \subset (c, c_{S_{R(k)}})$. So $S_k + S_{Q(k+1)-1}$ is the co-cutting time $\tilde{S}\langle S_k + S_{Q(k+1)}\rangle = \tilde{S}\langle S_{k+1}\rangle$. Also $S_{k+1} - \tilde{S}\langle S_{k+1}\rangle = S_k + S_{Q(k+1)} - (S_k + S_{Q(k+1)-1}) = S_{Q^2(k+1)}$ is indeed a cutting time: $R(k+1) = Q^2(k+1)$.

iii) $Q(k+1) = Q(R(k)+1)$. Now $(c, c_{S_k}) \subset (c, c_{S_{R(k)}}) \subset (c, c_{-S_{Q(k+1)-1}})$. In this case the next co-cutting time is larger than the next cutting time. So $\tilde{S}\langle S_{k+1}\rangle = \tilde{S}\langle S_k\rangle$, and $S_{k+1} - \tilde{S}\langle S_{k+1}\rangle = S_{Q(k+1)} + S_k - \tilde{S}\langle S_k\rangle = S_{Q(R(k)+1)} + S_{R(k)} = S_{R(k)+1}$ is again a cutting time: $R(k+1) = R(k)+1$.

With these adjustments, the proof of A3 is complete. □

5. Restrictive intervals and periodic attractors.

An interval $J \ni c$ is called *restrictive* if $f^n(J) \subset J$ for some integer n. So J is an n-periodic interval, see e.g. [CE,G]. Let us take n minimal and J maximal for this property. Then $f^n_{|J}$ is unimodal, $f^n(\partial J) \subset \partial J$ and $f^j(J) \cap J$ has empty interior if $j < n$. If f has a restrictive interval of period $n > 1$, then f is *renormalizable.* The unimodal map $f^n_{|J}$ is the *renormalization.* We will classify renormalizable maps with respect to the existence of periodic attractors.

Let J be a restrictive interval of period n. There are three cases:

i) $f^n(J) \not\ni c$. Then there is an orientation preserving n-periodic attractor.
ii) $f^n(J) \ni c$, and there is an orientation reversing n-periodic attractor.
iii) $f^n(J) \ni c$, but there is no n-periodic attractor. However, there may be a periodic attractor with a multiple of n as period.

In the following proposition we only consider periodic attractors that are visible in the kneadings. So we assume that a periodic attractor attracts the critical point.

Proposition 1.

i) If f has an orientation preserving n-periodic attractor, then $n = \tilde{S}_l$ for some l and $\tilde{S}_{l+1} = \infty$.
ii) If f has an orientation reversing n-periodic attractor, then $n = S_k$ for some k and $S_{k+1} = \infty$.
iii) f has a restrictive interval J such that $f^n(J) \ni c$ if and only if $n = S_k$ and $Q(m) \geq k$ for every $m > k$.

Remark 1. The well-known Feigenbaum map, which is so to say renormalizable as often as possible, has kneading map $Q(k) = k - 1$.

Remark 2. If f is renormalizable with period S_k, then $\check{f}^n(D_i) \subset \bigsqcup_{i > S_k} D_i$ for every $n > 0$. So points in D_{S_k} are trapped in $\bigsqcup_{i > S_k} D_i$.

Remark 3. For almost restrictive intervals of Johnson's example [J], the situation is as follows: Let J be an almost restrictive interval of period n, then $n = S_k$ for some k and there exists some large integer j_0 such that $Q(k+j) = k$ for $0 < j < j_0$ and $Q(k + j_0) < k$.

If the central branch of f^n is almost tangent to the diagonal, we have an almost saddle node bifurcation, see e.g. [B1]. In that case $n = \tilde{S}_l$ for some l and $\tilde{Q}(l + 1)$ is very large, but not infinite.

Proof of i). Let q be the orientation preserving n-periodic attractor and J the corresponding restrictive interval. Since c_n and q lie on the same side of c, $\nu = \sigma^n(\nu) = \nu(q)$, so ν is n-periodic. $f^n(J) \not\ni c$ and c_n is a closest return, so $n = \tilde{S}_l$ is a co-cutting time. Because $f^j((c_n, c)) \not\ni c$ for all $j \geq 1$, $\tilde{Q}(l + 1) = \infty$.

Proof of ii). Let this time q be the orientation reversing n-periodic attractor. Again c_n and q lie on the same side of c, so $\nu = \sigma^n(\nu) = \nu(q)$. Since $f^n(J) \ni c$ and $f^j(J) \not\ni c$ for $0 < j < n$, $n = S_k$ is a cutting time. $f^j((c_n, c)) \not\ni c$ for all $j \geq 1$, so $Q(k + 1) = \infty$.

Proof of iii). Let J be the restrictive interval of period n. From i) and ii) it follows that n is a cutting time if and only if $f^n(J) \ni c$. Let $J = [p, \hat{p}]$, where p is the orientation preserving periodic boundary point of J. As $f^j(J) \not\ni c$ for $0 < j < n$, $\nu(p) = e_1e_2...e'_{S_k}e_1e_2....e'_{S_k}e_1....$ Also $p \in A_k$. As $D_{S_k+1} \subset f(J)$, and J is restrictive, the subsequent cutting time can only occur at multiples of S_k. Because $J \subset (c_{-S_{k-1}}, \hat{c}_{-S_{k-1}})$, $Q(k+j) \geq k$ for every $j \geq 1$.

On the other hand, suppose that $Q(k+j) \geq k$ for all $j \geq 1$. Let p be the S_k-periodic point with the (admissable) itinerary $e_1 e_2 ... e'_{S_k} e_1 e2 ... e'_{S_k} e_1 ....$ It is easy to check that $\nu \succeq \nu(p)$, and because $f(x) \mapsto \nu(x)$ is increasing, $c_{S_k+1} \geq f(p)$. So $c_{S_k} \in [p, \hat{p}]$ and $[p, \hat{p}]$ is a restrictive interval. $\square$

Example 2. As an illustration of Proposition 1, we will follow the kneading invariant of the quadratic family $x \mapsto a\,x(1-x)$ as it undergoes the saddle node bifurcation and the subsequent period doubling cascade. We took the saddle node bifurcation creating the 3-periodic point with itinerary 101101101....

	a	ν **Table 1**
1. before restr. interval	3.818	1.0.11.0.11.0.11.0.1010.11....
2. saddle node	3.82842...	1.0.11.0.11.0.11.0.11.0.11.0....
3.	3.83	1.0.11.0.11.0.11.0.11.0.11.0....
4. c periodic	3.83187...	1.0.C.10C10C10C10C10C...
5. before per. doubl.	3.84	1.0.0.100100100100100...
6. past per. doubl.	3.844	1.0.0.100100100100100...
7. c periodic	3.84456...	1.0.0.10C.10010C10010C...
8.	3.847	1.0.0.101.100101100101...
9. Feigenbaum case	3.84944...	1.0.0.101.100100.100101100101....
10.	3.85680...	1.0.0.101.101.101.101....
11. past restr. interval	3.86	1.0.0.101.101.101.101.11....

Dots indicate the cutting times. Close before the saddle node bifurcation (1), ν has only a periodic initial part. The length of this initial part increases until, at the saddle node bifurcation (2), ν is periodic. At this point the restrictive interval emerges. $Q(k)$ is still defined for all k, but $\tilde{Q}(1) = \infty$ (case i) in Proposition 1). ν remains the same past the saddle node (3), but changes when c becomes periodic (4). At this point, the orientation of the attractor switches, so as in case ii), $Q(3) = \tilde{Q}(3) = \infty$. Notice that ν does not change if the map undergoes the first period doubling bifurcation (from (5) where case ii) applies to (6) where case i) applies for $n = 6$). At (7) c becomes periodic again, but this time of period 6. The second restrictive interval arises. Close to (8) we get the next period doubling bifurcation, and this procedure repeats itself until we arrive at the infinitely renormalizable case (9). At this point the first renormalization of f is conjugate to the Feigenbaum map. At (10) the restrictive interval is about to disappear. c is eventually periodic and the renormalization is conjugate to the map $x \mapsto 4x(1-x)$. Past this point (11) case iii) of Proposition 1 is no longer valid, because $Q(7) = 1 < 2 = Q(3)$.

6. The topology of $\omega(\mathbf{c})$.

The topological structure of the critical orbit is an interesting thing to study, not only for its own sake, but also because this structure plays a major role in the metric properties of unimodal map. The critical omega-limit set $\omega(c)$ can be (cf. [BL1,G])

i) a finite set, if c is (eventually) periodic, or attracted to a periodic orbit.
ii) the finite union of intervals,
iii) a Cantor set. For instance $\omega(c)$ is a Cantor set if f is infinitely often renormalizable.

The topological structure of $\omega(c)$ can have consequences for the metric structure of a unimodal map. For example, every S-unimodal map for which $\omega(c)$ is a Cantor set and which is not infinitely often renormalizable, has a σ-finite absolutely continuous invariant measure [Ma,HK2]. Also for the existence of an absorbing Cantor set, a certain rigid structure of the critical orbit is essential (persistent recurrence, see below, cf. [BL2,GJ]).

In this section we will formulate conditions on the kneading map that ensure that $\omega(c)$ is nowhere dense. It is well-known that $\omega(c)$ is nowhere dense if f is a Misiurewicz map: f is a *Misiurewicz map* if it has a non-recurrent critical point (i.e. $c \notin \omega(c)$) and no periodic attractor. The next results show that f is a Misiurewicz map if and only if the co-kneading map is bounded.

Lemma 4.
If $\tilde{Q}$ is bounded, then Q is bounded.
If $\lim_{k\to\infty} Q(k) = \infty$, then $\lim_{l\to\infty} \tilde{Q}(l) = \infty$.

Proof. Suppose $\limsup_k Q(k) = \infty$. Assume by contradiction that there exists an upper bound $B < \infty$ of $\tilde{Q}$. Choose a such that $Q(a+1) > B$ and b minimal such that $Q(b+1) > a$. $c_{S_b} \in (c, c_{-S_{Q(b+1)-1}}) \subset (c, c_{-S_a})$. By minimality of b, $\tilde{D}_{S_b} \supset (c, c_{-S_a})$. Hence $S_b + S_a = \tilde{S}_l$ is a co-cutting time, and $c_{\tilde{S}_l} \in f^{S_a}((c, c_{-S_a}) = (c_{S_a}, c)$. By definition of a and (6), $(c_{-S_B}, \hat{c}_{-S_B}) \supset (c_{S_a}, c) \ni c_{\tilde{S}_l}$. So $\tilde{Q}(l+1) > B$, a contradiction.

For the second statement, we first have to prove that $\tilde{D}_{S_k} \subset (c_{S_{Q(k)}}, c_{S_{Q^2(k)}})$ for every $k \geq 1$. As in Lemma 2, let $H_n \ni c_1$ be the maximal interval such that $f^{n-1}_{|H_n}$ is monotone. We know that $f^{n-1}(H_n) = \tilde{D}_n$. $c_{S_{k-1}} \in (c_{-S_{Q(k)-1}}, c)$ (or $(c, \hat{c}_{-S_{Q(k)-1}})$). This is a maximal interval on which $f^{S_{Q(k)}}$ is monotone. Hence $f^{S_{k-1}-1}(H_{S_k}) \subset (c, c_{-S_{Q(k)-1}})$ and thus $f^{S_k-1}(H_{S_k}) \subset f^{S_{Q(k)}}((c, c_{-S_{Q(k)-1}})) = (c_{S_{Q(k)}}, c_{S_{Q^2(k)}}))$.

Assume by contradiction that $\liminf_l \tilde{Q}(l) = B < \infty$. As $Q(k) \to \infty$, there exists a such that $Q(k) > B$ for every $k \geq a$. Secondly, there exists b such that $Q(Q^2(k)+1) > a$ for every $k \geq b$. Choose l such that $\tilde{S}_l > S_b$ and $\tilde{Q}(l+1) \leq B$. Then by (6), $c_{\tilde{S}_l} \notin (c_{-S_B}, \hat{c}_{-S_B})$. $\tilde{D}_{\tilde{S}_l} = (c_{S_{\tilde{Q}(l)}}, c_{\tilde{S}_l - S\langle \tilde{S}_l \rangle})$, and by Lemma 2, property vi), $\tilde{S}_l - S\langle \tilde{S}_l \rangle$ is a cutting time, say S_r. Moreover, as $c_{\tilde{S}_l} \in (c, c_{S_r})$, $c_{S_r} \notin (c_{-S_B}, \hat{c}_{-S_B})$. So by definition of a, $r < a$. Let $S_k = S\langle \tilde{S}_l \rangle$. By the choice of l, $k \geq b$. So by (6) and the definition of b, $c_{Q^2(k)} \in (c_{-S_a}, \hat{c}_{-S_a})$. It follows that $c \in \tilde{D}_{S_l} = f^{\tilde{S}_l - 1}(H_{\tilde{S}_l}) = f^{S_r}(f^{S_k-1}(H_{\tilde{S}_l}) \subset f^{S_r}((c_{S_{Q^2(k)}}, c)) \subset f^{S_r}((c_{-S_a}, \hat{c}_{-S_a}))$ On the other hand, as $a > r$, $f^{S_r}((c_{-S_a}, \hat{c}_{-S_a})) \not\ni c$. This contradiction shows that $\liminf_l \tilde{Q}(l) = \infty$. $\square$

Lemma 5. *f is Misiurewicz map if and only if the co-kneading map is bounded.*

Proof. If f is a Misiurewicz maps, then there exists a neighbourhood $U \ni c$ such that $c_n \notin U$ for all $n \geq 1$. In terms of kneadings, this means that $\#(n)$ is bounded. So in particular, the co-kneading map is bounded.

On the other hand, if the co-kneading is bounded, then, by Lemma 4, the kneading map is bounded too. Since closest returns of c occur either at cutting or at co-cutting times, this implies that $\#(n)$ is bounded. If f has a periodic attractor, then either $Q(k) = \infty$ for some k or $\tilde{Q}(l) = \infty$ fore some l. This is not the case, so f must be a Misiurewicz map. □

If only the kneading map is bounded, then f need not be a Misiurewicz map, cf. [B1]. Still

Lemma 6. *If Q is bounded, then $\omega(c)$ is nowhere dense.*

Proof. Assume that f is not renormalizable; otherwise we consider the first return map on the smallest restrictive interval. If $\omega(c)$ is finite, then there is nothing to prove. Therefore we may assume that Q is defined for all $k \in \mathbb{N}$ and $Q(k) < B$.

Recall that c_n is an odd return if $\vartheta(n)$ is odd. If c_n is a closest odd return, then $D_n \ni c$, so n is a cutting time. If $Q(k) < B$ for all k, an odd return $c_n \in (c_{-S_B}, \hat{c}_{-S_B})$ is not possible.

Let $V \subset (c_{-S_B}, c)$ be an interval adjacent to c_{-S_B} such that $f^{S_B}(V) \subset (c_{-S_B}, c)$. Now take n minimal such that $c_n \in V$. c_n must be an even return, so $D_n \ni c_{-S_B}$. But then $D_{n+S_B} \ni c$, so $n+S_B$ is a cutting time, say S_k. Moreover $c_{S_k} \in f^{S_B}(V) \subset (c_{-S_B}, c)$. So by (6), $Q(k+1) > B$. This proves the lemma. □

Presently, we will discuss the notions of minimality, uniform recurrence and persistent recurrence. $\omega(c)$ is *minimal* if for every $x \in \omega(c)$, $\omega(x) = \omega(c)$. In particular, if $\omega(c)$ is minimal, $\omega(c)$ is nowhere dense. The critical point c is *uniformly recurrent* if for every neighbourhood $U \ni c$, there exists $N = N(U) \in \mathbb{N}$ such that whenever $c_n \in U$, $c_{n+j} \in U$ for some $0 < j < N$.

Lemma 7. *$\omega(x)$ is minimal if and only if x is uniformly recurrent.*

Proof. See for example [BC Chapter V, Proposition 5]. □

Following Milnor [Mi], c is called *persistently recurrent* if the critical point is recurrent, and for every neighbourhood $U \ni c$, there exist only finitely many iterates n such that for some $V(n) \ni c_1$, $f^n(V_n) \supset U$, and $f^n_{|V_n}$ is monotone. In other words, for each neighbourhood $U \ni c$ there are only finitely cutting levels $\tilde{D}_n$ in the extended tower such that $U \subset \tilde{D}_n$. (The original definition of persistence recurrence comes from complex dynamics. It states that Yoccoz' τ-function of the critical tableau tends to infinity. [Mi].)

Lemma 8. *Persistent recurrence implies uniform recurrence.*

Proof. Recall that x is nice if $f^n(x) \notin (x, \hat{x})$ for every $n \geq 1$. Assume by contradiction that c is not uniformly recurrent, and let $U \ni c$ be a neighbourhood that allows arbitrary large return times. Without loss of generality we can assume that ∂U consists of nice points. If $c_n \in U$, let b_n be the first return time of c_n to U, and let W_n be the smallest neighbourhood of c_n such that $f^{b_n}(\partial W_n) \subset \partial U$. As ∂U

is nice, $f^{b_n}_{|W_n}$ is monotone, whenever $W_n \not\ni c$. Moreover, the sets W_n and W_m are disjoint if $b_n \neq b_m$. Hence $f^{b_n}(W_n) = U$ unless $W_n = W_0$.

Let $V_n \ni c_1$ be such that $f^{n-1}(V_n) = W_n$. If n is indeed the smallest integer such that $c_n \in W_n$, then $f^{n-1}_{|V_n}$ is monotone. Because c is not uniformly recurrent, there are infinitely many different sets W_n, and therefore infinitely many different sets V_n which are mapped monotonically onto U by f^{n+b_n-1}. So f is not persistently recurrent. $\square$

Proposition 2. *If* $\lim_{k\to\infty} Q(k) = \infty$, *then* f *is persistently recurrent.*

Proof. Lemma 4 shows that both $Q(k)$ and $\tilde{Q}(l)$ tend to infinity. Choose $U \ni c$ arbitrary. Let H_n be as in Lemma 2. Choose n such that $c_n \in U$ and $f^{n-1}(H_n) \supset U$. Clearly n is either a cutting or a co-cutting time. If $n = S_k$, then $f^{n-1}(H_n) = (c_{S_{Q(k)}}, c_{S_k - \tilde{S}\langle S_k\rangle})$. As $Q(k) \to \infty$, there are only finitely many choices of n such that $c_{S_{Q(k)}} \notin U$. If $n+1 = \tilde{S}_l$, then $f^n(V) = (c_{S_{\tilde{Q}(l)}}, c_{\tilde{S}_l - S\langle \tilde{S}_l\rangle})$. Again, as $\tilde{Q}(l) \to \infty$, there are only finitely many choices of n such that $c_{S_{\tilde{Q}(l)}} \notin U$. So f is persistently recurrent. $\square$

References

[B1] H. Bruin, *Topological conditions for the existence of invariant measures for unimodal maps*, Erg. Th. and Dyn. Sys **14** (1994), 433-451.

[B2] H. Bruin, *Invariant measures for interval maps*, Thesis, Delft (1994).

[BC] L. Block, W.A. Coppel, *Dynamics in one dimension*, Springer-Verlag, Berlin, 1992.

[BKNS] H. Bruin, G. Keller, T. Nowicki, S. van Strien, *Absorbing Cantor sets in dynamical systems: Fibonacci maps*, Preprint Stonybrook (1994/2).

[BL1] A. M. Blokh, M. Ju. Lyubich, *Attractors of the interval*, Banach Center Publ. **23** (1986), 427-442.

[BL2] A. M. Blokh, M. Ju. Lyubich, *Measurable dynamics of S-unimodal maps of the interval*, Ann. Scient. Éc. Norm. Sup. **4e série 24** (1991), 545-573.

[CE] P. Collet, J.-P. Eckmann, *Iterated maps of the interval as dynamical systems*, Birkhauser, Boston, 1980.

[G] J. Guckenheimer, *Sensitive dependence on initial conditions for unimodal maps*, Commun. Math. Phys. **70** (1979), 133-160.

[GJ] J. Guckenheimer, S. Johnson, *Distorsion of S-unimodal maps*, Ann. of Math. **132** (1990), 71-130.

[H1] F. Hofbauer, *The topological entropy of a transformation* $x \mapsto ax(1-x)$, Monath. Math. **90** (1980), 117-141.

[H2] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. of Math. **34** (1979), 213-237.

[HK1] F. Hofbauer, G. Keller, *Quadratic maps without asymptotic measure*, Commun. Math. Phys. **127** (1990), 319-337.

[HK2] F. Hofbauer, G. Keller, *Some remarks on recent results about S-unimodal maps*, Ann. Inst. Henri Poincaré, Physique théorique **53** (1990), 413-425.

[J] S. D. Johnson, *Singular measures without restrictive intervals*, Commun. Math. Phys. **110** (1987), 185-190.

[K1] G. Keller, *Exponents, attractors and Hopf decompositions for interval maps*, Erg. Th. and Dyn. Sys. **10** (1990), 717-744.

[K2] G. Keller, *Lifting measures to Markov extensions*, Monath Math. **108** (1989), 183-200.

[KN] G. Keller, T. Nowicki, *Fibonacci maps re(al)-visited*, Preprint Univ. of Erlangen. (1992).

[LM] M. Lyubich, *Combinatorics, geometry and attractor of quasi-quadratic maps*, Ann. of Math. (1994).

[LM] M. Lyubich, J. Milnor, *The Fibonacci unimodal map*, Preprint Stony Brook. (1991).

[Ma] M. Martens, *The existence of σ-finite measures, applications to 1-dimensional dynamics*, Preprint IMPA. (1991).

[Mi] J. Milnor, *Local connectivity of Julia sets: Expository lectures*, Preprint StonyBrook **1992/11**.

[MMS] M. Martens, W. de Melo, S. van Strien, *Julia-Fatou-Sullivan theory for real 1-dimensional dynamics*, Acta Math. **168** (1992), 273-318.

[MT] J. Milnor, W. Thurston, *On iterated maps of the interval*, in Lect. Notes in Math. **1342** (1988), 465-563.

[NS] T. Nowicki, S. van Strien, *Absolutely continuous measures under a summability condition*, Invent. Math. **93** (1988), 619-635.

[S] D. Sands, *Topological conditions for positive Lyapunov exponent in unimodal maps*, Thesis, Cambridge (1994).

[T] H. Thunberg, *Recurrence of the critical point*, Preprint KTH Stockholm (1993).