COMBINATORICS OF (FIBONACCI-LIKE) UNIMODAL MAPS

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Abstract. These notes were used as part of the Spring School "Ecole Plurithématique de Théorie Ergodique II " Luminy, April 2006. They focus on the properties of unimodal maps, their description in terms of kneading maps, and the behavior of unimodal maps restricted to the critical omega-limit set $\omega(c)$ if these are Cantor sets. Major references are [2, 7, 8, 21].

1. Combinatorics of Unimodal Maps

A unimodal map $f : I \to I$ on the interval is a continuous map having a unique point c, the *critical point*, such that f is increasing to the left and decreasing to the right of c. Let $c_n = f^n(c)$ be the n-th image of the critical point. It is convenient to scale f such that the interval coincides with the *core*: $I = [c_2, c_1]$, and unless $c_2 < c < c_1$ and $c_2 \leq c_3$, the dynamics of f are not very interesting.

The results that we state here hold for the family of unimodal maps

$$f_a(x) = 1 - a|x|^\ell, \qquad a \in [0, 2].$$

Here $c = 0$ is the critical point and ℓ is the order of the critical point. If $\ell = 1$, then f_a is the tent family; if $\ell = 2$ then f_a is the quadratic family. The core of f_a is the interval $I = [c_2, c_1] = [1 - a, 1]$. If $a \in [1, 2]$, then f_a is onto on this interval; if $a < 1$, then every point in $[-1, 1]$ is attracted to a fixed point.

1.1. **Symbolic dynamics.**

The system (I, f) can be described symbolically by a subshift of $\{0, \star, 1\}^{\mathbb{N}}$ where each $x \in I$ is assigned an *itinerary* $i(x) = i_0(x)i_1(x)i_2(x)\dots$ where

$$i_k(x) = \begin{cases} 0 & \text{if } f^k(x) \in [c_2, c), \\ \star & \text{if } f^k(x) = c, \\ 1 & \text{if } f^k(x) \in (c, c_1]. \end{cases}$$

If Σ is the collection of all itineraries, and σ is the left-shift, then the below diagram commutes.

Take $x \notin \text{orb}^-(c) := \cup_{j \geq 0} f^{-j}(c)$, $i(x) \in \{0, 1\}^{\mathbb{N}}$. For each k, the set

$$J_k(x) := \{y \in I : i_0(y) \dots i_{k-1}(y) = i_0(x) \dots i_{k-1}(x)\}$$

Date: Version of July 5, 2007.

These notes were written while HB was supported by EPSRC grant GR/S91147/01.

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "i"'] & I \arrow[d, "i"] \\ \Sigma \arrow[r, "\sigma"'] & \Sigma \end{tikzcd}$$

is an open interval; it is a maximal open neighborhood on which f^k is monotone. It can happen that there are several points with the same itinerary. In this case, $H = \cap_k J_k(x)$ is a non-degenerate interval; it is called a *homterval*, because $f^k : H \to f^k(H)$ is a homeomorphism for every k. If f is a *non-flat* (*i.e.,* the critical order is finite) C^2-map, then any homterval is attracted to a periodic orbit or interval, [21]. This has the following convenient consequence:

Lemma 1. *If f has no wandering intervals or periodic attractor, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if J is an interval of length $|J| > \delta$, then $|f^n(J)| > \varepsilon$ for all $n \geq 0$.*

Proof. See [21, Chapter IV], where this property is called the *Contraction Principle*, although *Non-contraction Principle* seems a better word. □

The *kneading invariant* is defined as the itinerary of the critical point, leaving out the initial $\star$:

$$\nu = \nu_f = \nu_1 \nu_2 \nu_3 \dots$$

Two unimodal maps are *combinatorially equivalent* if they have the same kneading invariant. If f and g are topologically conjugate, then they are combinatorially equivalent, but the converse is not true. The kneading invariant fails to notice:

- *Inessential periodic attractors*, *i.e.,* periodic attractors that don't attract the critical point. Recall that if f has negative Schwarzian derivative, or more precisely $Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\frac{f'''(x)}{f'(x)} \leq 0$ for every non-critical point x, then every periodic attractor attracts a critical point or boundary point, see [22]. In our case, we restricted the map to the core, so the boundary points belong to the critical orbit. Hence in this setting, every periodic attractor is essential.
- Wandering intervals, which however don't exist if f is non-flat and C^2, see [21, Chapter IV].
- The precise period of a periodic attractor generated in a period doubling bifurcation. For example, if $a_1 < a_2$ are parameters just before and after the first period doubling bifurcation creating a periodic attractor of period 2. Then in both cases $\nu_f = 1111\dots$, regardless whether $\omega(c)$ consist of a single or two points in $(c, c_1]$. The kneading invariant indicates this difference in period only when one of these period 2 points passes through c, as parameter a increases.

Two itineraries i and $i^\flat$ can be compared in *parity lexicographical order* $\prec_p$. First set $0 < \star < 1$. If $k = \min\{j \geq 0 : i_j \neq i^\flat_j\}$ then

$$i \prec_p i^\flat \text{ if } \begin{cases} i_k < i^\flat_k & \text{and } \#\{j < k : i_j = 1\} \text{ is even,} \\ i_k > i^\flat_k & \text{and } \#\{j < k : i_j = 1\} \text{ is odd.} \end{cases}$$

Lemma 2 (See [8])**.** *The map $i : I \to \Sigma$, $x \mapsto i(x)$ is order preserving.*

Corollary 1. *Given a unimodal map f with kneading invariant ν,*

$$\sigma(\nu) \preceq_p i(x) \preceq_p \nu \text{ for all } x \in I. \tag{1}$$

and

$$\sigma(\nu) \preceq_p \sigma^n(\nu) \preceq_p \nu \text{ for all } n \geq 0. \tag{2}$$

Conversely, we have:

Lemma 3.
- *Fix $f : I \to I$ with kneading invariant ν. If $e \in \{0,1\}^{\mathbb{N}}$ is a sequence such that* (1) *holds, then there is $x \in I$ such that $i(x) = e$.*
- *If $\nu \in \{0,1\}^{\mathbb{N}}$ is a sequence such that* (2) *holds, then there is a unimodal map f such that $\nu = \nu_f$.*

For this reason, equation (2) is called the *admissibility condition* for kneading invariants. A map f is *renormalizable* if there is an interval $J \ni c$ and period p such that $f^p(J) \subset J$ and $f^i(J)$ and $f^j(J)$ have disjoint interiors for $0 \leq i < j < p$. In this case, the map $f^p : J \to J$ is a new unimodal map, which can be renormalizable itself. Continuing inductively, we can arrive at *infinitely renormalizable* maps which have an infinite sequence of nested periodic interval $J_n \ni c$ of periods $p_n \to \infty$. The best known example is the *Feigenbaum-Coullet-Tresser map* (usually called *Feigenbaum map*) which has a periodic interval J_n of period 2^n for each $n \in \mathbb{N}$.

Renormalizability can be seen from the structure of the kneading invariant by the fact the ν has the structure of a *star-product*.

Proposition 1. *Let f have a p-periodic interval J such that the itinerary of c starts with $\star i_1 \dots i_{p-1}$. Let $f^p : J \to J$ be a unimodal map with kneading invariant $\tilde\nu$. Then the kneading invariant of f itself is*

$$\nu = \begin{cases} i_1 \dots i_{p-1}\tilde\nu_1 i_1 \dots i_{p-1}\tilde\nu_2 i_1 \dots i_{p-1}\tilde\nu_3 \dots & \text{if } \#\{j < k : i_j = 1\} \text{ is even,} \\ i_1 \dots i_{p-1}\tilde\nu'_1 i_1 \dots i_{p-1}\tilde\nu'_2 i_1 \dots i_{p-1}\tilde\nu'_3 \dots & \text{if } \#\{j < k : i_j = 1\} \text{ is odd.} \end{cases} \tag{3}$$

Here $\tilde\nu'_k = 1, \star, 0$ if $\tilde\nu_k = 0, \star, 1$ respectively.

The sequence ν defined by (3) is known as the *star-product* of $\star i_1 \dots i_{p-1}$ and $\tilde\nu$, and written as $\nu = (\star i_1 \dots i_{p-1}) * \tilde\nu$, see [8].

1.2. **Cutting times.** If J is a maximal (closed) interval on which f^n is monotone, then $f^n : J \to f^n(J)$ is called a *branch*. If $c \in \partial J$, $f^n : J \to f^n(J)$ is a *central branch*. Obviously f^n has two central branches, and they have the same image if n is sufficiently large. Denote this image (or the largest of the two) by D_n.

If $D_n \ni c$, then n is called a *cutting time*. Denote the cutting times by $\{S_i\}_{i \geq 0}$, $S_0 < S_1 < S_2 < \dots$ For interesting unimodal maps (such as tent maps with slope > 1 or f_a with $a \in (1, 2]$) $S_0 = 1$ and $S_1 = 2$.

Lemma 4. *Let $\beta(n) = n - \max\{S_k : S_k < n\}$. Then*

$$D_n = [c_n, c_{\beta(n)}] \text{ or } [c_{\beta(n)}, c_n] \qquad \text{for all } n \geq 2, \tag{4}$$

and $D_n \subset D_{\beta(n)}$.

Proof. For simplicity write $[x, y]$ for the interval with endpoints x and y, even if $y < x$. We prove (4) by induction. Since $D_2 = [c_2, c_1]$, it holds for $n = 2$. Next assume that (4) holds for n. If $D_n \not\ni c$ (so n is not a cutting time), then $D_{n+1} = f(D_n) = [c_{n+1}, c_{1+\beta(n)}]$. But $\beta(n+1) = n+1-\max\{S_k : S_k < n+1\} = n+1-\max\{S_k : S_k < n\} = 1+\beta(n)$. So the above interval is $[c_{n+1}, c_{\beta(n+1)}]$. If on the other hand $D_n \ni c$, then $D_{n+1} = [c_{n+1}, c_1]$, but $\beta(n + 1) = 1$, so (4) holds for $n + 1$. This proves the first statement.

The second statement holds for $n = 2$ because $D_2 = [c_2, c_1] = D_1 = D_{\beta(2)}$. Proceeding by induction, *i.e.,* assuming that $D_n \subset D_{\beta(n)}$, we have three cases:

- Neither n nor $\beta(n)$ is a cutting time. Then $D_{n+1} = f(D_n) \subset f(D_{\beta(n)}) = D_{\beta(n)+1} = D_{\beta(n+1)}$.
- $\beta(n)$ is a cutting time, but n is not. Then $D_{n+1} = f(D_n) = [c_{n+1}, c_{1+\beta(n)}] \subset [c_1, c_{1+\beta(n)}] = D_{\beta(n+1)}$.
- n and hence $\beta(n)$ is a cutting time. Then $D_{n+1} = [c_{n+1}, c_{1+\beta(n)}] \subset [c_2, c_1] = D_{\beta(n+1)}$.

This completes the proof. □

Lemma 5. *The sequence of cutting times completely determines the kneading invariant, and vice versa, using the rule:*

$$S_k = \min\{n > S_{k-1} \mid \nu_n \neq \nu_{n-S_{k-1}}\}.$$

Proof. Left to the reader. □

Corollary 2. *If cutting times S_k are defined for all k, then the difference between two consecutive cutting time is again a cutting time. In other words, there is a function: $Q : \mathbb{N} \to \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that*

$$S_k - S_{k-1} = S_{Q(k)}. \tag{5}$$

Proof. If $n = S_k$ is a cutting time, then $\beta(n) = S_k - S_{k-1}$ by definition of $\beta(n)$. But also $D_{\beta(n)} \supset D_n \ni c$, so $\beta(n)$ is a cutting time. □

The map Q is called the *kneading map* of f. Obviously, Q determines the sequence of cutting times, which in turn determine the kneading invariant, and vice versa.

The following *admissibility condition* (equivalent to (2)) characterizes the possible kneading maps:

Proposition 2 ([13, 3])**.** *A map $Q : \mathbb{N} \to \mathbb{N}_0$ is the kneading map of some unimodal map if and only if*

$$\{Q(k+j)\}_{j\geq 1} \succeq \{Q(Q^2(k)+j)\}_{j\geq 1}, \tag{6}$$

where $\succeq$ denotes the lexicographical ordering. The only exception is when the critical point is attracted to an orientation reversing periodic attractor, and $Q(k)$ is only defined for finitely many k.

One can see from this admissibility condition that every non-decreasing kneading map is admissible. Two famous unimodal maps can be easily described by kneading map and/or cutting times[1]:

$Q(k) = k-1$	$S_k = 2^k$	Feigenbaum map
$Q(k) = \max\{0, k-2\}$	$S_k = k$-th Fibonacci number	Fibonacci map

Let us call f *Fibonacci-like* if there is N such that $k - Q(k) \leq N$ for all $k \in \mathbb{N}$.

Proposition 3 ([3])**.** *Assume a unimodal map f has no (essential) periodic attractor. Then f is renormalizable with period n if and only if $n = S_k$ for some k and $Q(k+j) \geq k$ for all $j \geq 1$. In this case, the kneading map $\tilde{Q}$ of the renormalization $f^n : J \to J$ satisfies $\tilde{Q}(j) = Q(k+j) - k$ for all $j \geq 1$.*

Proof. Let $J \ni c$ be a periodic interval with period n. Then $c_n \in J$ and unless f has a periodic attractor of period n, $f^n(J) \ni c$. Therefore n is a cutting time, say $n = S_k$. Because $f^m(J)$ and J have disjoint interiors unless m is a multiple of n, $D_m \not\ni c$ for if m is not a multiple of n. Hence all cutting times S_{k+j}, $j \geq 1$, are multiples of n. It follows from (5) that $Q(k+j) \geq k$ for all $j \geq 1$.

Conversely, if $Q(k+j) \geq k$ for all $j \geq 1$, then all cutting times S_{k+j} are multiples of $n := S_k$. Therefore $D_m \not\ni c$ for $m > n$ unless $m = rn$. Take $J = \cup_r D_{rn}$. Then $f^n(J) \subset J$ as required.

Finally, if f has indeed an n-periodic interval $J \ni c$, then r is a cutting time of f^n only if rn is a cutting time of f. It follows that $n\tilde{S}_j = S_{k+j}$ for all $j \geq 0$ and $\tilde{Q}(j) = Q(k+j) - k$ follows immediately from this. □

1.3. **The canonical Markov extension (Hofbauer tower).** Canonical Markov extensions were first described for intervals maps by Hofbauer and Keller (e.g. [13, 14, 15, 16, 17]), as a way to build the "most economical" extension satisfying the *Markov property*, *i.e.,* there exists a partition of the space that is preserved by the dynamics. This means that the dynamics can be expressed symbolically as (the infinite alphabet equivalent of) a subshift of finite type.

[1]The Feigenbaum map might as well be called Coullet-Tresser map, see [9].

In this section we only look at the unimodal setting. Then the *canonical Markov extension* of (I, f) is the disjoint union $\hat{I} = \sqcup_{n \geq 2} D_n$ equipped with a map $\hat{f} : \hat{I} \to \hat{I}$ defined below[2].

Remark: It would be natural (also in view of Lemma 4) to take the disjoint union starting with D_1, but since D_1 and D_2 are the same interval, it is convenient to identify D_1 and D_2 in the disjoint union and start at $n = 2$.

Let $\pi : \hat{I} \to I$ be the projection (or inclusion map); points $\hat{x} \in \hat{I}$ can be written as $\hat{x} = (x, D_n)$ where $x \in D_n$ and $\pi(\hat{x}) = x$. Then the map $\hat{f} : \hat{I} \to \hat{I}$ is defined as

$$\hat{f}(\hat{x}) = \hat{f}(x, D_n) := (f(x), D_{n'})$$

where

$$n' = \begin{cases} n+1 & \text{if } D_n \not\ni c \text{ or } D_n \ni c \text{ and } x \in [c, c_n]; \\ \beta(n) + 1 & \text{if } D_n \ni c \text{ and } x \in [c_{\beta(n)}, c]. \end{cases}$$

If n is not a cutting time, then $c \notin D_n$ and $n' = n + 1$. If n is a cutting time, say $n = S_k$, then $\beta(n) = S_k - S_{k-1} = S_{Q(k)}$, and $n' = n + 1$ or $n' = \beta(n) + 1$, depending on which side of c the point $\pi(\hat{x})$ lies. If $\pi(\hat{x}) = c$, then $\hat{f}$ is two-valued, but obviously this applies to a countable set only, so it is negligible for many purposes.

The projection π satisfies $\pi \circ \hat{f} = f \circ \pi$, so the below diagram commutes:

$$\begin{tikzcd} \hat{I} \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & \hat{I} \arrow[d, "\pi"] \\ I \arrow[r, "f"'] & I \end{tikzcd}$$

Figure 1 shows the Markov extensions (Hofbauer towers) for the Feigenbaum and Fibonacci map. Arrows indicate the way $\hat{f}$ connects the levels D_n; as you can see, for the Feigenbaum map, there are no arrows leading from any domain D_n, $n \geq 2^k + 1$ to a level D_m, $m < 2^k + 1$. Hence $\sqcup_{n \geq 2^k+1} D_n$ is an absorbing subgraph of the tower for each k; this is a consequence of the fact that the Feigenbaum map is renormalizable.

We have seen before that the Feigenbaum is infinitely renormalizable, and that therefore $\omega(c)$ is a *(solenoidal) attractor*[3]. For the Fibonacci map $\omega(c)$ can also be an attractor, namely if the critical order is sufficiently large, see [6]. It should be noted, however, that $\omega(c)$ is only an attractor in the sense of Lebesgue measure: for Lebesgue-a.e. $x \in I$, $\omega(x) = \omega(c)$. In contrast, there are points $x \in I$ which have a dense orbit in I. These orbits come arbitrarily close to $\omega(c)$, but are not attracted to it. An attractor of this type is called a *wild attractor*: it attracts a set of positive (and in our setting

[2] Some authors take the components of $D_n \setminus \{c\}$ as partition of $\hat{I}$, which differs from ours only at cutting times. This is not the coarsest Markov partition, but has the advantage that $\hat{f}$ is continuous on every partition element.

[3] The *suspension* over $\omega(c)$, *i.e.,* the topological space $\omega(c) \times [0, 1]/\sim$, where $(x, 1) \sim (f(x), 0)$ is a solenoid, hence the name.

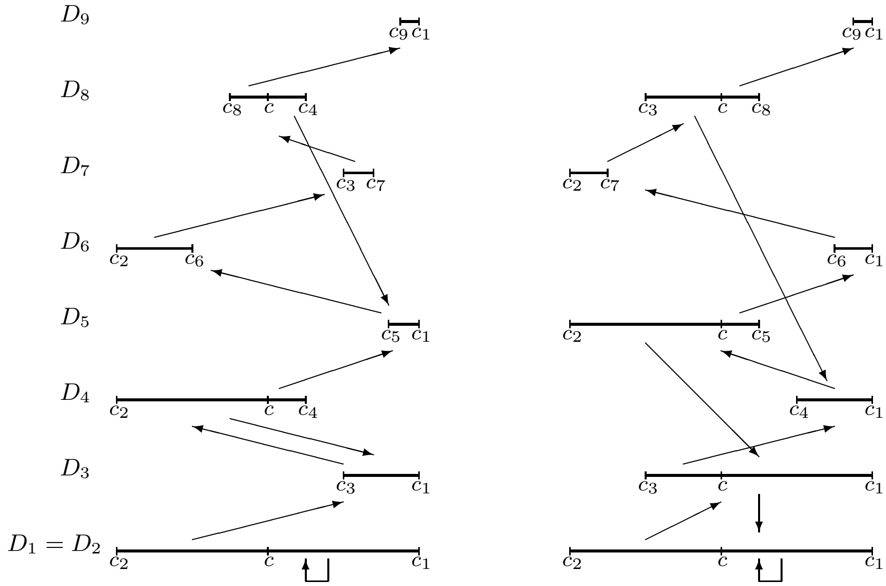

FIGURE 1. Levels D_2 to D_9 of the Hofbauer tower of the Feigenbaum map (left) and Fibonacci map (right).

full) Lebesgue measure, but a set of second Baire category has a dense orbit[4]. The full result in this direction is:

Theorem 1.
- *If f is a quadratic unimodal map (of Fibonacci combinatorics or not), then f has no wild attractor,* [19, 12]*;*
- *If f is a Fibonacci map with sufficiently large critical order ℓ, then $\omega(c)$ is a wild attractor* [6].
- *If f is Fibonacci-like*[5] *with sufficiently large critical order, then f has a wild attractor,* [5].

We will not comment on the proofs of these results, except that the existence of points with dense orbits is not hard to see from the fact that the Hofbauer tower of the Fibonacci map is a *primitive Markov graph*, *i.e.,* you can go from any level in the tower to any (other) level in a finite number of steps. Therefore it is possible to find orbits, represented by infinite paths in the tower, that contain any possible finite path. Such paths correspond to points x with a dense orbit.

Recall that a dynamical system (X, T) is *minimal* if every orbit is dense in X. The condition $Q(k) \to \infty$ has many implications.

[4]Or more precise: for a second Baire category set of points, $\omega(x)$ is a finite union of intervals, cyclically permuted by f.

[5]and satisfies some additional technical, and probably superfluous, conditions,

Theorem 2. *If $Q(k) \to \infty$, then*

(1) $|D_n| \to 0$ *as* $n \to \infty$*;*
(2) c *is recurrent and* $\omega(c)$ *is a minimal Cantor set;*
(3) *The topological entropy* $h_{top}(f|_{\omega(c)}) = 0$*;*
(4) c *is* persistently recurrent*, i.e., for every neighborhood* $U \ni c$ *there are only finitely many* n *such that* $f^{n-1} : V \to U$ *is diffeomorphic for a neighborhood* $V \ni c_1$*;*
(5) $r_n(x) \to 0$ *uniformly over all* $x \in \omega(c)$*, where* $r_n(x)$ *indicates the length* $|f^n(J)|$ *of the branch of* f^n *containing* x*.*
(6) $\limsup_{n\to\infty} \frac{1}{n} \log |Df(x)| \leq 0$ *for every* $x \in \omega(c)$*, so in particular (take* $x = c_1$*),* f *is not a* Collet-Eckmann map*. (A map is called* Collet-Eckmann *if* $|Df^n(f(c))|$ *grows exponentially for every critical point.)*

Remark: It can be proved that c is persistently recurrent if $r_n(c_1) \to 0$, so Statement 5 implies Statement 4. Statement 4 in turn implies minimality of $\omega(c)$, see e.g. [1]. Statement 4 is also instrumental in proving Statement 6. Indeed, by an idea of Ledrappier [18], for any invariant measure μ with positive Lyapunov exponent, typical points should have nontrivial unstable backward manifolds. Therefore, for any $n \geq 0$ and almost every preimage $y \in f^{-n}(x) \cap \omega(c)$ of a typical point x, there should be a neighborhood $U \ni x$ such that $f^n : U \to f^n(U)$ contains a fixed non-degenerate interval. However, this contradicts the fact that $r_n(x) \to 0$ for all $x \in \omega(c)$, see [4].

Proof. We only prove the first two statements, as they are used later on in these notes.

1. First we claim that $c_{1+S_k} \to c_1$. Indeed, if this were not the case, then (recalling from Lemma 4 that $D_{1+S_k} = [c_{1+S_k}, c_1]$) there would be a subsequence $(k_i)_i$ such that $B := \cap_i D_{1+S_{k_i}}$ is a nondegenerate interval. But $f^m(D_{1+S_k}) \not\ni c$ for $m = 1, \dots, S_{k+1} - S_k - 1$ and because $S_{k+1} - S_k - 1 = S_{Q(k+1)} - 1 \to \infty$, we find that B is actually a wandering interval. Thus non-existence of wandering intervals proves the claim.

Consequently, $|D_{S_k}| = |c_{S_k} - c_{S_{Q(k)}}| \to 0$ as $k \to \infty$.

Now let $\varepsilon > 0$ be arbitrary and let δ be as in Lemma 1. Take K such that $|D_{S_k}| < \delta$ for all $k > K$. If $n > S_K$, then there is j such that $f^j(D_n) \subset D_{S_k}$ for some $k \geq K$, so $|D_n| < \varepsilon$. This proves statement 1.

2. By definition, $\mathrm{orb}(c)$ is dense in $\omega(c)$, and since $|c - c_{S_k}| < |D_{S_k}| \to 0$, c is recurrent as well. If x is such that $\mathrm{orb}(x)$ accumulates on c, then $\omega(c) \subset \omega(x)$. Therefore $(\omega(c), f)$ is minimal if and only if $c \in \omega(x)$ for every $x \in \omega(c)$.

Suppose by contradiction that $(\omega(c), f)$ is not minimal, and that $x \in \omega(c)$ and $\varepsilon > 0$ are such that $\mathrm{orb}(x) \cap B(c; \varepsilon) = \emptyset$. We know that $|D_n| \to 0$ as $n \to \infty$, so there is K such that $D_{S_k} \subset B(c; \varepsilon)$ for all $k \geq K$.

For each $m < S_K$, let $\gamma(m) \geq S_K$ be such that $\beta(\gamma(m)) = m$ and $D_{\gamma(m)}$ is the largest interval among all D_n with $n \geq S_K$ and $\beta(n) = m$. We claim that $\omega(c) \subset \cup_{m<S_K} D_{\gamma(m)}$. Indeed, if $n \geq S_K$, then there is a nested sequence of intervals $D_n \subset D_{\beta(n)} \subset \cdots \subset D_{\beta^r(n)}$, where $r \geq 1$ is minimal such that $m := \beta^r(n) < S_K$. But then $D_n \subset D_{\beta^{r-1}(n)} \subset$

$D_{\gamma(m)}$. Therefore

$$\omega(c) \subset \overline{\cup_{n \geq S_K} D_n} \subset \overline{\cup_{m < S_K} D_{\gamma(m)}} = \cup_{m < S_K} D_{\gamma(m)},$$

where the last equality follows because $\cup_{m<S_K} D_{\gamma(m)}$ is the finite union of closed sets. It follows that $x \in D_{\gamma(m)}$ for some $m < S_K$, and because $\gamma(m) \geq S_K$, there is $j \geq 0$ such that $f^j(x) \in f^j(D_{\gamma(m)}) \subset B(c;\varepsilon)$. This contradiction proves statement 2. $\square$

2. Unimodal Maps and Enumeration Scales

We have seen that maps with $Q(k) \to \infty$ have minimal Cantors sets $\omega(c)$, the Lyapunov exponent of $x \in \omega(c)$ is nonpositive (in fact, it is 0), and $\omega(c)$ can even be an attractor. Hence maps with $Q(k) \to \infty$ are exceptional from the point of view of the customary classification of unimodal maps in Collet-Eckmann maps and maps with periodic attractor, which applies to Lebesgue-a.e. parameter for the quadratic family. However, maps such as the Fibonacci map are interesting from a different (number-theoretic) viewpoint too. Lyubich & Milnor showed that if f is a Fibonacci map, then $(\omega(c), f)$ is semi-conjugate to the golden ratio circle rotation $(\mathbb{S}^1, R_\gamma)$, where $\gamma = \frac{1+\sqrt{5}}{2}$. In this section, we present a unified way, developed in [7], for proving this, which also deals with many other cases. Using so-called *enumeration scales* cf. [11], we can make the connection between $(\omega(c), f)$ and several other ways of describing minimal Cantor sets such as substitution shifts and adic transformations on Bratteli diagrams.

2.1. **Enumeration scales.**

An *enumeration scale* (from the French: *échelle de numération*) is an adding machine-like number system based on a strictly increasing sequence of non-negative integers $\{S_k\}_{k\geq 0}$ with $S_0 = 1$. Any non-negative integer n can be written by a *greedy algorithm* a sum of S_k's.

1. First set all e_j equal to 0.
2. Take the largest number $S_j \leq n$ and add 1 to e_j.
3. Replace n by $n - S_j$. If this number > 0, go back to step 2.
4. Otherwise, the procedure stops, and we have $n = \sum_j e_j S_j$.

Assuming that $S_k \leq 2S_{k-1}$ for all $k \geq 1$, one can check that

$$e_j = \begin{cases} 1 & \text{if } j = \max\{k; S_k \leq n - \sum_{k>j} e_k S_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular $e_j = 0$ if $S_j > n$. In this way we can code the non-negative integers $\mathbb{N}_0$ as zero-one sequences with a finite number of ones: $n \mapsto \langle n \rangle \in \{0,1\}^{\mathbb{N}_0}$. Let $E_0 = \langle \mathbb{N}_0 \rangle$ be the set of such sequences, and let E be the closure of E_0 in the product topology.

In our case, we let $\{S_k\}_k$ be the cutting time of a unimodal map, and assume that $Q(k) \to \infty$. This results in

$$E = \{e \in \{0,1\}^{\mathbb{N}_0}; e_i = 1 \Rightarrow e_j = 0 \text{ for } Q(i+1) \leq j < i\},$$

because if $e_i = e_{Q(i+1)} = 1$, then this should be rewritten to $e_i = e_{Q(i+1)} = 0$ and $e_{i+1} = 1$. It follows immediately that for each $e \in E$ and $j \geq 0$,

$$e_0 S_0 + e_1 S_1 + \cdots + e_j S_j < S_{j+1}. \tag{7}$$

There exists the standard addition of 1 by means of 'add and carry'. Given an $e \in E$ proceed as follows:

add 1 : Only one of the first two digits of e can be one. If the first digit is 0, replace it by 1. If it is 1, shift it to the right (*i.e.,* replace 10 by 01). In general the result will be no element of E anymore.

carry : Beginning with the smallest l such that $e_l = 1$ take the following steps: There is at most one $k \in \mathbb{N}$ with $Q(k+1) = l$ and $e_k = 1$. If such a k exists replace e_k and e_l by 0 and e_{k+1} by 1. (In this case k equals $\min\{i > l : e_i = 1\}$) Then restart the carry operation with $l = k+1$. If there is no such k, then the procedure stops.

Denote this action by g. As intended, $g(\langle n \rangle) = \langle n+1 \rangle$.

Example: Adding 1 to 33 when the S_k are the Fibonacci numbers looks like this:

	S_k :	1	2	3	5	8	13	21	34	55	89	$\cdots\cdots$
$\langle 33 \rangle$	=	1	0	1	0	1	0	1	0	0	0	$\cdots\cdots$
$+ \langle 1 \rangle$	=	1	0	0	0	0	0	0	0	0	0	$\cdots\cdots$
	=	0	1	1	0	1	0	1	0	0	0	$\cdots\cdots$
	=	0	0	0	1	1	0	1	0	0	0	$\cdots\cdots$
	=	0	0	0	0	0	1	1	0	0	0	$\cdots\cdots$
$\langle 34 \rangle$	=	0	0	0	0	0	0	0	1	0	0	$\cdots\cdots$

Lemma 6. *For a sequence $e \in E$, let $\{q_j\}_{j \geq 0}$ be the sequence of indices (in increasing order) such that $e_{q_j} = 1$. We have $g(e) = \langle 0 \rangle$ if and only if $e \notin E_0$, $Q(q_0+1) = 0$ and $Q(q_j+1) = q_{j-1}+1$ for $j \geq 1$. Moreover, $g^{-1}(\langle 0 \rangle) \neq \emptyset$.*

Proof. This follows immediately from the add and carry construction, because the condition on $\{q_j\}$ is the only way the addition of 1 carries 'to infinity'. By applying the pigeon hole principle, one can find an infinite sequence $(q_j)_{j \geq 0}$ such that indeed $Q(q_j+1) = q_{j-1}+1$ for all j. $\square$

Example: If $Q(k) = \max\{0, k-d\}$ for some fixed $d \geq 1$, then we obtain a number system (E, g) where $\langle 0 \rangle$ has exactly d preimages; namely the sequences where all 1s are exactly d entries apart map to $\langle 0 \rangle$.

Lemma 7. *If $Q(k) \to \infty$, then g extends uniquely to a continuous map $g : E \to E$. Moreover g^{-1} is well defined on $E \setminus \langle 0 \rangle$, g is surjective, and (E, g) is minimal.*

Proof. First we claim that $g : E_0 \to \omega(c)$ is uniformly continuous. Take M arbitrary and N so large that $Q(n) \geq M$ for all $n \geq N$. If $e, \tilde{e} \in \mathbb{N}_0$ are such that $e_i = \tilde{e}_i$ for all $i \leq N$. Then $\sum_{i \leq M} e_i S_i = \sum_{i \leq M} \tilde{e}_i S_i < S_{M+1}$. Apply g to e and $\tilde{e}$. There are three possibilities:

- There is a carry beyond entry M for both e and $\tilde{e}$. That means that $\sum_{i \leq M} e_i S_i = \sum_{i \leq M} \tilde{e}_i S_i = S_{Q(n+1)} - 1$ for some $n \geq M$ and $g(e)_i = g(\tilde{e})_i = 0$ for $i \leq n$.
- There is no carry beyond entry M for neither e nor $\tilde{e}$. Then $\sum_{i \leq M} e_i S_i = \sum_{i \leq M} \tilde{e}_i S_i < S_{Q(n+1)} - 1$ for all $n \geq M$ and $g(e)_i = g(\tilde{e})_i$ for all $i \leq M$.
- There is a carry beyond entry M for e but not for $\tilde{e}$. That means that $\sum_{i \leq M} e_i S_i = \sum_{i \leq M} \tilde{e}_i S_i = S_{Q(n+1)} - 1$ for some $n \geq M$ and $e_n = 1$ but $\tilde{e}_n = 0$. By assumption on e and $\tilde{e}$, $n > N$, but then $Q(n+1) \leq M$ contradicts the choice of N. Hence this case is impossible.

This proves the claim, so g can be extended uniquely to a continuous map $g : E \to E$. If $\tilde{e} \in E$, then there is a sequence $(e^n)_{n \in \mathbb{N}} \subset E_0$ such that $e^n \to e$ and $g(e^n) \to \tilde{e}$. If there is another sequence $(\tilde{d}^n)_{n \in \mathbb{N}} \subset E_0$ such that $\tilde{d}^n \to \tilde{d} \neq \tilde{e}$ and $g(\tilde{d}) = e$. Now unless $e = \langle 0 \rangle$ there is j such that

$$\sum_{i \leq j} g(\tilde{e})_i S_i, \ \sum_{i \leq j} g(\tilde{d})_i S_i < S_{j+1}.$$

Taking j sufficiently large, we can assume that $\sum_{i \leq j} \tilde{e}_i S_i \neq \sum_{i \leq j} \tilde{d}_i S_i$, and therefore $\sum_{i \leq j} \tilde{e}^n_i S_i \neq \sum_{i \leq j} \tilde{d}^n_i S_i$ for n sufficiently large. But that means that $\sum_{i \leq j} g(\tilde{e}^n)_i S_i \neq \sum_{i \leq j} g(\tilde{d}^n)_i S_i$ But then $\sum_{i \leq j} g(\tilde{e}^n)_i S_i \neq \sum_{i \leq j} g(\tilde{d}^n)_i S_i$ and taking the limit $n \to \infty$ we get $g(\tilde{e}) \neq g(\tilde{d})$, a contradiction. This shows that $g^{-1} : E \setminus \langle 0 \rangle \to E$ is well defined. By Lemma 6, there is a sequence $e \in E$ such that $g(e) = \langle 0 \rangle$.

Finally, $\mathrm{orb}(\langle 0 \rangle) = E_0$ is dense in E and for each $e \in E$, and $j \geq 1$, we have $g^k(e)_i = 0$ for all $i \leq j$ when $k = S_{j+1} - \sum_{i \leq j} e_i S_i$. This proves that (E, g) is minimal. □

Lemma 8. *If $Q(k) \to \infty$, then (E, g) factorizes over $(\omega(c), f)$,* i.e., *there is a continuous map $\pi : E \to \omega(c)$ such that $\pi \circ g = f \circ \pi$.*

Proof. First define $\pi(\langle n \rangle) = c_n$. Obviously $f \circ \pi = \pi \circ g$. Let us show that $\pi : E_0 \to \omega(c)$ is uniformly continuous. Take $\varepsilon > 0$ arbitrary. Because $|D_n| \to 0$, there exists R such that $|D_n| < \varepsilon/2$ for all $n \geq S_R$. Let m be arbitrary, and let U be an ε-neighborhood of $c_m = \pi(\langle m \rangle)$. Take $i \in \mathbb{N}$ minimal such that $\beta^i(m) =: k < S_R$. So $\langle k \rangle$ coincides with $\langle m \rangle$ up to entry $R-1$, and has only 0's elsewhere. If $m < S_R$ (that is: $i = 0$), then simply $k = m$, and $c_m \in D_{k+S_R}$. If $m \geq S_R$, then $\beta^{i-1}(m) = k + S_r$ for some $r \geq R$. It follows by Lemma 4 that $c_m \in \beta^{i-1}(D_m) = D_{k+S_r}$.

Let $\langle n \rangle$ be such that it coincides with $\langle m \rangle$ up to the $(R-1)$–st position. Then there exists i' such that $\beta^{i'}(n) = \beta^i(m) = k$. This means that also $\beta^{i'-1}(n) = k + S_{r'}$ for some $r' \geq R$. By Lemma 4, $\beta^{i'-1}(D_n) = D_{k+S_{r'}}$ and $c_n \in D_{k+S_{r'}}$. So $c_m \in D_{k+S_r}$ and $c_n \in D_{k+S_{r'}}$ for $r, r' \geq R$. Since D_{k+S_r} and $D_{k+S_{r'}}$ have c_k as common boundary point, this yields $c_n \in U$. Hence π is uniformly continuous. The existence and continuity of the extension follows.

A geometric way of picturing $\pi : E \to \omega(c)$ is as follows: Recall that for $n \in (S_{k-1}, S_k]$, we defined $\beta(n) = n - S_{k-1}$. It is easy to check that $\langle \beta(n) \rangle$ is $\langle n \rangle$ with the last non-zero entry changed to 0. Recall from Lemma 4 that $D_n \subset D_{\beta(n)}$ for all $n \geq 2$, and that D_n and $D_{\beta(n)}$ have the boundary point $c_{\beta(n)}$ in common. Define (with $\{q_j\}_{j \geq 0}$ the sequence of indices from Lemma 6)

$$b(i) := \sum_{j \leq q_i} e_j S_j.$$

We have $b(i) \geq S_{q_i}$ by definition of q_i and $b(i) < S_{q_i+1}$ by (7). It follows that $\beta(b(i)) = b(i-1)$. By Lemma 4 and the fact that $\beta(b(i)) = b(i-1)$, the domains $D_{b(i)}$ are nested, and because $Q(k) \to \infty$ implies that $|D_n| \to 0$ (see Theorem 2) each nested sequence defines a unique point $x = \cap_i D_{b(i)} \in \omega(c)$. Therefore the following projection (see [7]) makes sense:

$$\pi(\langle n \rangle) = c_n \text{ and } \pi(e) = \cap_i D_{b(i)} \text{ if } e \notin E_0. \tag{8}$$

This is indeed the extension of π to E. □

2.2. **Enumeration scales and circle rotations.**

Let $\rho \in [0, 1]$ and $R_\rho : \mathbb{S}^1 \to \mathbb{S}^1$ be the rotation by angle ρ. Define $\Pi_\rho : \Omega \to \mathbb{S}^1$ by

$$\Pi_\rho(e) = \sum_{k \geq 0} e_k S_k \rho \pmod 1$$

For $x \in \mathbb{R}$, let $\|x\|$ denote the distance of x to the closest point in $\mathbb{Z}$.

Lemma 9.
(1) *If $\sum_k \|\rho S_k\| < \infty$, then Π_ρ is well defined, continuous and $\Pi_\rho \circ g = R_\rho \circ \Pi_\rho$.*
(2) *If $\varphi : (E, g) \to (\mathbb{S}^1, R_\rho)$ is a nonconstant continuous homomorphism and if $\varphi(\langle 0 \rangle) = 0$, then $\sum_k \|\rho S_k\| < \infty$ and $\varphi = \Pi_\rho$.*

Proof. The first assertion is obvious. We prove the second one: Let $\{n_j\}_j$ be any strictly increasing sequence of integers such that $\langle n_j \rangle \to \langle 0 \rangle$ as $j \to \infty$. Then by continuity of φ,

$$\sum_k \langle n_j \rangle_k S_k \rho = n_j \rho = R_\rho^{n_j}(0) = R_\rho^{n_j}(\varphi \langle 0 \rangle) = \varphi(g^{n_j} \langle 0 \rangle) = \varphi(\langle n_j \rangle) \to \varphi(\langle 0 \rangle) = 0$$

as $j \to \infty$. As this is true for any such sequence $\{n_j\}_j$, it follows that $\sum_k \|\rho S_k\| < \infty$. In particular, Π_ρ is well defined. Furthermore, as $(\varphi - \Pi_\rho) \circ g^n = \varphi - \Pi_\rho$ for all n, $\varphi - \Pi_\rho$ is constant by continuity and topological transitivity of (E, g), and as $\varphi \langle 0 \rangle = 0 = \Pi_\rho \langle 0 \rangle$, we conclude that $\varphi = \Pi_\rho$. □

Lemma 10. *The map $\pi : E \to \omega(c)$ is such that $\pi^{-1}(c) = \langle 0 \rangle$.*

Proof. Suppose $\langle 0\rangle \neq e = e_0e_1\cdots \in E$ and $\pi(e) = c$. Let r, s be the first two integers such that e_r, $e_s = 1$. Clearly s exists, otherwise $\pi(e) = c_r \neq c$. It follows from (8), that $\pi(e) \in D_{S_r+S_s}$. So if $\pi(e) = c$, then $c \in D_{S_r+S_s}$, and $S_r + S_s$ is a cutting time, say S_t. But then $s = t-1$ and $r = Q(t)$, which is forbidden by the construction of E. $\square$

The final goal of this section is to define a factor map $\pi_\rho : \omega(c) \to \mathbb{S}^1$ which makes the following diagram commute:

$$\begin{tikzcd} & (E,g) \arrow[dl, "\pi"'] \arrow[dr, "\Pi_\rho"] & \\ (\omega(c), f) \arrow[rr, "\pi_\rho"] & & (\mathbb{S}^1, R_\rho) \end{tikzcd}$$

Theorem 3. *Suppose that*

$$\sum_k \|\rho S_k\| < \infty \tag{9}$$

(whence Π_ρ is well defined and continuous by Lemma 9). Then $\#\Pi_\rho(\pi^{-1}\{x\}) = 1$ for all $x \in \omega(c)$, and if the unique element of this set is denoted by $\pi_\rho(x)$ this defines a continuous factor map $\pi_\rho : (\omega(c), f) \to (\mathbb{S}^1, R_\rho)$.

Proof. Let $M := \{x \in \omega(c) : \#\Pi_\rho(\pi^{-1}\{x\}) > 1\}$. We claim that if $x \in M$ then the closure of the orbit of x is contained in M. Since $(\omega(c), f)$ is minimal it follows that M is either empty or equal to $\omega(c)$. Finally, since $c \notin M$ by Lemma 10, this proves that π_ρ is well defined.

Now let us prove the claim. Suppose $x \in M$. Then there exist $e, e' \in E$ such that: $\pi(e) = \pi(e') = x$ and $\Pi_\rho(e) \neq \Pi_\rho(e')$. Then

$$\begin{array}{ccccccc} \Pi_\rho \circ g^n(e) & = & R_\rho^n \circ \Pi_\rho(e) & \neq & R_\rho^n \circ \Pi_\rho(e') & = & \Pi_\rho \circ g^n(e') \\ \pi \circ g^n(e) & = & f^n \circ \pi(e) & = & f^n \circ \pi(e') & = & \pi \circ g^n(e') \ , \end{array}$$

whence $\mathrm{orb}(x) \subset M$. Now let y be any limit point of $\mathrm{orb}(x)$ and a sequence $(n_k)_{k\in\mathbb{N}}$ be given with $f^{n_k}(x) \to y$. Since E is compact we can assume that $g^{n_k}(e)$ and $g^{n_k}(e')$ converge simultaneously to some e_0 resp. e_0' in E. From continuity of π follows that $\pi(e_0) = \pi(e_0') = y$ and from continuity of Π_ρ together with the fact that R_ρ is an isometry we get $|\Pi_\rho(e_0) - \Pi_\rho(e_0')| = |\Pi_\rho(e) - \Pi_\rho(e')| \neq 0$. This shows that $y \in M$ proving the claim.

Since the map $\pi : E \to \omega(c)$ is continuous, surjective and closed (E is compact), $\omega(c)$ carries the final topology of E with respect to π. So continuity of π_ρ follows from continuity of $\Pi_\rho = \pi \circ \pi_\rho$. $\square$

2.3. **Unimodal maps for each rotation.** Since the Fibonacci numbers can be written as $S_k = C_1\gamma^k + C_2\hat{\gamma}^k$ for $-1 < \hat{\gamma} := \frac{1-\sqrt{5}}{2} < 1 < \gamma := \frac{1+\sqrt{5}}{2}$ and some $C_1, C_2 \in \mathbb{R}$, we get

$$\gamma S_k = C_1\gamma^{k+1} + C_2\gamma\hat{\gamma}^k = S_{k+1} + C_2\hat{\gamma}^k(\gamma - \hat{\gamma}),$$

and therefore $\|\gamma S_k\| = |C_2 \hat{\gamma}^k(\gamma - \hat{\gamma})|$ is exponentially small in k. Therefore Theorem 3 applies and $(\mathbb{S}^1, R_\gamma)$ is a continuous factor of (*i.e.,* semi-conjugate to) $(\omega(c), f)$.

Can you do this for other irrational rotations $(\mathbb{S}^1, R_\rho)$? In other words, for some $\rho \in (0,1)\setminus\mathbb{Q}$, can you find some sequence of cutting times $(S_k)_k$, such that $\sum_k \|\rho S_k\| < \infty$? The first idea is to use the convergents p_n/q_n of the continuous fraction of ρ. If

$$\rho = [a_1, a_2, a_3, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

then the truncated continued fraction $[a_1, a_2, \dots, a_n] = \frac{p_n}{q_n}$, where

$$\begin{cases} p_0 = 0, & p_1 = 1, \quad p_{n+1} = a_{n+1} p_n + p_{n-1}, \\ q_0 = 1, & q_1 = a_1, \quad q_{n+1} = a_{n+1} q_n + q_{n-1}. \end{cases}$$

The numbers q_n grow exponentially, and it is known that $|\rho - \frac{p_n}{q_n}| \leq \frac{1}{q_n q_{n+1}}$. So

$$\|\rho q_n\| \leq |\rho q_n - p_n| = q_n |\rho - \frac{p_n}{q_n}| \leq \frac{1}{q_{n+1}},$$

which is summable. The problem is that, unless $a_n = 1$ for all $n \in \mathbb{N}$ (and $\rho = \gamma$ is the golden ratio), $q_{n+1} > 2q_n$ for some n, and the numbers q_n cannot be used as cutting times.

The solution to this is to use the *Farey convergents*. These Farey convergents insert numbers in between the usual convergents (underbraced) in the following way:

$$\begin{aligned} &\underbrace{\frac{0}{1} = \frac{p_0}{q_1}}, \ \frac{1}{2}, \ \frac{1}{3}, \dots, \underbrace{\frac{1}{a_1} = \frac{p_1}{q_1}}, \\ &\qquad \frac{p_1 + p_0}{q_1 + q_0}, \ \frac{p_1 + p_0}{q_1 + q_0}, \ \frac{p_1 + p_0}{q_1 + q_0}, \dots\dots, \underbrace{\frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{p_2}{q_2}}, \\ &\qquad \frac{p_2 + p_1}{q_2 + q_1}, \ \frac{2p_2 + p_1}{2q_2 + q_1}, \ \frac{3p_2 + p_1}{3q_2 + q_1}, \dots\dots, \underbrace{\frac{a_3 p_2 + p_1}{a_3 q_2 + q_1} = \frac{p_2}{q_2}}, \\ &\qquad \frac{p_3 + p_2}{q_3 + q_2}, \ \frac{2p_3 + p_2}{2q_3 + q_2}, \ \frac{3p_3 + p_2}{3q_3 + q_2}, \dots\dots, \underbrace{\frac{a_4 p_3 + p_2}{a_4 q_3 + q_2} = \frac{p_4}{q_4}}, \\ &\qquad \frac{p_4 + p_3}{q_4 + q_3}, \dots \end{aligned}$$

Then $|\rho - \frac{ip_n + p_{n-1}}{iq_n + q_{n-1}}| \leq \frac{2}{iq_n q_{n+1}}$, so

$$\sum_{i=1}^{a_{n+1}} \|\rho(iq_n + q_{n-1})\| \leq \sum_{i=1}^{a_{n+1}} (iq_n + q_{n-1}) \left| \rho - \frac{ip_n + p_{n-1}}{iq_n + q_{n-1}} \right| \leq \sum_{i=1}^{a_{n+1}} \frac{2(iq_n + q_{n-1})}{iq_n q_{n+1}} \leq \frac{4}{q_{n+1}}.$$

Therefore

$$\sum_n \sum_{i=1}^{a_{n+1}} \|\rho(iq_n + q_{n-1})\| \leq \sum_n \frac{4}{q_{n+1}} < \infty.$$

The denominators of the Farey convergents *are* suitable as cutting times. Indeed, if $q_{n-1} = S_{k'}$ and $q_n = S_k$, then

$$\begin{array}{rcl} S_{k+1} & = & q_n + q_{n-1} = S_k + S_{k'} \\ S_{k+2} & = & 2q_n + q_{n-1} = S_{k+1} + S_k \\ S_{k+3} & = & 3q_n + q_{n-1} = S_{k+2} + S_k \\ \vdots & = & \vdots \\ S_{k+a_{n+1}} & = & a_{n+1}q_n + q_{n-1} = S_{k+a_{n+1}-1} + S_k. \end{array}$$

So the difference between any pair of subsequent cutting times is again a cutting time (whence the kneading map can be defined), and it is not hard to check the admissibility condition (6). This proves the following proposition:

Proposition 4. *For every $\rho \in (0,1)\backslash\mathbb{Q}$, there exists a unimodal map f with $Q(k) \to \infty$ such that $(\omega(c), f)$ factorizes continuously over (*i.e., *is semi-conjugate to) the circle rotation $(\mathbb{S}^1, R_\rho)$.*

Definition 1. *The dynamical system (X, T) is* topologically weakly mixing *if there is no continuous function $\varphi : X \to \mathbb{C}$ and $\lambda \neq 1$ such that $\varphi \circ T = \lambda \cdot \varphi$.* [6]

The examples in Proposition 4 are *not* topologically weakly mixing because if $\varphi = \exp(2\pi i \cdot \pi_\rho)$, then

$$\varphi \circ \pi_\rho = \exp(2\pi i \pi_\rho \circ f) = \exp(2\pi i R_\rho \circ \pi_\rho) = \exp(2\pi i(\pi_\rho + \rho)) = e^{2\pi i \rho} \cdot \varphi$$

2.4. **Maps onto higher dimensional tori $\mathbb{T}^N$.** One question to ask regarding Theorem 3 is whether one sequence of cutting times can satisfy (9) for more than one ρ. If this were the case, say for $\rho_1, \dots \rho_N$, then

$$\begin{array}{rcl} \pi_{\vec{\rho}} & : & \omega(c) \to \mathbb{T}^N \\ & & x \mapsto (\pi_{\rho_1}(x), \dots \pi_{\rho_N}(x)) \end{array}$$

is a continuous map such that $\pi_{\vec{\rho}} \circ f = R_{\vec{\rho}} \circ \pi_{\vec{\rho}}$ for the rotation $R_{\vec{\rho}} : \mathbb{T}^N \to \mathbb{T}^N$ over vector $\vec{\rho} = (\rho_1, \dots \rho_N)$. This is interesting if $1, \rho_1, \dots, \rho_N$ are rationally independent; otherwise $\mathbb{T}^N$ decomposes into $R_{\vec{\rho}}$ –invariant subtori.

In the Fibonacci case, we had $\gamma S_k = S_{k+1} + \mathcal{O}(\hat{\gamma}^{k+1})$, so $\|\rho S_k\|$ is exponentially small in k. Similarly $\gamma^2 S_k = S_{k+2} + \mathcal{O}(\hat{\gamma}^{k+2})$. However, γ is a solution of $\gamma^2 = \gamma + 1$, so $R_{\gamma^2} = R_\gamma$, and no new phenomena are obtained.

[6] If no such measurable φ (w.r.t. the σ-algebra of measurable sets $\mathcal{B}$ on X) exists, then $(X, \mathcal{B}, T)$ is called *weakly mixing*.

For different recursive relations we have more success. Take $Q_d(k) = \max\{0, k-d\}$, then the corresponding cutting times satisfy the recursive relation

$$S_k = S_{k-1} + S_{k-d}, \tag{10}$$

with characteristic equation

$$x^d = x^{d-1} + 1. \tag{11}$$

Let $\rho_1, \dots \rho_d$ be the roots of this equation, where $\rho_1 > 1$ is the leading root. Then there are $C_i \in \mathbb{R}$, $i = 1, \dots, d$, such that $S_k = \sum_{i=1}^d C_i \rho_i^k$. It follows that

$$\rho_1^m S_k = \sum_{i=1}^d C_i \rho_i^{k+m} + \sum_{i=2}^d C_i \rho_i^k (\rho_1^m - \rho_i^m) = S_{k+m} + \sum_{i=2}^d C_i \rho_i^k (\rho_1^m - \rho_i^m),$$

and hence

$$\|\rho_1^m S_k\| \leq \sum_{i=2}^d |C_i \rho_i^k (\rho_1^m - \rho_i^m)|.$$

Now if $|\rho_i| < 1$ for each $i \geq 2$, then the right hand side of this inequality is exponentially small in k, and hence $\|\rho_1^m S_k\|$ is summable over k.

Definition 2. *A number $\rho > 1$ is a* Pisot-Vijayaraghavan *number if it is algebraic and all its algebraic conjugates have absolute value < 1.*

It is known that the leading root ρ_1 of (11) is Pisot-Vijayaraghavan for $d = 2, 3$ and 4, but no longer for $d \geq 5$. This gives an insight why the following theorem holds.

Theorem 4. *Let f_d be a unimodal map with kneading map $Q_d(k) = \max\{0, k-d\}$.*

- *For $d = 2, 3$ and 4, the factor map $\pi_{\vec{\rho}} : \omega(c) \to \mathbb{T}^{d-1}$ is a metric isomorphism onto the $d-1$–dimensional torus.*
- *For $d \geq 5$, $(\omega(c), f_d)$ is weakly mixing.*

Proof. For $d = 2$, this is just the result from Lyubich & Milnor [20]. The rest of this theorem is discussed in [7], where the full argument for $d = 3$ and $d = 4$ rely on work in [23, 11]. $\square$

Example: If $d = 3$, then the characteristic equation $x^3 = x^2 + 1$ has leading root $\rho_1 > 1$ and two complex conjugate roots of modulus < 1. Let $\rho_2 = \rho_1^2$. Then $\pi_{\vec{\rho}} : E \to \mathbb{T}^2$ is defined as $\pi_{\vec{\rho}} = (\sum_i e_i \|\rho_1^i S_i\| \pmod 1, \sum_i e_i \|\rho_2^i S_i\| \pmod 1)$. Leaving out the (mod 1), we obtain a map from $\pi_{\vec{\rho}} : E \to \mathbb{R}^2$, the image of which is shown in Figure 2 (left). Some features of this map:

- ◇ The different shades correspond to the images of different cylinder sets $E^k := \{e \in E \ : \ e_k = 1\}$ for $k = 0, 1, 2$. They are similar to each other, and to the whole set $\pi_{\vec{\rho}}(E)$. In fact, $\pi_{\vec{\rho}}(E)$ is very much like the Rauzy fractal, which is based on the characteristic equation $x^3 = x^2 + x + 1$.

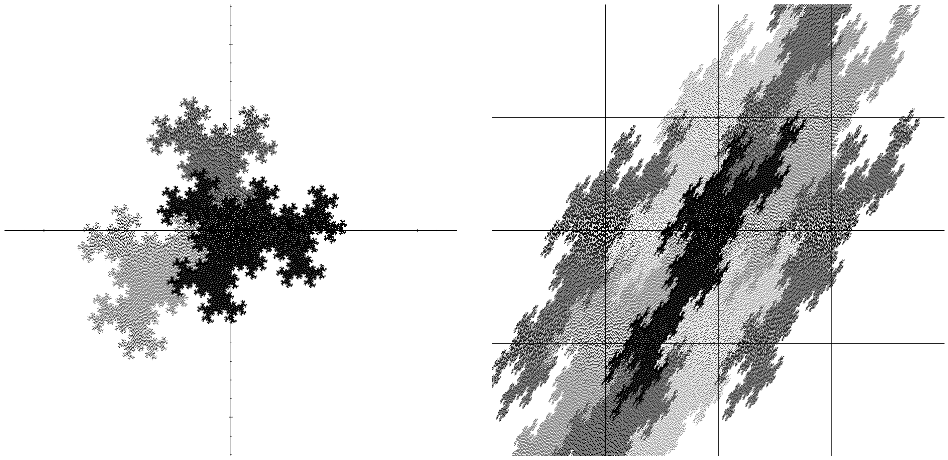

FIGURE 2. The image $\pi_{\rho_1} \times \pi_{\rho_2}(\omega)$ (left) resembles the Rauzy-fractal, and tiles the plane (right).

- ◇ The sets $\pi_{\vec{\rho}}(E)$ are simply connected, *i.e.,* have no holes. By estimating the Lebesgue measure of $\pi_{\vec{\rho}}(E^k)$, we find that $\sum_{k=0}^{2} \mathrm{Leb}(\pi_{\vec{\rho}}(E^k)) = 1 = \mathrm{Area}(\mathbb{T}^2)$. This implies that the images of the cylinder sets have overlap of Lebesgue measure 0.
- ◇ A corollary of this is that you can tile the plane with copies of $\pi_{\vec{\rho}}(E)$, see Figure 2 (right).
- ◇ Analogous properties hold for $d = 4$, but the proof that $\pi_{\vec{\rho}}(E)$ is doubly connected and that copies of $\pi_{\vec{\rho}}(E)$ tile $\mathbb{R}^3$ is much more involved, see [7].

More results can be proved by these methods (cf. [7]):

- For each dimension N, there is a recursive relation (that can be mimicked by cutting times) such that its characteristic equation has a Pisot-Vijayaraghavan number ρ as leading root. Therefore, for each $N \in \mathbb{N}$, there is a transitive rotation $R_{\vec{\rho}} : \mathbb{T}^N \to \mathbb{T}^N$ for $\vec{\rho} = (\rho, \rho^2, \dots, \rho^N)$ that is semi-conjugate to $(\omega(c), f)$ for some unimodal map with $Q(k) \to \infty$.
- We don't know if there is any example of a unimodal map with $Q(k) \to \infty$ such that $(\omega(c), f)$ factorizes over an infinite dimensional torus.
- If f is infinitely renormalizable, and $(p_i)_{i\in\mathbb{N}}$ is the sequence of periods of the periodic intervals, then (9) holds for each $1/p_i$. Therefore Theorem 3 gives another proof that the (p_i)-adic adding machine is a factor of $(\omega(c), f)$ for this infinitely renormalizable unimodal map.
- There is a unimodal map with cutting times $(S_k)_{k\in\mathbb{N}}$ such that (9) holds for 2^{-i} for each $i \in \mathbb{N}$ and also for $\gamma = \frac{1+\sqrt{5}}{2}$. In this case $(\omega(c), f)$ factorizes over a rotation on a solenoid. (A solenoid can be obtained as quotient space $C \times [0,1]/\sim$ where C is a Cantor set and $(x, 0) \sim (h(x), 1)$ for some minimal homeomorphism $h : C \to C$.)

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