QUASI-SYMMETRY OF CONJUGACIES BETWEEN INTERVAL MAPS.

Henk Bruin

Abstract. A homeomorphism $h : [0,1] \to [0,1]$ is quasi-symmetric if there exist $K \geq 1$ such that for every $x \in [0,1]$ and $\varepsilon > 0$, $\frac{1}{K} \leq \frac{|h(x+\varepsilon)-h(x)|}{|h(x)-h(x-\varepsilon)|} \leq K$. In this paper we demonstrate a topological condition that prohibits a tent-map to be quasi-symmetrically conjugate to any C^2 unimodal map. This topological condition is so weak, that almost every tent-map satisfies it. We show in a similar way that typically a C^2 degree 2 circle map with a critical point cannot be quasi-symmetrically conjugate to the angle doubling map.

We discuss another topological condition (persistent recurrence of the critical point), which is almost complementary to the first. We show that a C^2 unimodal map f with a persistently recurrent critical point does not satisfy the Collet-Eckmann condition and (if f is non-flat as well), is not quasi-symmetrically conjugate to a tent-map.

1 Introduction

Unimodal maps are often topologically conjugate to tent-maps. So in a topological sense, such a unimodal map is the same as a tent-map. In a metric sense however, the difference is clear. Still some metric similarities occur, when the conjugacy satisfies certain constraints. In this paper we will consider quasi-symmetry. A homeomorphism $h : I \to I$ on the interval $I = [0,1]$ is *quasi-symmetric* if there exists $K \geq 1$ such that for all $x \in I$ and all $\varepsilon > 0$, $\frac{1}{K} \leq \frac{|h(x+\varepsilon)-h(x)|}{|h(x)-h(x-\varepsilon)|} \leq K$. Quasi-symmetry turns out to be a strong property for conjugacies between unimodal maps.

The notions of quasi-symmetry gained importance since Sullivan [Su] proved the following result: Let $f_a(x) = 1-ax^2$. If f_a and f_b are quasi-symmetrically conjugate and do not have a periodic attractor, then $a = b$. Sullivan used the rigid structure of quasi-conformal complex mappings. Note that if $f : \mathbb{C} \to \mathbb{C}$ is quasi-conformal and preserves the real line, then $f : \mathbb{R} \to \mathbb{R}$ is quasi-symmetric.

In this paper we concentrate on the conjugacy between C^2 unimodal maps (see below) and tent-maps. We will prove

---

1991 *Mathematics Subject Classification.* 58F03, 58H20.

Supported by the Netherlands Organization for Scientific Research (NWO)

Typeset by $\mathcal{AMS}$-TEX

Theorem 1. *Let T_a be the tent-map with slope $\pm a$. There exists a set $A \subset [1, 2]$ of full Lebesgue measure such that for every $a \in A$ the following property holds: If f is a C^2 unimodal map whose critical point c has order $\ell > 1$, then f is not quasi-symmetrically conjugate to T_a.*

Since the topological entropy $h_{top}(T_a) = \log a$, we might as well say: For a.e. $h \in [0, \log 2]$, no C^2 unimodal map f with $h_{top}(f) = h$ is quasi-symmetrically conjugate to $T_{\exp(h)}$.

According to a result of Nowicki and Przytycki [NP], every non-renormalizable S-unimodal Collet-Eckmann map is Hölder conjugate to a tent-map. A map f is called *Collet-Eckmann*, if $\liminf \frac{1}{n}|Df^n(c_1)| > 0$. ($c_1 = f(c)$ is the critical value.) Sands [Sa] proved that Collet-Eckmann maps are abundant in the topological sense: For a.e. $h \in [0, \log 2]$ it holds that if f is S-unimodal and conjugate to $T_{\exp(h)}$, then f is a Collet-Eckmann map. So quasi-symmetry is a much stronger property than Hölder continuity.

The proof of Theorem 1 is based on a topological condition, stated in Proposition 1. In short the condition reads: There is a sequence n_i such that $f^{n_i}(c) \to c$, while the image of the central branch of f^{n_i} does not cover c and a has length bounded away from 0. If a C^2 map f satisfies it, then f cannot be quasi-symmetrically conjugate to a tent-map. In section 4 it is shown that a.e. tent-map satisfies this condition.

If the lengths of the images of the central branches tend to 0, Proposition 1 is indecisive. In section 5 we present a topological condition that deals with this case: *persistent recurrence* of the critical point. The notion of persistent recurrence originally comes from complex dynamics: Yoccoz' τ-function tends to infinity. We will use an equivalent definition for the interval, which has been used by e.g. Blokh and Lyubich [BL]. An example of a map with a persistently recurrent critical point is the Fibonacci map. Fibonacci maps are characterized by a certain combinatorial pattern. They have been studied frequently in the past few years, especially in connection with so-called absorbing Cantor sets [BKNS,HK,KN,LM]. We will show

Theorem 2. *If f is C^2 and has a persistently recurrent critical point, then f does not satisfy the Collet-Eckmann condition.*

and

Theorem 3. *If f is non-flat, C^2 and has a persistently recurrent critical point, then f is not quasi-symmetrically conjugate to a tent-map.*

We remark that Sands [Sa] proved very similar results in the S-unimodal case.

In spite of all this, there exist C^2 unimodal maps that are quasi-symmetrically conjugate to tent-maps. *Misiurewicz maps*, i.e. maps with a non-recurrent critical point and no periodic attractor, have this property. For the proof see [J,St]. But apart from Misiurewicz maps there are other maps that don't have a persistently recurrent critical point, and to which Proposition 1 does not apply.

Question. *Are there C^2 non-Misiurewicz maps that are quasi-symmetrically conjugate to tent-maps?*

The technique presented in Proposition 1 can be used in a more general setting. We are convinced that statements similar to Theorem 1 can be proved for piecewise monotone maps on the interval or the circle. We have worked out some of the details for a certain class of circle maps:

Theorem 4. *Let $T : S^1 \to S^1$ be angle doubling map $x \mapsto 2x \mod 1$. There exists a set $X \subset S^1$ of full Lebesgue measure such that for every $x \in X$ the following holds: Let g be a C^2 degree 2 circle map conjugate to T $(h \circ T = g \circ h)$. If $h(x)$ is a critical point of g, then h is not quasi-symmetric.*

Quasi-symmetry of degree $d \geq 2$ circle maps can play a role in the construction of complex mappings whose Julia sets contain quasi-circles. A *quasi-circle* is the image of the unit circle under a quasi-conformal map. Ma proved, using van Strien's results on Misiurewicz maps, that monotone degree d maps with non-recurrent critical points are quasi-symmetrically conjugate to the linear degree d circle map [Ma]. So the above question can also be posed for these kind of circle maps.

We want to thank Sebastian van Strien and Gerhard Keller for many discussions and also Duncan Sands. The proof of Lemma 2 is based on his remarks. We also thank the referee for his suggestions.

2 Preliminaries

Let $\mathbb{N} = \{0, 1, 2, ...\}$ and let (x, y) be the interval with end-points x and y, also if $y < x$. If $J \subset I$, then $|J|$ denotes the Lebesgue measure of J.

The mapping $f : I \to I$, $I = [0, 1]$ is *unimodal* if f has a unique critical point c, i.e. a unique point where Df vanishes or does not exist. Assume that $f(c)$ is a maximum and that $f(0) = f(1) = 0$. f^n is the n-fold composition of f. The forward images of the critical point will be denoted as $c_n = f^n(c)$. The critical point c has order ℓ if there exists a diffeomorphism h such that $h(0) = 0$ and $f(x) = f(c) - |h(x - c)|^\ell$. So if f is C^2, ℓ is at least 2. f is called *non-flat* if $\ell < \infty$. A corollary of this is that there exist $0 < O_1 < O_2 < \infty$ such that

$$O_1 |x - c|^{\ell - 1} \leq |Df(x)| \leq O_2 |x - c|^{\ell - 1} \tag{1}$$

for all x. If x is close to c, then O_2/O_1 can be taken close to 1.

Let us turn to the Koebe Principle. This principle gives bounds for the distortion on branches of f^n. If $f^n|J$ is C^1, then the *distortion* is defined as

$$dis(f^n, J) = \sup_{x,y \in J} \frac{|Df^n(x)|}{|Df^n(y)|}.$$

Let $J \subset T$ be intervals, and let $f^n|T$ be monotone. $f^n(T)$ contains a *δ-scaled neighbourhood* of $f^n(J)$ if both components of $f^n(T \setminus J)$ are longer than $\delta |f^n(J)|$. Usually the Koebe Principle is stated for maps with negative Schwarzian derivative. There exists however a C^2 version.

Koebe Principle. *Suppose f is C^2 and $J \subset T$ are intervals such that $f^n|T$ is monotone. If $f^n(T)$ contains a δ-scaled neighbourhood of $f^n(J)$, then*

$$dis(f^n, J) \leq B = B(\delta, \varepsilon, L).$$

The numbers ε, L and B are defined as follows: $\varepsilon = \max\{|f^i(T)| \mid 0 \leq i \leq n\}$. $L = \sum_{i=0}^n |f^i(J)|$ and $B(\delta, \varepsilon, L) = (\frac{\delta+1}{\delta})^2(1 + \rho(\varepsilon)L)$. Here $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $\rho(0) = 0$. ρ only depends on the smoothness of f.

In this paper we will use a one-sided version of the Koebe Principle:

One-sided Koebe Principle. *Let f, $J \subset T = (x, y)$, ε, L and B be as in the Koebe Principle. If $f^n((x, y))$ contains a one-sided δ-scaled interval of $f^n(J)$ at the side of $f^n(x)$, then $|Df^n(z)| \geq \frac{1}{B}|Df^n(y)|$ for every $z \in J$.*

The proofs of these Koebe Principles can be found in [MMS,MvS]. If f has negative Schwarzian derivative, then $B = (\frac{\delta+1}{\delta})^2$ already suffices. In the C^2 case the number $\varepsilon = \max\{|f^i(T)| \mid 0 \leq i \leq n\}$ is important. The next lemma shows that ε can be estimated from $|f^n(T)|$. A non-degenerate interval $J \subset I$ is called a *homterval* if $f^n|J$ is a homeomorphism for every $n \geq 0$.

Lemma 1. *If f admits no homterval, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f^n(J)| > \delta$ whenever $|J| > \varepsilon$ and $f^n|J$ is monotone.*

For the proof we refer to [BL] or [MvS, Contraction Principle]. Clearly the existence of a homterval prohibits the existence of a conjugacy with any tent-map. So in the sequel we can assume that the conclusion of Lemma 1 is true. (In fact, no non-flat C^2 unimodal map without periodic attractor admits a homterval [MMS,MvS].) For our purpose it suffices to observe that $\varepsilon \to 0$ as $|f^n(T)| \to 0$.

The *tent-map* $T_a : I \to I$ with slope a is defined as $T_a(x) = \min(ax, a(1-x))$. It is well-known that every unimodal map f is semi-conjugate to a tent-map with the same topological entropy [MT]. An interval J, $J \ni c$, is called *restrictive* if there exists some $n > 1$ such that $f^n(J) \subset J$ and $f^j(J) \not\subset J$ for $0 < j < n$. In this case $f^n|J$ is again a unimodal map, called a *renormalization* of f. In many cases the above semi-conjugacy is a conjugacy: If f admits no restrictive interval, and no homterval, then f is conjugate to a tent-map T_a, with slope $a > \sqrt{2}$.

If f is conjugate to a tent-map T_a with $a \in (1, \sqrt{2}]$, then both f and T_a are finitely renormalizable. In that case we can consider the deepest renormalizations of f and T_a.

As we said before, a homeomorphism h is quasi-symmetric if there exists $K > 1$ such that for all $x \in I$ and all $\varepsilon > 0$, $\frac{1}{K} \leq \frac{|h(x+\varepsilon)-h(x)|}{|h(x)-h(x-\varepsilon)|} \leq K$. Let us give some corollaries of quasi-symmetry:

Lemma 2. *If h is quasi-symmetric, then h is Hölder continuous.*

Proof. Let $h : I \to I$ be quasi-symmetric with constant K. Let $\alpha = \frac{\log \frac{1+K}{K}}{\log 2}$ and $C = 2^{1+\alpha}$. We will show that

$$|h(x) - h(y)| \leq C|x - y|^\alpha.$$

Assume that $x < y$ and $\frac{1}{2} \leq y$. If $y < \frac{1}{2}$, then we consider $x \mapsto 1 - h(1-x)$ instead of $x \mapsto h(x)$. Define $w_i \in I$ as follows: $w_0 = 0$, $w_1 = \frac{y}{2}$, and in general $w_{n+1} = \frac{w_n+y}{2}$. Then there exists N such that $w_N < x \leq w_{N+1} < y$. Clearly $2^{-(N+1)} < \frac{|y-x|}{|y-0|}$. By quasi-symmetry, $|h(w_i) - h(y)| \leq \frac{K}{1+K}|h(w_{i-1}) - h(y)|$ for all

$i \geq 1$. So

$$\begin{aligned}
|h(x) - h(y)| &\leq |h(w_N) - h(y)| \\
&\leq \frac{K}{1+K}|h(w_{N-1}) - h(y)| \leq (\frac{K}{1+K})^N |h(0) - h(y)| \\
&\leq 2(\frac{K}{1+K})^{N+1} \leq 2\left(2^{\frac{\log \frac{K}{1+K}}{\log 2}}\right)^{N+1} \leq 2\left(2^{-(N+1)}\right)^{\frac{\log \frac{1+K}{K}}{\log 2}} \\
&\leq 2\left(\frac{|y-x|}{|y-0|}\right)^{\frac{\log \frac{1+K}{K}}{\log 2}} \leq C|y-x|^\alpha.
\end{aligned}$$

This concludes the proof. We remark that in the more general case $h : I_1 \to I_2$, for some intervals I_1 and I_2, the constant C also depends on the ratio $|I_2|/|I_1|$. □

Lemma 3. *If h is quasi-symmetric, then for every $A > 0$ there exists $B > 0$ such that if J_1 and J_2 are adjacent intervals, and $\frac{|J_1|}{|J_2|} \leq A$, then $\frac{|h(J_1)|}{|h(J_2)|} \leq B$.*

Proof. Similar to the proof of Lemma 2. □

For an interval $J \subset I$, let $\tau(J) = \max\{i \mid f^i|J \text{ is monotone}\}$.

Lemma 4. *Assume that f is quasi-symmetrically conjugate to the tent-map T_a with slope $a > \sqrt{2}$. Then there exists M such that for every interval $J \subset I$,*

$$L(J) = \sum_{i=0}^{\tau(J)} |f^i(J)| \leq M.$$

Proof. Let h be the conjugacy ($h \circ T_a = f \circ h$), and h has quasi-symmetry constant K. Let α and $C = 2^{1+\alpha}$ be as in Lemma 2. In particular h^{-1} is Hölder with constants α and C. Take M such that $\frac{1}{m^2} > C2^{\alpha(1-m/2)}$ for every $m \geq M-2$. If there exists J such that $L(J) > M$, then first of all $\tau(J) > M$. Also there exists i such that $\tau(J) - i = m \geq M-2$ and $|f^i(J)| > \frac{1}{m^2}$. Indeed, if this is not the case, then

$$L(J) < M - 2 + \sum_{i=0}^{\tau(J)-M-2} \frac{1}{(\tau(J)-i)^2} < M - 2 + \frac{\pi^2}{6} < M.$$

Because T_a expands distances with a factor $a > \sqrt{2}$ it follows that $h(f^i(J)) < 2^{-m/2}$.

Assume for simplicity that $c = h(c) = \frac{1}{2}$ and that $f^i(J) = (x, y) \subset [\frac{1}{2}, 1]$. Then also $h((x, y)) \in [\frac{1}{2}, 1]$. By the choice of M it follows that

$$|x - y| \geq \frac{1}{m^2} > C2^{\alpha(1-m/2)} \geq C|h(x) - h(y)|^\alpha.$$

So h^{-1} cannot be Hölder, contradicting the assumptions. □

Let us continue with a few combinatorial notions. Let $f : I \to I$ be a unimodal map. For each $x \neq c$, the *symmetric point* is defined as the point $\hat{x} \neq x$ such that $f(\hat{x}) = f(x)$. A point $x \in I$ is called a *closest precritical point* if $f^n(x) = c$ for some

$n \geq 1$ and $f^j((x,c)) \not\ni c$ for $0 < j < n$. Clearly x and $\hat{x}$ are closest precritical points simultaneously. If f admits no periodic attractor or wandering interval, then the closest precritical points accumulate on c. Let $\{z_k\}_{k \geq 0}$ denote the closest precritical points left of c. So $f^{-1}(c) = \{z_0, \hat{z}_0\}$, and

$$z_0 < z_1 < z_2 < ... < c < ... < \hat{z}_2 < \hat{z}_1 < \hat{z}_0.$$

Let S_k be such that $f^{S_k}(z_k) = f^{S_k}(\hat{z}_k) = c$. The iterates $\{S_k\}_k$ are called the *cutting times*. Let $U_i \ni c$ be the maximal interval such that $f^{i-1}|f(U_i)$ is monotone. It is not hard to see that if $S_{k-1} < i \leq S_k$, then $U_i = (z_{k-1}, \hat{z}_{k-1})$. Hence $f^{S_{k-1}}(U_{S_k}) = f^{S_{k-1}}((z_{k-1}, \hat{z}_{k-1})) = (c, c_{S_{k-1}}]$. By definition of closest precritical point, $f^{S_{k-1}}(z_k) \in (c, c_{S_{k-1}})$ is again a closest precritical point, which we will denote as $z_{Q(k)}$ or $\hat{z}_{Q(k)}$. It follows that

$$S_{Q(k)} = S_k - S_{k-1}. \tag{2}$$

The map Q is called the kneading map. The kneading map was introduced by Hofbauer, e.g. [H]. It determines the combinatorics of a unimodal map completely. It can be proved that if f has no periodic attractor, then $Q(k) < k$ for all $k \geq 1$. Additionally, an integer map $Q : \mathbb{N} \to \mathbb{N}$ is an *admissible* kneading map, i.e. Q appears as the kneading map of some unimodal map, if and only if

$$\{Q(k+j)\}_{j \geq 1} \succeq \{Q(Q \circ Q(k) + j)\}_{j \geq 1}$$

for all k. $\succeq$ denotes lexicographical order. For the proof of these statements, and for more details on Q we refer to [H,B].

Finally, notice that $f^m(U_n) \ni c$ if and only if $n = m$ is a cutting time. More precisely,

$$f^{S_k}(U_{S_k}) = [c_{S_k}, f^{S_k}(z_{k-1})) = [c_{S_k}, c_{S_k - S_{k-1}}) = [c_{S_k}, c_{S_{Q(k)}}). \tag{3}$$

3 The main topological condition

The proof of Theorem 1 relies on the existence of certain close returns of the critical point to itself. In the smooth case, the effect of these returns is that $dis(f^{n-1}, f(U_n))$ becomes arbitrarily large for appropriate iterates n. More precisely, we will define points $y_k \in (c, z_k)$ such that the quantity

$$R_f(k) = \frac{|z_k - y_k|}{|z_k - c|}$$

becomes arbitrarily small for appropriate values of k. However, for the conjugate tent-map T, $R_T(k)$ is bounded away from 0 for the same values of k. By Lemma 3, this prohibits the conjugacy to be quasi-symmetric.

Proposition 1. *Let T be a tent-map, and f a C^2 unimodal map conjugate to T. If there exists $\beta > 0$ and a sequence $\{n_i\}_i$ such that*
i) $f^{n_i}(c) \to c$ as $i \to \infty$,
ii) $f^{n_i}(U_{n_i}) \not\ni c$, and
iii) $|f^{n_i}(U_{n_i})| \geq \beta$,

then the conjugacy between f and T is not quasi-symmetric.

After reading the proof, the reader may conclude that condition iii) can be weakened. $|f^{n_i}(U_{n_i})|$ need not be bounded away from 0; it should not tend to 0 too fast. Condition i) is essential; it excludes Misiurewicz maps.

Proof of Proposition 1. Let us first assume by contradiction that f and T are quasi-symmetrically conjugate with constant K. Without loss of generality, we can assume that the slope of T is larger than $\sqrt{2}$. By Lemma 4, there exists M such that $M \geq L(J)$ for every $J \subset I$.

As f is C^2, the order ℓ of the critical point is larger than 1. So by (1), there exists $1 < \ell' < \ell$ and a neighbourhood $V \ni c$ with the property: For every $x \in V$ and every $y \in (c, x)$ such that $f(y) \in (f(x), \frac{f(c)+f(x)}{2})$,

$$\frac{|x-y|}{|x-c|} \leq \frac{1}{\ell'} \frac{|f(x)-f(y)|}{|f(x)-f(c)|}. \tag{4}$$

Let $B > 1$ and $\zeta > 1$ be so small that

$$\frac{\zeta B}{\ell'} \leq r < 1.$$

Let $V'' \subset V'$ be a small neighbourhoods of c to be specified shortly. Let ε be such that if T is an interval such that $f^n(T) \subset V'$ and $f^n|T$ is monotone, then $\max\{|f^i(J)| \mid 0 \leq i \leq n\} \leq \varepsilon$. By Lemma 1 it follows that $\varepsilon \to 0$ as $|V'| \to 0$. Let $\delta = \frac{|V'|}{2|V''|}$. Recall that $(\frac{\delta+1}{\delta})^2(1+\rho(\varepsilon)M)$ is the distortion bound corresponding to Koebe-space δ in the (One-sided) Koebe Principle. So let V' and V'' be so small that $(\frac{\delta+1}{\delta})^2(1+\rho(\varepsilon)M) < B$.

We inductively construct a subsequence $\{m_i\}_i \subset \{n_j\}_j$ and another integer sequence $\{k_i\}_i$ as follows: Let $m_1 = n_1$ and let k_1 be so large that $z_{k_1} \in V''$. Let y_{k_1} be such that $R_f(k_1) = \frac{|z_{k_1}-y_{k_1}|}{|z_{k_1}-c|} \leq \frac{1}{3\ell'}$.

Now for $i > 1$, assume by induction that

$$R_f(k_i) = \frac{|z_{k_i}-y_{k_i}|}{|z_{k_i}-c|} \leq \frac{1}{3\ell'}. \tag{5}$$

Suppose that also m_i and k_i are known. Choose m_{i+1} ($m_i < m_{i+1} = n_j$ for some j) so large that $c_{m_{i+1}} \in (z_{k_i}, \hat{z}_{k_i})$ and

$$\frac{|z_{k_i}-c|}{|z_{k_i}-c_{m_{i+1}}|} \leq \zeta,$$

if $c_{m_{i+1}} \in (z_{k_i}, c)$ and

$$\frac{|\hat{z}_{k_i}-c|}{|\hat{z}_{k_i}-c_{m_{i+1}}|} \leq \zeta,$$

if $c_{m_{i+1}} \in (\hat{z}_{k_i}, c)$. See figure 1.

We pull back the point z_{k_i} by the branch $f^{m_{i+1}-1}|f(U_{m_{i+1}})$, obtaining a point x' close to $f(c)$. We claim that x' is the image of a closest precritical point $x < c$. Because z_{k_i} is a closest precritical point, x is also a precritical point. So $f^a(x) = c$

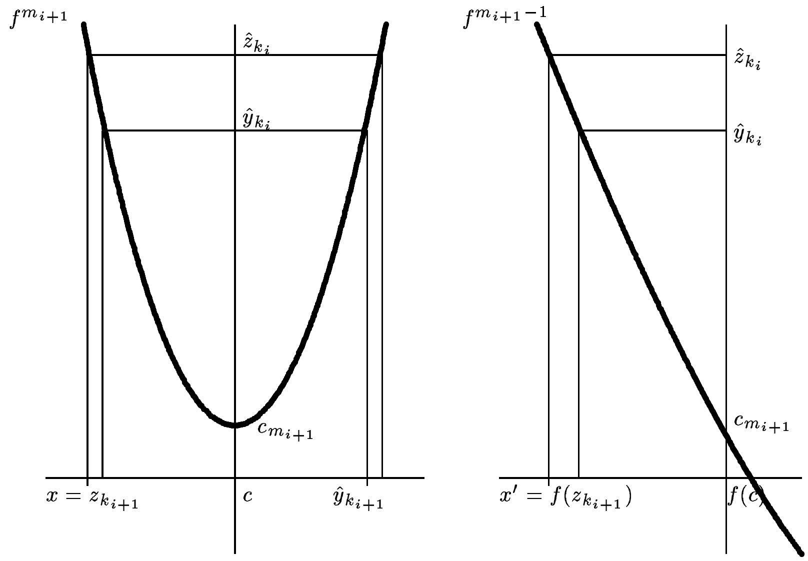

Figure 1

for $a = m_{i+1} + S_{k_i}$. If $f^b((x,c)) \ni c$ for some $b < a$, then $f^{m_{i+1}}((x,c)) \subset (z_{k_i}, c)$ contains a point in $f^{-S_{k_i}+a-b}(c)$, contradicting that z_{k_i} is a closest precritical point. Hence x is a closest precritical point.

Let k_{i+1} be such that $z_{k_{i+1}} = x$. Then $f^{m_{i+1}}(z_{k_{i+1}}) = z_{k_i}$. Now let $y_{k_{i+1}} \in (z_{k_{i+1}}, c)$ be such that

$$f^{m_{i+1}}(y_{k_{i+1}}) = y_{k_i}.$$

Now let us check that we can use the One-sided Koebe Principle. Because $m_{i+1} \in \{n_j\}_j$, the branch $f^{m_{i+1}-1}|f(U_{m_{i+1}})$ is long: $|f^{m_{i+1}}(U_{m_{i+1}})| \geq \beta > |V'|$. Let $U'_{m_{i+1}} \subset (0,c) \cap U_{m_{i+1}}$ be the maximal interval such that $f^{m_{i+1}}(U'_{m_{i+1}}) \subset V'$. By definition of ε and V', $\max\{|f^i(U'_{m_{i+1}})| \mid 0 \leq i \leq m_{i+1}\} \leq \varepsilon$. Also $L(U'_{m_{i+1}}) \leq M$. Because $z_{k_i} \in V''$, the branch $f^{m_{i+1}-1}|f(U'_{m_{i+1}})$ has relative Koebe-space of length $\geq \frac{|V'|}{2|V''|} = \delta$ at the side of z_{k_i}. So the One-sided Koebe Principle can be applied. We obtain:

$$\frac{|f(z_{k_{i+1}}) - f(y_{k_{i+1}})|}{|f(z_{k_{i+1}}) - f(c)|} \leq B \frac{|z_{k_i} - y_{k_i}|}{|z_{k_i} - c_{m_{i+1}}|} \leq \zeta B \frac{|z_{k_i} - y_{k_i}|}{|z_{k_i} - c|}.$$

It follows

$$\begin{aligned} R_f(k_{i+1}) &= \frac{|z_{k_{i+1}} - y_{k_{i+1}}|}{|z_{k_{i+1}} - c|} \\ &\leq \frac{1}{\ell'} \frac{|f(z_{k_{i+1}}) - f(y_{k_{i+1}})|}{|f(z_{k_{i+1}}) - f(c)|} \\ &\leq \frac{\zeta B}{\ell'} \frac{|z_{k_i} - y_{k_i}|}{|z_{k_i} - c|} \\ &\leq r \frac{|z_{k_i} - y_{k_i}|}{|z_{k_i} - c|} = r R_f(k_i). \end{aligned}$$

For the first inequality we have used (4) and (5). Because $r < 1$, (5) holds for k_{i+1}. So we can continue the induction. It follows that $R_f(k_i) \to 0$ as $i \to \infty$.

On the other hand, f is the conjugate tent-map T. Let h be the conjugacy. $T^{m_{i+1}}$ maps $(h(z_{k_{i+1}}), h(c))$ monotonically into $(h(z_{k_i}), h(c))$ or $(h(c), h(\hat{z}_{k_i}))$. Let $M_i = m_2 + m_3 + ... + m_i$, then T^{M_i} maps $(h(z_{k_i}), h(c))$ monotonically into $(h(z_{k_1}), h(c))$ or into $(h(\hat{z}_{k_1}), h(c))$. Also $T^{M_i}(h(y_{k_{i+1}})) = h(y_{k_1})$ or $h(\hat{y}_{k_1})$.

As $T^{M_i}|h(U_{M_i})$ is linear, $R_T(k_i) = \frac{|h(z_{k_i})-h(y_{k_i})|}{|h(z_{k_i})-h(c)|} \geq \frac{|h(z_{k_1})-h(y_{k_1})|}{|h(z_{k_1})-h(c)|} = R_T(k_1) > 0$ for all i. So by Lemma 3, h cannot be quasi-symmetric. □

4 The proof of Theorem 1

Let T_a be the tent-map with slope $\pm a$ and $\varphi_n(a) = T_a^n(c)$. For each $a \in [\sqrt{2}, 2]$ we denote the subsequent cutting times of T_a by $\{S_k(a)\}_k$. The same applies to $c_n(a)$ and $z_n(a)$.

Proposition 2. *Let $\varepsilon > 0$ be arbitrary and let*

$$Y(a, \varepsilon) = (z_3(a), z_3(a) + \varepsilon) \cup (\hat{z}_3(a) - \varepsilon, \hat{z}_3(a)).$$

Then for a.e. $a \in [\sqrt{2}, 2]$, $\varphi_{S_k(a)}(a) \in Y(a, \varepsilon)$ for some $k \in \mathbb{N}$.

Before proving this proposition, we need some preliminary results.

Lemma 5. *Let $a > \sqrt{2}$ be arbitrary. Suppose that $n = S_k(a)$ is a cutting time. Then there exists an interval $J \ni a$ such that $\varphi_n|J$ is monotone and $\varphi_n(J) \ni c$.*

Sketch of proof. a is a turning point of φ_n if and only if $\varphi_m(a) = c$ for some $m < n$. Hence the domain J of a branch of φ_n is a maximal interval J such that $\varphi_m(J) \not\ni c$ for every $m < n$. Let J be the branch domain of φ_n containing a.

Define the *itinerary* $\nu_a(x) = e_1(x)e_2(x)e_3(x)...$, where

$$e_i(x) = \begin{cases} 0 & \text{if } T_a^i(x) \in [0, c), \\ 1 & \text{if } T_a^i(x) \in (c, 1], \\ C & \text{if } T_a^i(x) = c. \end{cases}$$

The *kneading invariant* $\nu_a = e_1e_2e_3... = \nu_a(c)$ is the itinerary of the critical point. As $n = S_k$ is a cutting time, i.e. $T_a^n(U_n) \ni c(a)$, there are points in U_n whose itinerary coincides with ν_a up to the S_k-th entry. In particular, the itinerary of c_{-S_k} is $e_1e_2...e_{S_k-1}Ce_1e_2...$. Therefore, there exists a tent-map $T_{a'}$ whose kneading invariant is periodic: $\nu_{a'} = e_1e_2...e_{S_k-1}Ce_1e_2...e_{S_k-1}C....$ As $\nu_{a'}$ coincides with ν up to entry $S_k = n$, $a' \in J$. It is also clear that $\varphi_n(a') = c$. This concludes the sketch. A more comprehensive treatment of the functions φ_n can be found in [Sa] and [BM]. □

Lemma 6. *For a.e. $a \in [\sqrt{2}, 2]$, the critical orbit of the tent-map T_a is dense in the dynamical core $[c_2, c_1]$.*

Proof. See [BM]. □

Lemma 7. *Let T be a tent-map with cutting times S_k and kneading map Q. If $\liminf Q(k) \geq 2$, then $orb(c)$ is not dense in $[c_2, c_1]$.*

Proof. Assume there exists k_0 such that $Q(k) \geq 2$ for all $k \geq k_0$. Let p be the orientation reversing fixed point of T. We will show that p is not approximated by the forward orbit of c.

Let V be a neighbourhood of p such that $c_i \notin V$ for $0 \leq i \leq S_{k_0}$. Let $T^v(V) \ni c$ for some minimal integer v, then c_{-v} is a precritical point closest to p. Since p is a repelling fixed point, $c_{-v-1} \in T^{-v-1}(c)$, $c_{-v-2} \in T^{-v-2}(c)$,... can be chosen to be precritical points closest to p. If we assume that $p < c_{-v}$, then

$$(*) \qquad c_{-v-1} < c_{-v-3} < ... < p < ... < c_{-v-2} < c_{-v}.$$

Assume by contradiction that $p \in \omega(c)$, then there exists n minimal such that $c_n \in (c_{-v-3}, c_{-v-2})$. Suppose $T^n(U_n)$ covers two adjacent precritical points c_{-w} and c_{-w-2} of $(*)$, where w is taken minimal. Then $T^{w+n}(U_n)$ contains a component of $(z_1, \hat{z}_1) \setminus \{c\}$. So $n+w$ is a cutting time, say S_l, and as $c_{n+w} \notin (z_1, \hat{z}_1)$, $Q(l+1) \leq 1$.

Hence $\{c_n\}$ can only approximate p stepwise: $T^n(U_n)$ can only contain one preimage from $(*)$ at the time. Assume, without loss of generality, that $c_n \in (c_{-v-4}, c_{-v-2})$ and $c_{-v-2} \in T^n(U_n) = [c_n, c_m) \subset (c_{-v-4}, c_{-v})$. By minimality of n, $T^m(U_m) \subset (c_{-v-2}, c_{-v})$. $T^{v+2}|T^n(U_n)$ is monotone and $T^{v+2+n}(U_n) \ni c$, so $n+v+2$ is a cutting time S_k and $m+v+2 = S_{Q(k)}$. On the other hand $T^{v+2+m}|U_m$ is monotone and $T^{v+2+m}(U_m) \not\ni c$. So $m+v+2$ is not a cutting time. This contradiction shows that $orb(c) \cap (c_{-v-3}, c_{-v-2}) = \emptyset$. $\square$

Finally, we need some estimates on the branches of $\varphi_n(a)$

Lemma 8. *For every $\varepsilon > 0$ there exist N such that for every $n \geq N$, and every $U \subset [\sqrt{2}, 2]$ on which $\varphi_n|U$ is monotone, $dis(\varphi_n, U) \leq 1 + \varepsilon$.*

Proof. See [BM]. $\square$

Now we are ready to prove Proposition 2.

Proof of Proposition 2. Choose $\varepsilon \in (0,1)$ arbitrary. By Lemmas 6 and 7 it follows that for a.e. $a \in [\sqrt{2}, 2]$, $\liminf Q_a(k) \leq 1$. Let

$$A_\varepsilon = \{a \in [\sqrt{2}, 2] \mid \liminf Q_a(k) \leq 1 \text{ and } T_a^{S_k(a)}(c) \notin Y(a, \varepsilon) \text{ for all } k\}.$$

Let N be so large that $dis(\varphi_n, J) \leq 1 + \varepsilon/2$ for every $n \geq N$ and every interval J of monotonicity of φ_n. By Lemma 8, this is possible.

Suppose by contradiction that $|A_\varepsilon| > 0$. Then there exists a density point a of A_ε and an integer k such that

i) $S_k(a) \geq N$,

ii) $Q_a(k+1) \leq 1$, so $T_a^{S_k(a)}(c) \notin (z_3(a), \hat{z}_3(a))$,

iii) there is a one-sided neighbourhood $J \ni a$, such that $\varphi_{S_k(a)}(J) = (c, T_a^{S_k(a)}(c))$ and

iv) $\frac{|J \cap A_\varepsilon|}{|J|} \geq 1 - \varepsilon/4$.

Property iii) follows from Lemma 5. In particular, $S_k(a) = S_k$ is constant on J. By definition, $\varphi_n(a) \notin Y(a,\varepsilon)$ for every $a \in A_\varepsilon$. Due to the boundedness of distortion,

$$\begin{aligned} 1 - \frac{\varepsilon}{4} \le \frac{|J \cap A_\varepsilon|}{|J|} &\le (1 + \frac{\varepsilon}{2}) \frac{|\varphi_{S_k}(J \cap A_\varepsilon)|}{|\varphi_{S_k}(J)|} \\ &\le (1+\frac{\varepsilon}{2}) \frac{|(c, \varphi_{S_k}(a)) \setminus Y(a,\varepsilon)|}{|(c, \varphi_{S_k}(a))|} \\ &\le (1+\frac{\varepsilon}{2})(1-\varepsilon) < (1-\frac{\varepsilon}{2}). \end{aligned}$$

This contradicts the existence of the density point a. So $|A_\varepsilon| = 0$. $\square$

Proof of Theorem 1. Suppose $a \in [\sqrt{2}, 2]$. By Proposition 2, A_ε has zero measure for every $\varepsilon > 0$. So $A = \{a \in [\sqrt{2}, 2] \mid \liminf Q_a(k) \le 1\} \setminus \bigcup_i A_{\frac{1}{i}}$ has full measure in $[\sqrt{2}, 2]$. Pick $a \in A$, then there exists a sequence k_i such that $T_a^{S_{k_i}(a)}(c) \in Y(a, \frac{1}{i})$ for every i. As $T_a^{S_3(a)}(Y(a,\frac{1}{i})) = (c, c + \frac{a^{S_3(a)}}{i})$ or $(c - \frac{a^{S_3(a)}}{i}, c)$, and $S_3(a) \le 8$

$$T_a^{S_{k_i}(a)+S_3(a)}(c) \in (c - \frac{2^8}{i}, c) \text{ or } (c, c+\frac{2^8}{i}). \tag{6}$$

So

$$\begin{aligned} T_a^{S_{k_i}(a)+S_3(a)}(U_{S_{k_i}(a)+S_3(a)}) &= T_a^{S_3(a)}(c, T_a^{S_{k_i}(a)}(c)) \\ &= (T_a^{S_3(a)}(c), T_a^{S_{k_i}(a)+S_3(a)}(c)) \not\ni c. \end{aligned} \tag{7}$$

As (6) and (7) are true for every i, the conditions of Proposition 1 are fulfilled. This proves Theorem 1 for non-renormalizable maps. If $a \le \sqrt{2}$, i.e. T_a is renormalizable, we pass to the smallest restrictive interval. $\square$

5 Maps with persistently recurrent critical points

As before, let f be C^2 unimodal. Proposition 1 does not cover maps for which the height of the central branches tends to 0. Another, more or less complementary, topological condition ensures that in that case f cannot fulfill the Collet-Eckmann condition, and, if additionally f is non-flat, f is not quasi-symmetrically conjugate to a tent-map.

Let us be more precise: Let $H_n(x) \ni x$ be the maximal interval on which f^n is monotone, and let $M_n(x) = f^n(H_n(x))$. So $H_n(x)$ is the domain of the branch f^n at x, and $M_n(x)$ is the co-domain. Notice that $\partial M_n(x) \subset orb(c) \cup \{0, 1\}$. Let

$$r_n(x) = d(f^n(x), \partial M_n(x)).$$

We call c *persistently recurrent* if

$$r_n(c_1) \to 0 \text{ as } n \to \infty.$$

We will prove slightly different versions of Theorems 2 and 3.

Theorem 2'. *Let f be a C^2 unimodal map with a persistently recurrent critical point. Then*
$$\limsup_n \frac{1}{n} \log |Df^n(c_1)| \leq 0.$$

Theorem 3'. *Let f be a non-flat C^2 unimodal map, such that*
$$\limsup_n \frac{1}{n} \log |Df^n(c_1)| \leq 0.$$
Then f is not quasi-symmetrically conjugate to a tent-map.

Sands [Sa, Theorem 49] has found similar, and probably stronger topological conditions that prohibit the Collet-Eckmann condition in the S-unimodal case. He also proved [Sa, Theorem 31] that S-unimodal maps not satisfying the Collet-Eckmann condition cannot be quasi-symmetrically conjugate to tent-maps. So Theorem 3' is an improvement on the part of the smoothness condition. But the condition $\limsup_n \frac{1}{n} \log |Df^n(c_1)| \leq 0$ is stronger than the negation of the Collet-Eckmann condition.

Persistent recurrence of the critical point is quite a restrictive condition. We will see in Lemma 11 that the $\omega(c)$ (i.e. the set of accumulation points of the orbit of c,) must be minimal. On the other hand, every map for which the kneading map Q tends to infinity has a persistently recurrent critical point [B]. In particular, the Fibonacci unimodal map cannot be Collet-Eckmann. Fibonacci maps are unimodal maps with kneading map $Q(k) = \max(k-2, 0)$. They have been studied by many authors in the past few years, e.g. [BKNS,HK,KN,LM]. So non-flat C^2 Fibonacci maps are not quasi-symmetrically conjugate to the Fibonacci tent-map.

The proof of Theorem 2' heavily depends on a result by Ledrappier. What we need is, in our notation:

Lemma 9. *Let μ be an ergodic non-atomic invariant measure, and suppose that $\int \log |Df| d\mu > 0$. Then for μ-a.e. $x \in I$, there exists a neighbourhood $U \ni x$ and sequences $\{n_i\} \subset \mathbb{N}$ and $\{y_i\} \subset supp(\mu)$ such that for every i, $f^{n_i}(y_i) = x$ and $M_{n_i}(y_i) \supset U$.*

Proof. Follows directly from [L, Proposition 7 and Theorem 8]. $\square$

Define the *nice* points as $\mathcal{N} = \{x \in I \mid f^i(x) \notin (x, \hat{x}) \text{ for all } i \geq 0\}$. Nice points are used to estimate distortion of branches of f^n. For instance, Martens uses them in [Mr]. We list a couple of properties of nice points, using Martens' notation.

Clearly $\mathcal{N}$ is closed. Every periodic orbit contains a nice point, and because the periodic points are dense under our assumptions, c is an accumulation point of $\mathcal{N}$. We call V a *nice* interval if it is symmetric and its boundary points are nice. There are arbitrarily small nice intervals.

Let c_r be the first return of c to a nice interval V. Then $\psi(V)$ is defined to be the maximal neighbourhood of c such that $f^r(\psi(V)) \subset V$. Because f is not renormalizable, $\psi(V) \subsetneq V$, and $orb(\partial\psi(V)) \cap V = \emptyset$.

Lemma 10. *Let V be a nice interval, and $U = \psi(V)$. Let $x \in [c_2, c_1] \setminus U$ and let r be the smallest integer such that $f^r(x) \in U$ (assuming it exists). Then*
$$M_r(x) \supset V.$$

Furthermore, if $x \in V \setminus U$, then pull-back of V along the orbit $x, f(x), ..., f^r(x)$ is contained in $V \setminus U$.

Proof. Can be found in [Mr], but it is short enough to give it here. Let $(y,z) = H_r(x)$. By definition of $H_r(x)$, there exists $a, b < r$ such that $c = f^a(y) = f^b(z)$. So $f^a((y,x))$ and $f^b((x,z))$ both contain a point in ∂U. It follows that $f^r((y,x))$ and $f^r((x,z))$ contain points in $orb(\partial U)$, and therefore $f^r((y,z)) = M_r(x) \supset V$.

Now for the second statement, assume J is the pull-back. If $J \cap \partial V \neq \emptyset$, then $V = f^r(J)$ contains a point from $orb(\partial V)$ in its interior. Hence ∂V is not nice. This same argument works for ∂U. $\square$

Lemma 11. *If $r_n(c_1) \to 0$, then $r_n(x) \to 0$ uniformly on $\omega(c)$.*

Proof. We will only carry out the proof for maps having no homtervals, allowing us to use Lemma 1. (Proofs for the maps with homtervals are somewhat tedious, but not substantially more difficult.) The Lemma is clear if f is infinitely renormalizable. By passing to the smallest restrictive interval, we can assume that f is not renormalizable at all.

We may also assume that $c \in \omega(x)$ for every $x \in \omega(c)$. Indeed, suppose $x \in \omega(c)$ and $c \notin \omega(x)$. Let $\varepsilon = d(c, orb(x)) > 0$. Then by Lemma 1 we can find δ such that $r_n(x) \geq \delta$ for all n. Next take n arbitrary, and let m be minimal such that $c_m \in H_n(x)$. Then $M_{m-1}(c_1) \supset H_n(x)$, and $M_{m+n-1}(c_1) \supset M_n(x)$. So $|M_{m+n-1}(c_1)| \geq |H_n(x)| \geq \delta$, contradicting the assumption $r_i(c_1) \to 0$.

In particular we can conclude that $\omega(c)$ is minimal and $c \in \omega(c)$.

Assume by contradiction that there exists $\{x_n\}_n \subset \omega(c)$ with the property that $\limsup r_n(x_n) \geq \varepsilon > 0$. Let $\delta > 0$ be as in Lemma 1. Let V be a nice interval such that $|V| \leq \delta$, and let $U = \psi(V)$. Let δ' be the length of the smallest component of $V \setminus U$.

Take n such that $r_n(x_n) \geq \frac{\varepsilon}{2}$. Let $m_1(n) = \min\{k \geq 0 \mid f^{n+k}(x_n) \in V\}$. Because $r_n(x_n) \geq \frac{\varepsilon}{2}$, i.e. $M_n(x_n) \geq \varepsilon$, it follows from Lemmas 1 and 10 that there exists an interval J_1 such that $f^n(x_n) \in J_1 \subset M_n(x_n)$ and $f^{m_1(n)}$ maps J_1 monotonically onto V. Next let $J_2 \subset H_n(x_n)$ be the pull-back of J_1 along the orbit $x_n, f(x_n), ..., f^n(x_n)$. So $f^{n+m_1(n)}$ maps J_2 monotonically onto V.

Let $m_2(n) = \min\{k \mid c_{k+1} \in J_2\}$. $x_n \in \omega(c)$, so $m_2(n)$ exists. Because $\partial M_m(c_1) \subset orb(c) \cup \{0,1\}$, $M_{m_2(n)}(c_1) \supset J_2$. Let J_3 be the pull-back of J_2 along the orbit $c_1, c_2, ..., c_{1+m_2(n)}$. Then $f^{n+m_1(n)+m_2(n)}$ maps J_3 monotonically onto V.

Finally, take $m_3(n) = \min\{k \geq 0 \mid c_{1+n+m_1(n)+m_2(n)+k} \in U\}$, and abbreviate $N(n) = n + m_1(n) + m_2(n) + m_3(n)$. By Lemma 10, there exists J_4 such that $c_{1+n+m_1(n)+m_2(n)} \in J_4 \subset V\}$ and $f^{m_3(n)}$ maps J monotonically onto V. Pulling back J_4 along the orbit $c_1, c_2, ..., c_{N(n)+1}$, we obtain an interval $J_5 \ni c_1$ which is mapped monotonically onto V by $f^{N(n)}$. As $c_{N(n)+1} \in U$, $r_{N(n)}(c_1) \geq \delta'$. Since this is true for arbitrarily large numbers n, $r_k(c_1) \not\to 0$. This proves Lemma 11. $\square$

Proof of Theorem 2'. Let f have a persistently recurrent critical point. Suppose by contradiction that $\limsup_n \frac{1}{n} \log |Df^n(c_1)| \geq \varepsilon > 0$. Then we can find a subsequence $\{n_i\}_i$ such that both $\lim_i \frac{1}{n_i} \log |Df^{n_i}(c_1)| = \varepsilon$ and $\frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{c_{j+1}}$ tends weakly to a measure μ. Here δ_x is a Dirac-measure in x. Clearly μ is invariant and lives on $\omega(c)$. As $\omega(c)$ is minimal, μ is non-atomic as well.

Let $\varphi = \log |Df|$ and for $L > 0$, let $\varphi_L = \max(\varphi, -L)$. Then φ_L is continuous, and by definition of weak convergence,

$$\int \varphi_L d\mu = \lim_i \frac{1}{n_i} \sum_{j=0}^{n_i-1} \varphi_L(c_{j+1}) \geq \lim_i \frac{1}{n_i} \sum_{j=0}^{n_i-1} \log |Df(c_{j+1})|$$
$$= \lim_i \frac{1}{n_i} \log \prod_{j=0}^{n_i-1} |Df(c_{j+1})| = \lim_i \frac{1}{n_i} \log |Df^n(c_1)| = \varepsilon,$$

for every L. As $\varphi_L \searrow \varphi$ as $L \to \infty$, also $\int \varphi d\mu \geq \varepsilon$. Using the ergodic decomposition, we can also find an ergodic measure ν, with $\int \varphi d\nu \geq \varepsilon$. Indeed, let $\mathcal{I}$ be the algebra $\{A \mid f^{-1}(A) = A\}$ mod $\mu-$nullsets. And let $\mathcal{A}$ be the set of atoms of $\mathcal{I}$. Then for a non-empty $A \in \mathcal{A}$, $\mu_A = \frac{1}{\mu(A)} \mu|A$ is a ergodic invariant probability measure. Because

$$\int \varphi d\mu = \sum_{\emptyset \neq A \in \mathcal{A}} \mu(A) \int \varphi d\mu_A,$$

$\int \varphi d\mu_A \geq \varepsilon$ for at least one $A \in \mathcal{A}$.

Hence we can apply Lemma 9 on μ_A. Take $x \in \omega(c)$, $U \ni x$ and $y_i \in f^{-n_i}(x) \cap supp(\mu_A) \subset \omega(c)$ as in Lemma 9. Then $r_{n_i}(y_i) \geq d(x, \partial U) > 0$ for every i, contradicting Lemma 11. So $\limsup \frac{1}{n} |Df^n(c_1)| \leq 0$. $\square$

In order to prove Theorem 3', we need the following lemma.

Lemma 12. *Let f be a non-flat C^2 unimodal map, and suppose f is quasi-symmetrically conjugate to a tent-map. Then there exists $\delta > 0$ and a sequence of nice intervals $\{V_n\}_n$ with $|V_n| \to 0$, such that V_n contains a δ-scaled neighbourhood of $\psi(V_n)$.*

Proof. If f is S-unimodal, then Lemma 12 is just Theorem 3.4 of [Mr]. In that theorem, negative Schwarzian derivative is only used for the Koebe Principle. Due to our assumptions, we can use the Koebe Principle stated in section 2. In Martens' proof, the Koebe Principle is only applied to pull-backs of the intervals $V_n \supset \psi(V_n)$. In our version, the distortion bound is $B = B(\delta, \varepsilon, L)$. For large n, V_n is small, and therefore the ε can be taken small too. By Lemma 4, L is uniformly bounded. $\square$

Proof of Theorem 3'. Let f be a non-flat C^2 unimodal map with the property that $\limsup_n \frac{1}{n} \log |Df^n(c_1)| \leq 0$. In particular, f can not be a Misiurewicz map, because all C^2 Misiurewicz maps are Collet-Eckmann [St]. If the critical point is periodic, or attracted to a periodic orbit, then f is not even conjugate to a tent-map. So we can assume that c is recurrent, but not periodic. As before, we can assume that f is not renormalizable.

Suppose by contradiction that f is quasi-symmetrically conjugate to the tent-map T_a, for some $a > \sqrt{2}$. So we can apply Lemmas 4 and 12. By Lemma 2, the conjugacy h ($f \circ h = h \circ T_a$,) is also Hölder. Let $C, \alpha > 0$ be the Hölder constants.

Let $V_n \supset \psi(V_n) = U_n$ be nice intervals as in Lemma 12. Let W_n'' be a $\frac{\delta}{2}$-scaled neighbourhood of U_n. Then V_n contains an $\frac{\delta}{4}$-neighbourhood of W_n''. We will use the Koebe Principle for these intervals, yielding a distortion bound $B(\frac{\delta}{4}, \varepsilon, M)$. M is taken as in Lemma 4, and $\varepsilon \to 0$ as $n \to 0$. Therefore, there exists B such that $B \geq B(\frac{\delta}{4}, \varepsilon, M)$ for every $n \in \mathbb{N}$.

Let t_n be the smallest positive integer such that $f^{t_n}(c) \in U_n$. So by Lemma 10, there exists an interval $J \ni c_1$ such that $f^{t_n - 1}$ maps J monotonically onto

W_n''. Let $W_n' = J \cap (c, c_1)$, then $f^{t_n - 1}(W_n')$ contains a component of $W_n'' \setminus U_n$, so $|f^{t_n-1}(W_n')| \geq \frac{\delta}{2}\frac{1}{1+\delta}|W_n''|$. Let also $W_n = f^{-1}(W_n')$. Clearly $W_n \subsetneq W_n''$, because otherwise $f^{t_n}(W_n) \subset W_n$, and f would be renormalizable. Using non-flatness and the Koebe Principle, we obtain

$$\begin{aligned} |W_n| < |W_n''| &\leq \frac{2(1+\delta)}{\delta}|f^{t_n-1}(W_n')| \leq 2B\frac{1+\delta}{\delta}|Df^{t_n-1}(c_1)|\,|W_n'| \\ &\leq 4B\frac{1+\delta}{\delta}|Df^{t_n-1}(c_1)|\,|W_n|^{\ell}. \end{aligned}$$

Hence

$$|W_n| > \left\{\frac{1}{4B\frac{1+\delta}{\delta}|Df^{t_n-1}(c_1)|}\right\}^{\frac{1}{\ell-1}}.$$

On the other hand, $f^{t_n-1}|f(W_n)$ and therefore $T_a^{t_n-1}|T\circ h^{-1}(W_n)$ are monotone. So

$$|h^{-1}(W_n)| \leq \frac{2}{a^{t_n}}.$$

As h is Hölder, $|W_n| \leq C|h^{-1}(W_n)|^{\alpha}$, which results in

$$\left\{\frac{1}{4B\frac{1+\delta}{\delta}|Df^{t_n-1}(c_1)|}\right\}^{\ell-1} \leq C\left(\frac{2}{a^{t_n}}\right)^{\alpha},$$

whence

$$\frac{1}{t_n}\log|Df^{t_n-1}(c_1)| + \frac{\log C + \alpha\log 2}{t_n(\ell-1)} + \frac{\log 4B\frac{1+\delta}{\delta}}{t_n} \geq \frac{\alpha}{\ell-1}\log a.$$

Taking the limit $n \to \infty$, we get $\limsup_t \frac{1}{t}\log|Df^t(c_1)| > 0$ after all. $\square$

6 Circle maps

In this section we prove a result similar to Theorem 1 in a class of increasing degree 2 circle maps. Generalizations to degree n circle maps are possible, but no more interesting than this class. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle and $T: S^1 \to S^1$ be the angle doubling map: $T(x) = 2x$. Fix $x \in S^1$, and let $g_x: S^1 \to S^1$ satisfy the following conditions:

i) g_x is an increasing degree 2 circle map, so g has at least one fixed point p.
ii) g_x is C^2.
iii) g_x has at least one critical point c.
iv) For every interval $J \subset S^1$, there exists $n > 0$ such that $g_x^n(J) \ni p$.

Condition iv) implies that p is the only fixed point. There are no periodic attractors, so in particular c is not periodic. Moreover, g_x is conjugate to T. Let h be the conjugacy, $h \circ T = g_x \circ h$.

v) The critical point $c = h(x)$.

Let us reformulate Theorem 4:

Theorem 4. *For a.e. $x \in S^1$ the following property holds: If g_x satisfies the above conditions, then h is not quasi-symmetric.*

The proof is completely analogous to the proof of Theorem 1. But we have to define the topological notions all over again. So let us start with this. For brevity, we drop the subscript x of g_x. Put an orientation on S^1 such that $p < c$.

- The left precritical points of c will be denoted as $\{z_k\}_k$. $z_{-1} = z_{-2} = c$, and $z_0 = g^{-1}(c) \cap (p, c)$.
- The right precritical points of c will be denoted as $\{\hat{z}_k\}_k$. $\hat{z}_{-1} = \hat{z}_{-2} = c$, and $\hat{z}_0 = g^{-1}(c) \cap (c, p)$.
- The left and right cutting times will be denoted as $\{S_i\}_i$ and $\{\hat{S}_i\}_i$. $S_{-1} = \hat{S}_{-1} = 0$ and $S_0 = \hat{S}_0 = 1$.
- $S_i = \min\{n \geq S_{i-1} \mid g^n(z_{i-1}, c) \ni c\}$ and $z_i \in (z_{i-1}, c)$ is the point closest to z_i such that $g^{S_i}(z_i) = c$.
- $\hat{S}_i = \min\{n \geq \hat{S}_{i-1} \mid g^n(c, \hat{z}_{i-1}) \ni c\}$ and $\hat{z}_i \in (c, \hat{z}_{i-1})$ is the point closest to $\hat{z}_i$ such that $g^{\hat{S}_i}(\hat{z}_i) = c$.

It can happen that $S_i = S_{i+1}$. In that case, $g^{S_i}((z_i, z_{i+1})) = (c, \hat{z}_{-1}) = S^1 \setminus \{c\}$. An analogous statement holds for $\hat{S}_i$.

- By definition of closest precritical point, $g^{S_{i-1}}(z_i) = \hat{z}_k$ for some $k \geq -1$. Hence it is possible to define the *left kneading map* $Q : \mathbb{N} \to \mathbb{N} \cup \{-1\}$ such that
$$S_i - S_{i-1} = \hat{S}_{Q(i)},$$
and
$$g^{S_{i-1}}(c) \in (\hat{z}_{Q(i)}, \hat{z}_{Q(i)-1}),$$
for all $i \geq 1$. If $Q(i) = 0$, this property degenerates to $g^{S_{i-1}}(c) \in S^1 \setminus \{c\}$. One must understand that in this case $g^{S_{n-1}}((z_{n-1}, c)) \supset S^1$
- For the right cutting times we have the analogous statements: $g^{\hat{S}_{i-1}}(\hat{z}_i) = z_k$ for some $k \geq -1$. The *right kneading map* $\hat{Q} : \mathbb{N} \to \mathbb{N} \cup \{-1\}$ satisfies
$$\hat{S}_i - \hat{S}_{i-1} = S_{\hat{Q}(i)},$$
and
$$g^{\hat{S}_{i-1}}(c) \in (z_{\hat{Q}(i)}, z_{\hat{Q}(i)-1}),$$
for all $i \geq 1$. If $\hat{Q}(i) = 0$, $g^{\hat{S}_{i-1}}(c) \in S^1 \setminus \{c\}$. As above this means that $g^{S_{n-1}}((c, \hat{z}_{n-1})) \supset S^1$

Let $U_n \ni c$ be the largest intervals such that $g^j(U) \not\ni c$ for all $0 < j < n$. Then $U_n = (z_k, \hat{z}_{k'})$, where $k = \max\{m \mid S_m < n\}$ and $k' = \max\{m \mid \hat{S}_m < n\}$. Consequently, $g^n(U_n) = (g^{n-S_k}(c), g^{n-\hat{S}_{k'}}(c))$.

As for kneading maps of unimodal maps, we have some admissibility constraints. First, if g has no periodic attractor, then $Q \circ \hat{Q}(i+1) \leq i$ and $\hat{Q} \circ Q(i+1) \leq i$ for all i. Additionally, a pair of integer maps Q and $\hat{Q}$ are admissible left and right kneading maps if and only if
$$\{Q(k+j)\}_{j \geq 1} \succeq \{Q(\hat{Q} \circ Q(k) + j)\}_{j \geq 1},$$
and
$$\{\hat{Q}(k+j)\}_{j \geq 1} \succeq \{\hat{Q}(Q \circ \hat{Q}(k) + j)\}_{j \geq 1},$$

for all k. Again $\succeq$ denotes the lexicographical order. Any pair of left and right kneading maps uniquely determines (the dynamics of) a point $x \in S^1$.

Remark: We did not define the kneading invariant for the circle map g_x. Much work in that direction was done by Bandt and Keller [BK1-2]. They noticed and exploited the fact that the kneading invariant does not uniquely determine the map. Therefore the left and right kneading map cannot be determined by the kneading invariant only: It is not visible from the kneading invariant if cutting times S_k with $Q(k) = -1$ (or $\hat{S}_k$ with $\hat{Q}(k) = -1$) occur.

We will prove none of the above statements, because they do not play a role in the proof of Theorem 4. However, we need the analogous version of Lemma 7.

Lemma 13. *If the orbit of c is accumulates on the fixed point of T, then*

$$\min(\liminf_{k\to\infty} Q(k), \liminf_{k\to\infty} \hat{Q}(k)) \leq 2.$$

Proof. We argue as in Lemma 7. Assume that c is not periodic, and not eventually fixed either. Let in this proof $<$ denote an ordering such that $\hat{z}_0 < p < z_0 < c$. Consider the precritical points closest to p:

$$\hat{z}_0 = c_{-1} < c_{-2} < c_{-3} < \dots < p < \dots < c_{-3} < c_{-2} < c_{-1} = z_0.$$

Assume that c accumulates on p. So take n such that $c_{-w} < c_n < c_{-w-1} < p$ and $c_m \notin (c_{-w}, p)$ for $m < n$. Then $g^n(U_n) \supset (c_{-w}, c_{-w-1})$, and $g^{n+w}(U_{n+w}) \supset (c, \hat{z}_0)$. Moreover $n + w = S_k$ is a left cutting time. If $c_{n+w} \in (\hat{z}_1, \hat{z}_0)$, then $Q(k+1) \leq 2$. If on the other hand $c_{n+w} \in (c, \hat{z}_1)$, then $n + w + 1 = \hat{S}_{k'}$ is a right cutting time, and $c_{n+w+1} \in (c_1, z_0)$. So $\hat{Q}(k'+1) \leq 2$. This proves the lemma, if $orb(c)$ accumulates on p from the left . A similar argument is valid if $orb(c)$ accumulates on p from the right. $\square$

In order to express the dependence on x, we will write $S_k(x)$, $z_k(x)$, $Q_x(k)$ etc.

Proposition 3. *Choose $\varepsilon > 0$ arbitrary. For a.e. x the following holds: Let*

$$Y(x, \varepsilon) = (z_3(x), z_3(x) + \varepsilon)$$

and

$$\hat{Y}(x, \varepsilon) = (\hat{z}_3(x) - \varepsilon, \hat{z}_3(x)).$$

Then there exists k such that $T^{S_k(x)}(x) \in Y(x, \varepsilon)$ or $T^{\hat{S}_k(x)}(x) \in \hat{Y}(x, \varepsilon)$.

Proof. The proof is similar to the proof of Proposition 2. Lemma 1 can be easily generalized to expanding circle maps. Because $x \mapsto T^n(x)$ is affine, we do not need versions of Lemmas 5 and 8 here. $\square$

Proof of Theorem 4. We will use Proposition 1 in an adjusted form. Let V_n be the maximal interval adjacent to and to the left of x such that $T^m(V_n) \not\ni x$ for $m < n$. Suppose that g_x satisfies the following properties: There exist $\beta > 0$ and a sequence of iterates $\{n_i\}_i$ such that

i) $g^{n_i}(c) \to c$ from the left as $i \to \infty$,

ii) $g^{n_i}(V_{n_i}) \not\ni c$, and

iii) $|g^{n_i}(V_{n_i})| \geq \beta$.
Then h_x is not quasi-symmetric. (The analogous statement is true if $g^{n_i} \to c$ from the right.) Let us show that for a.e. x, the combinatorics of g_x satisfy the above conditions.

The angle doubling map T preserves Lebesgue measure. So by Birkhoff's Ergodic Theorem, $p \in \omega(x)$ a.e. By Lemma 13, either $\liminf_{k\to\infty} Q_x(k) \leq 2$ or $\liminf_{k\to\infty} \hat{Q}_x(k)) \leq 2$. Let $Y(x,\varepsilon)$ and $\hat{Y}(x,\varepsilon)$ be as in Proposition 3. Let

$$X(\varepsilon) = \{x \in S^1 \mid T^{S_k}(x) \notin Y(x,\varepsilon) \text{ for every } k\}$$

and $X = S^1 \setminus \bigcup_n X(\frac{1}{n})$. Assume that X has full measure, otherwise we start a similar argument for sets $\hat{X}(\varepsilon)$ and $\hat{X}$. Pick $x \in X$, then there exists a sequence k_i such that $T^{S_{k_i}(x)}(x) \in Y(x, \frac{1}{i})$ for every i. Hence

$$(z_{\hat{Q}_x(4)}(x), x) \supset T^{S_{k_i}(x)+\hat{S}_3(x)}(V_{S_k(x)+\hat{S}_3(x)}) \supset (z_{\hat{Q}_x(4)}(x), x - \frac{1}{i}2^{\hat{S}_3(x)}).$$

Since this is true for every i, the conditions i) to iii) are satisfied. This proves the theorem. □

References

[B] H. Bruin, *Invariant measures of interval maps*, Thesis, Delft. (1994).

[BK1] C. Bandt, K. Keller, *Symbolic dynamics for angle-doubling on the circle, I. The topology of locally connected Julia sets*, in LNM. 1514 **1514** (1992), 1-23.

[BK2] C. Bandt, K. Keller, *Symbolic dynamics for angle-doubling on the circle, II. Symbolic description of the abstract Mandelbrot set*, Nonlinearity **6** (1993), 377-392.

[BKNS] H. Bruin, G. Keller, T. Nowicki, S. van Strien, *Absorbing Cantor sets in dynamical systems: Fibonacci maps*, Preprint StonyBrook **1994/2**.

[BL] A.M. Blokh, M.Yu. Lyubich, *Measurable dynamics of S-unimodal maps of the interval*, Ann. Scient. Éc. Norm. Sup. 4^e **série, 24** (1991), 545-573.

[BM] K. Brucks, M. Misiurewicz, *Trajectory of the turning point is dense for almost all tent maps*, Preprint Milwaukee (1993).

[H] F. Hofbauer, *The topological entropy of a transformation $x \mapsto ax(1-x)$*, Monath. Math. **90** (1980), 117-141.

[HK] F. Hofbauer, G. Keller, *Quadratic maps without asymptotic measure*, Commun. Math. Phys. **127** (1990), 319-337.

[J] Y. Jiang, *Geometry of geometrically finite one-dimensional maps*, Comm. Math. Phys. **156** (1993), 639-647.

[KN] G. Keller, T. Nowicki, *Fibonacci maps re(al)-visited*, Erg. Th. and Dyn. Sys. **15** (1992), 99-120.

[L] F. Ledrappier, *Some properties of absolutely continuous invariant measures on an interval*, Erg. Th. and Dyn. Sys. **1** (1981), 77-93.

[LM] M. Yu. Lyubich, J. Milnor, *The Fibonacci unimodal map*, Journ. A.M.S. **6** (1993), 425-457.

[Ma] L. Ma, Thesis, Cornell University (1992).

[Mr] M. Martens, *Distortion results and invariant Cantor sets of unimodal maps*, Erg. Th. and Dyn. Sys. **14** (1994), 331-349.

[MMS] M. Martens, W. de Melo, S. van Strien, *Julia-Fatou-Sullivan theory for real one-dimensional dynamics*, Acta Math. **168** (1992), 273-318.

[MT] J. Milnor, W. Thurston, *On iterated maps of the interval*, in Lect. Notes in Math. **1342** (1988), 465-563.

[MS] W. de Melo, S. van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematic und ihrer Grenzgebiete (1993), Springer Verlag, Berlin and New York.

[NP] T. Nowicki, F. Przytycki, *The conjugacy of Collet-Eckmann's map of the interval with the tent map is Hölder continuous*, Erg. Th. and Dyn. Sys. **9** (1989), 377-386.

[Sa] D. Sands, *Topological conditions for positive Lyapunov exponent in unimodal maps*, Thesis (1994).

[St] S. van Strien, *Hyperbolicity and invariant measures for general C^2 interval maps satisfying the Misiurewicz condition*, Commun. Math. Phys **128** (1990), 473-495.

[Su] D. Sullivan, *Quasiconformal homeomorphisms in dynamics, topology and geometry*, Proc. Int. Congress of Math. Berkeley (1986).

Mathematisches Institut University of Erlangen-Nürnberg Bismarckstrasse $1\frac{1}{2}$ D-91054 Erlangen Germany E-mail bruin@mi.uni-erlangen.de