Hausdorff dimension of biaccessible angles of quadratic Julia sets and of the Mandelbrot set

Henk Bruin (University of Surrey)

joint work with

Dierk Schleicher

(Jacobsuniversität Bremen)

Personae Dramatis:

- Quadratic polynomials $f_c(z) = z^2 + c$;
- Basin of ∞: $B_c(\infty) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$;
- Julia set $\mathcal{J}_c = \partial B_c(\infty)$;
- Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C} : \mathcal{J}_c \text{ is connected}\}$;
- Riemann map for dynamic plane $\varphi_c : B_c(\infty) \to \{z \in \mathbb{C} : |z| > 1\}$;
- $R_c(\vartheta) = \varphi_c^{-1}(\{re^{2\pi i\vartheta} : r > 1\})$ is dynamic ray of angle ϑ;
- Riemann map for parameter plane $\varphi : \mathbb{C} \setminus \mathcal{M} \to \{z \in \mathbb{C} : |z| > 1\}$;
- $R(\vartheta) = \varphi^{-1}(\{re^{2\pi i\vartheta} : r > 1\})$ is parameter ray of angle ϑ.

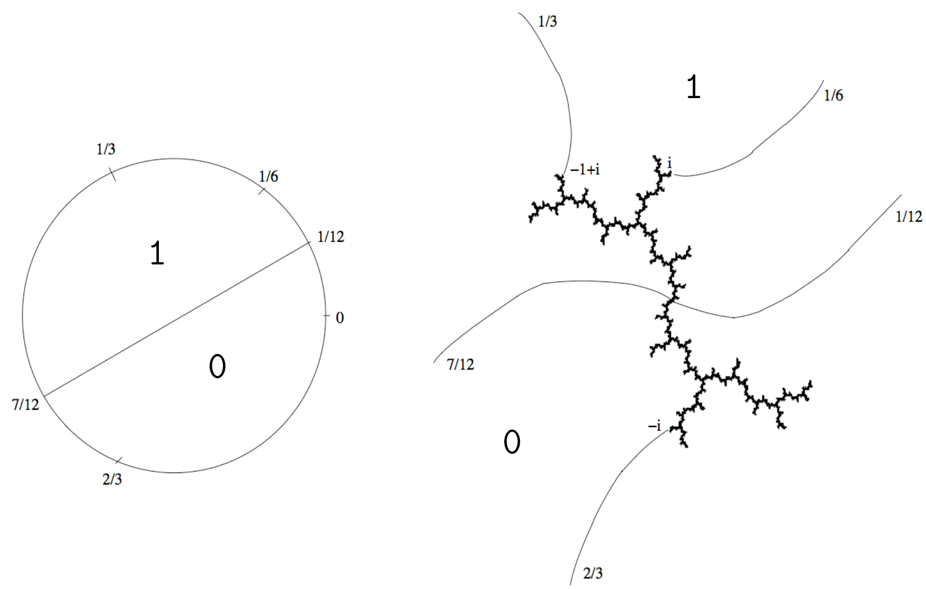

FIGURE 1. Left: the kneading sequence of an external angle ϑ (here $\vartheta = 1/6$) is defined as the itinerary of the orbit of ϑ under angle doubling, where the itinerary is taken with respect to the partition formed by the angles $\vartheta/2$, and $(\vartheta + 1)/2$. Right: in the dynamics of a polynomial for which the ϑ-ray lands at the critical value, an analogous partition is formed by the dynamic rays at angles $\vartheta/2$ and $(\vartheta + 1)/2$, which land together at the critical point.

Dendritic model for Julia sets

For every $c \in \mathcal{M}$, there is a model (J, f) that is combinatorially representing what happens on the true Julia set.

- J is a **dendrite**: compact, one-dimensional connected, locally connected space without loops.
- $f : J \to J$ is continuous and unicritical: there is a unique critical point 0 such that f is locally homeomorphic away from 0.
- Notation for critical orbit: $c_i = f^i(0)$.
- 0 separates J into two components $J_1 \ni c_1$ and J_0, and $f : \overline{J_e} \to J$ is onto for $e \in \{0, 1\}$.
- $f : J \to J$ is transitive (hence no superfluous arms).
- A point $x \in J$ has **valency** n if $J \setminus \{x\}$ has n components.

$$\begin{array}{lcl} & 1 & \text{end point} \\ \text{biaccessible} \left\{ \begin{array}{l} \\ \\ \end{array} \right. & 2 & \\ & \geq 3 & \text{branch point} \end{array}$$

Symbolic dynamics:

- Using the partition $J = J_0 \cup J_1 \cup \{0\}$ we can define symbolic dynamics. The **itinerary** of $z \in J$ is
$$e(z) = e_0 e_1 e_2 \dots \text{ where } e_i = \begin{cases} \mathtt{0} & \text{if } f^i(z) \in \mathcal{J}_0; \\ \mathtt{1} & \text{if } f^i(z) \in \mathcal{J}_1; \\ \star & \text{if } f^i(z) = 0; \end{cases}$$

- The **kneading sequence** $\nu = \nu_1 \nu_2 \nu_3 \dots$ is the itinerary of the critical point (neglecting the initial $\star$).

- The ρ-function is defined as
$$\begin{aligned} \rho &= \rho_\nu : \mathbb{N} \to \mathbb{N} \cup \{\infty\}, \\ \rho_\nu(n) &= \inf\{k > n : \nu_k \neq \nu_{k-n}\}. \end{aligned}$$

- The **internal address**
$$1 = S_0 \to S_1 \to S_2 \to \dots$$
is the ρ-orbit of 1.

- More generally, for fixed itinerary x and kneading sequence ν, define also
$$\rho_{\nu,x}(n) := \min\{k > n \;:\; x_k \neq \nu_{k-n}\} .$$

Biaccessibility:

- A point is **biaccessible** if at least two external rays land on it.
- An angle ϑ is **biaccessible** if there is another angle ϑ' so that their external rays land at the same point.
- This definition holds both for **dynamic** and **parameter** space.
- Modulo local connectivity,

  $c \in \mathcal{M}$ is biaccessible if and only if $c \in \mathcal{J}_c$ is biaccessible.

Theorem 1. *Let ν be the kneading sequence of a Julia set J and $z \in J$ have itinerary $e(z) = x$. Then the valency of J at z is equal to the number of grand $\rho_{\nu,x}$-orbits.*

Modulo local connectivity: the valency of c in $\mathcal{M}$ is equal to the number of grand ρ-orbits.

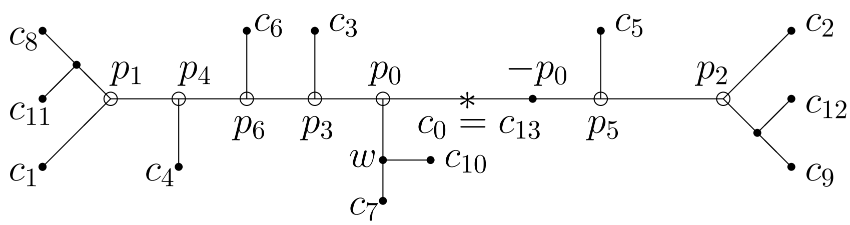

FIGURE 2. The Hubbard tree for $1 \to 2 \to 4 \to 5 \to 7 \to 13$ with $\text{orb}(p_0)$ drawn in and pre-periodic branch point $w \in [p_0, c_7]$; it maps to p_1 in three iterates.

Hubbard trees:

The **Hubbard tree** (T, f) of a Julia set (or more easily, its dendritic model) is the connected hull of the critical orbit.

- $f(T) = T$, but $f : T \to T$ is not globally 2-to-1 (except for $z \mapsto z^2 - 2$).

- c_1 is always an endpoint of T; so 0 has at most two branches in T.

- Every biaccessible point in J eventually maps into T.

Admissibility:

For every $\nu = \mathtt{1}\nu_2\nu_3 \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}_0}$, there is a dendritic model J and Hubbard tree T such that its kneading sequence is ν.

However, not very such model belongs to a true Julia set of a quadratic polynomial. This is not is much about local connectivity, but rather about the existence of **evil branch points**, that is: m-periodic branch points so that f^m does fix one of its arms, and permutes the others.

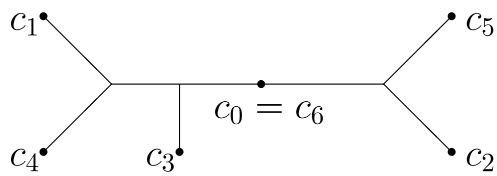

FIGURE 3. The Hubbard tree for $1 \to 2 \to 4 \to 5 \to 6$ contains an evil orbit of period 3.

The Admissibility Condition
A kneading sequence $\nu \in \{\mathtt{0},\mathtt{1}\}^{\mathbb{N}_0}$ *fails the admissibility condition (for period m)* if the following three conditions hold:

(1) the internal address of ν does not contain m;
(2) if $k < m$ divides m, then $\rho(k) \leq m$;
(3) $\rho(m) < \infty$ and if $r \in \{1, \dots, m\}$ is congruent to $\rho(m)$ modulo m, then $\operatorname{orb}_\rho(r)$ contains m.

Theorem 2. *The non-admissible kneading sequences are dense. The admissible kneading sequences have positive $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure.*

Corollary 1. *Restricting to admissible kneading sequences has no "real effect" on Hausdorff dimension of biaccessible parameter angles.*

Hausdorff estimate "near the antenna of $\mathcal{M}$"

Let

$$N := 1 + \min\{i > 1 \ : \ \nu_i = \mathtt{1}\}$$

$$L_1(N) := \begin{cases} \frac{\log\left(2^{\lfloor N/2 \rfloor - 1} - 1\right)}{\log 2^{\lfloor N/2 \rfloor - 1}} & \text{if } N \in \{6, 7, 8, \dots\} \\ 1/2 & \text{if } N = 5, \\ 0 & \text{if } N \in \{3, 4\} \end{cases}$$

$$U_1(N) := \frac{\log(2^N - 1)}{\log 2^N},$$

and $L_1(N) = U_1(N) = 1$ if $N = \infty$.

Theorem 3. *(***Julia set***)*
For every external parameter angle $\vartheta \in \mathbb{S}^1$, the set of biaccessible external dynamic angles $\varphi \in \mathbb{S}^1$ has Hausdorff dimension between $L_1(N)$ and $U_1(N)$ for $N = N(\vartheta)$.

In particular, the set of biaccessible external dynamic angles has Hausdorff dimension less than 1 unless $\vartheta = 1/2$.

Theorem 4. *(***Mandelbrot set***)*
The set of external parameter angles ϑ for with $N(\vartheta) = N$ has Hausdorff dimension in between $L_1(N)$ and $U_1(N)$.

In particular, the measure of biaccessible parameter angles (and this includes the real spine of $\mathcal{M}$) is zero.

Hausdorff estimate "near the main cardioid of $\mathcal{M}$"

Define

$$\kappa := \sup\{k \geq 1 : S_j \text{ is a multiple of } S_{j-1} \text{ for all } 1 \leq j \leq k\},$$

$$L_2(\kappa) := \frac{1}{S_{\kappa+1}}$$

$$U_2(\kappa) := \min\left\{1, \sqrt{\frac{7}{2(S_\kappa + 1)}}\right\} \ ,$$

and if $\kappa = \infty$, then $L_2(\kappa) = U_2(\kappa) = 0$; if $S_\kappa < S_{\kappa+1} = \infty$, then $0 = L_2(\kappa) < U_2(\kappa)$.

Theorem 5. *(***Julia set***)*
For every external parameter angle $\vartheta \in \mathbb{S}^1$, the set of biaccessible external dynamic angles $\varphi \in \mathbb{S}^1$ has Hausdorff dimension between $L_2(\kappa)$ and $U_2(\kappa)$ for $\kappa = \kappa(\vartheta)$.

In particular, the set of biaccessible external dynamic angles of the Feigenbaum map has Hausdorff dimension zero.

Theorem 6. *(***Mandelbrot set***)*
The set of external parameter angles ϑ for with $\kappa(\vartheta) = \kappa$ has Hausdorff dimension in between $L_2(\kappa)$ and $U_2(\kappa)$.

In particular, the Hausdorff dimension of angles of infinitely renormalizable maps is zero.

Theorem 7. *Let B_ϑ be the set of combinatorially biaccessible dynamic external angles for parameter angle ϑ, and (T_ϑ, f) is the corresponding (abstract) Hubbard tree. Then*

$$\dim_H(B_\vartheta) = \frac{h_{top}(f|_{T_\vartheta})}{\log 2}. \tag{1}$$

Furthermore, these numbers depend Hölder continuously on $\vartheta \in \mathbb{S}^1$, with Hölder exponent equal to $\dim_H(B_\vartheta)$.

Remarks: Monotonicity was proven in the Ph.D-theses of Chris Penrose (1994) and Tao Li (2007). (Monotonicity of $\vartheta \mapsto h_{top}(f|_{T_\vartheta})$ is in the sense that if ϑ and ϕ are biaccessible parameter angles such that $\phi \in (\vartheta, \vartheta')$, *i.e.*, ϕ belongs to the wake, or shorter arc, of ϑ and its companion angle ϑ' with the same kneading sequence, then $h_{top}(f|_{T_\phi}) \geq h_{top}(f|_{T_\vartheta})$.)

With Hölder exponent $\dim_H(B_\vartheta) = \frac{h_{top}(f|_{T_\vartheta})}{\log 2}$, Hölder continuity fails at the zero-entropy locus.

For estimates of $\vartheta \mapsto h_{top}(f_{c(\vartheta)}|_{T_\vartheta})$ along the the real antenna (and especially the Feigenbaum parameter), see also work by Carminati and Tiozzo.

A similar question which we don't solve here is the modulus of continuity of $c \mapsto h_{top}(f_c|_{T_\vartheta})$ along the the real antenna.

Steps in the proof:

I: $\dim_H(B_\vartheta) = h_{top}(f|_{T_\vartheta})/\log 2$ for finite trees:
By Variational Principle.

II. Argument is inconclusive for infinite trees:
The problem is potential non-compactness of infinite Hubbard trees. taking the closure can increase entropy.

III. Continuity of $\vartheta \mapsto \dim_H(B_\vartheta)$:
Exploit that for large N, up to a set of tiny Hausdorff dimension, biaccessibility of itineraries can be determined by the first N entries of ν.

IV. Monotonicity of $\vartheta \mapsto h_{top}(f|_{T_\vartheta})$:
Itineraries of Hubbard tree of the smaller parameter angle are represented as Cantor set within the Hubbard tree of the parameter larger.

V. The biaccessible Mandelbrot set:
Monotonicity + Continuity of Hausdorff dimension implies Continuity of Entropy as long as we have majorizing parameter angles. No majorized angles at “tips” of the Mandelbrot set.

VI. Zero entropy tips:
No possibility of jumps here.

VII. Continuity of $\vartheta \mapsto h_{top}(f|_{T_\vartheta})$ at tips:
Compare non-zero entropy tips with nearby Thurston-Misiurewicz tips.

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