Stochastic behaviour of dynamical systems

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Times series

Measurement at a unknown/partially known system in form of a **times series**:

$$X_1, X_2, X_3, X_4, \ldots$$

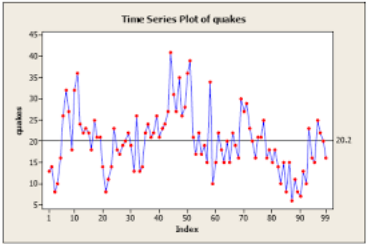


Figure: Some times series.

For simplicity we take $X_i \in \mathbb{R}$ (i.e., real numbers).

Random variables

Treating $(X_i)_{i\geq 1}$ as **stochastic process**: the X_i are i.i.d. random variables.

- independent:

$$\mathbb{P}(X_i \in A \text{ and } X_j \in B) = \mathbb{P}(X_i \in A) \cdot \mathbb{P}(X_j \in B) \qquad i \neq j.$$

- identically distributed: for each $A \subset \mathbb{R}$

$$\mathbb{P}(X_i \in A) \text{ is the same for all } i \geq 1.$$

We will assume that the first two moments exist, and then also the mean $\mu = \int X_i d\mathbb{P}$ and variance $\sigma^2 = \int (X_i - \mu)^2 d\mathbb{P}$.
(By independence, μ and σ^2 don't depend on i.)

Stochastic Laws

Under these condition we have for $S_n = X_1 + \cdots + X_n$:

- Weak Law of Large Numbers

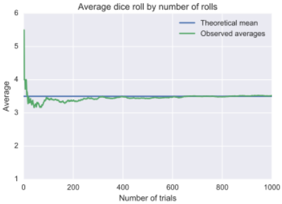
$$\lim_{n\to\infty} \mathbb{P}(|\frac{1}{n}S_n - \mu| > \varepsilon) = 0 \text{ for every } \varepsilon > 0.$$

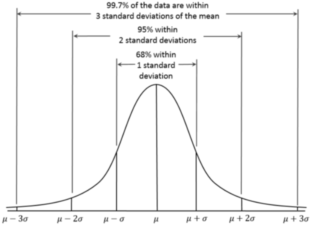
- Strong Law of Large Numbers

$$\mathbb{P}(\lim_{n\to\infty} \frac{1}{n}S_n = \mu) = 0.$$

- The Central Limit Theorem:

$$\lim_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{n}} = \mathcal{N}(0,1) \text{ in distribution.}$$





Stochastic Laws

Further laws exist for $M_n = \max\{X_1, \ldots, X_n\}$:

- Extremal Value Laws:

$$\lim_{n\to\infty} \mathbb{P}((M_n - a_n)/b_n \leq t) = G(t) \sim e^{-\alpha t^{-1/\alpha}} \text{ as } t \to \infty.$$

Depending on the tail of M_n, the parameter α varies. We have

$$G(t) = \begin{cases} \text{Weibull's Law} & \text{light tail - } M_n \text{ bounded} \\ \text{Gumbel's Law} & \text{exponential tail} \\ \text{Fréchet's Law} & \text{heavy tail} \end{cases}$$

Dynamical systems

Let $(\mathbb{R}^d, f)$ be a deterministic but chaotic dynamical system, given by

$$\begin{cases} \text{iteration: } z_{n+1} = f(z_n) & \text{discrete time} \\ \text{a flow: } z_t = f^t(z_0) & \text{continuous time} \end{cases}$$

Chaos means here: **sensitive dependence** on initial conditions or (stronger) existence of **positive Lyapunov exponents**.

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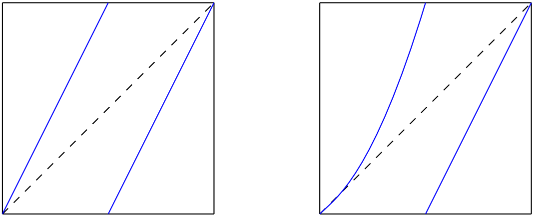
Let $v : \mathbb{R}^d \to \mathbb{R}$ be an observable.

Due to the chaos, precise predictions of $X_n := v \circ f^n$ are impossible.

If there is a "good" f-invariant measure μ, one can hope to prove **stochastic laws**.

Dynamical systems

The more expansion (hyperbolicity) in the system, the more chaos (more sensitivity), but also the better stochastic laws tend to work.



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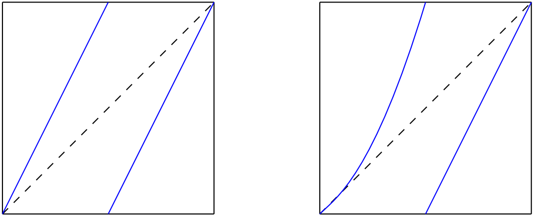

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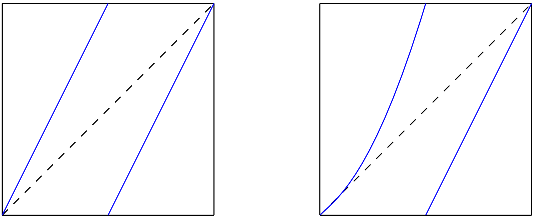

The **doubling map** $f(z) = 2z \bmod 1$ is uniformly expanding.

The **intermittent map**

$$f(z) = \begin{cases} z(1+(2z)^\alpha) & z \in [0, \frac{1}{2}]; \\ 2z - 1 & z \in (\frac{1}{2}, 1]. \end{cases} \qquad \alpha > 0.$$

has a neutral fixed point, where orbits linger.

Stochastic Laws

- Independence is replaced by asymptotic independence.
  This is called **mixing**, i.e., the **correlation coefficients**

  $$\rho_n(v,w) = \int v \cdot w \circ f^n \, d\mu - \int v \, d\mu \int w \, d\mu$$

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- Central Limit Theorem (CLT):

$$\frac{\sum_{j=0}^{n-1}(X_j - \mu)}{\sigma\sqrt{n}} \Rightarrow_d \mathcal{N}(0,1)$$

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- Extremal Value Laws, many more....

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Mixing with sufficiently good rate tends to imply other stochastic laws.

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In fact, this applies to virtually every uniformly expanding sufficiently smooth interval map.

Theorem

The intermittent map preserves a probability measure $\mu \sim$ Lebesgue measure provided $\alpha < 1$. In this case,

$$\rho_n(v, w) \sim \frac{1}{\bar{\tau}} n^{-1/\alpha} \int v \; d\mu \int w \; d\mu + O(d_n)$$

for some $\bar{\tau}$ and with known error terms $O(d_n)$.
If $\alpha < 1/2$, then the Central Limit Theorem holds.

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- Dimension higher than one is (when expanding) not an intrinsic problem, but definitely more technical.
- Contracting directions pose a serious problem. Results in this direction only from the last decade (or two decades in idealised settings).
- Continuous time systems (flows) are much harder to deal with. They have a neutral direction, which makes mixing rates and even mixing itself hard to prove. Results only from last half-decade.

Dynamical systems

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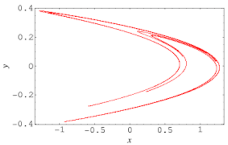
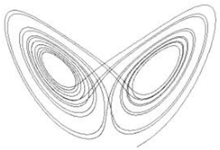


Hénon attractor

$$H(x, y) = (1 - ax^2 + y, x)$$



Lorenz attractor

$$\dot{x} = \sigma(x - y)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Takens' Reconstruction Theorem

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One way (exploited by Takens' Reconstruction Theorem) is by plotting $(X_n, X_{n+1}, \ldots, X_{n+k-1})$ in $\mathbb{R}^k$ and see if a pattern emerges.

Takens' Embedding Theorem

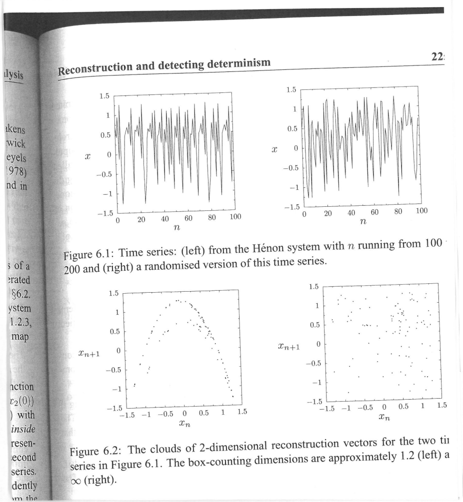

Figure 6.1: Time series: (left) from the Hénon system with n running from 100 200 and (right) a randomised version of this time series.

Figure 6.2: The clouds of 2-dimensional reconstruction vectors for the two ti series in Figure 6.1. The box-counting dimensions are approximately 1.2 (left) a ∞ (right).

Takens' Reconstruction Theorem

Theorem (Takens' Reconstruction Theorem)

Let M be an m-dimensional manifold and $k > 2m$. Then for a **generic** *dynamical system $f : M \to M$ and observable $v : M \to \mathbb{R}$, the map $R_k : M \to \mathbb{R}^k$ defined by*

$$x \mapsto (v(x), v \circ f(x), \ldots, v \circ f^{k-1}(x))$$

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We omitted some conditions on the smoothness (and invertibility) of the dynamical systems and observables.