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That monotonous thing called entropy

Mankato, July 2012

Coauthors

The results on complex dynamics are joint with
Dierk Schleicher (Jacobsuniversität, Bremen).

The results on multimodal maps are joint with
Sebastian van Strien (Imperial College, London).

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A yet more tractable definition for interval maps

$$X = [0,1], \quad f : [0,1] \to [0,1] \text{ continuous with finitely many laps}$$

is due to Misiurewicz & Szlenk (1980)

$$\begin{aligned} h_{top}(f) &= \max\{0, \lim_{n\to\infty} \frac{1}{n} \log \#\{x = f^n(x)\}\} && (1) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \#\{ \text{ laps of } f^n \} && (2) \\ &= \max\{0, \lim_{n\to\infty} \frac{1}{n} \log(\mathrm{Var}(f^n))\}. && (3) \end{aligned}$$

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(4) Analogous results hold for maps on finite trees.

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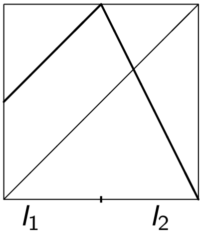
Entropy can be computed as the the logarithm of the largest (Perron-Frobenius) eigenvalue $\sigma(A)$ of the corresponding transition matrix A. Matrix A could be infinite.

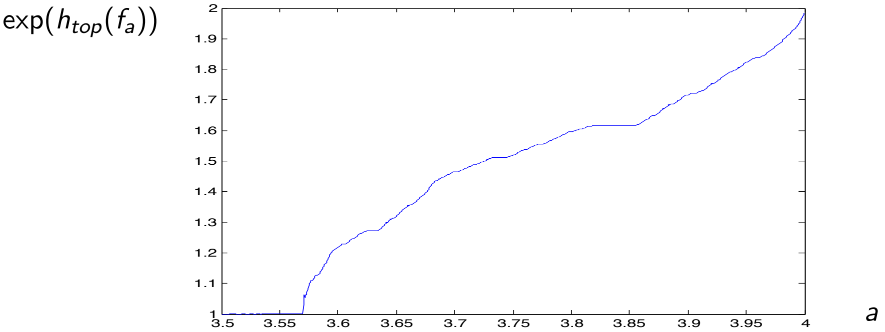


$$I_2 \circlearrowleft \quad \uparrow\downarrow \quad I_1$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\sigma(A) = \log \frac{1+\sqrt{5}}{2}$$

$h_{top}(f_a)$ for quadratic family $f_a(x) = ax(1-x)$



For $f_a(x) = ax(1-x)$, the entropy map $a \mapsto h_{top}(f_a)$ is

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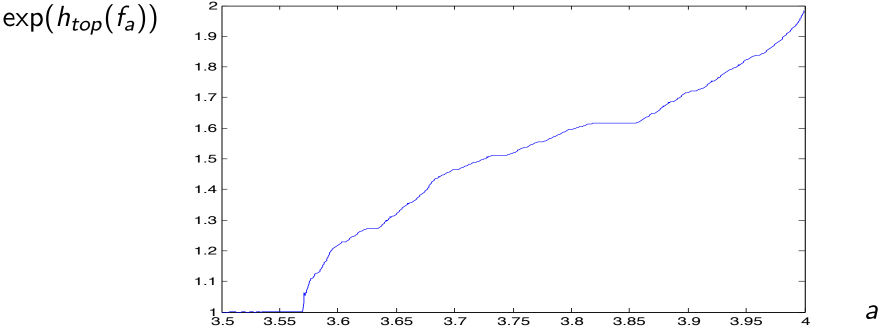

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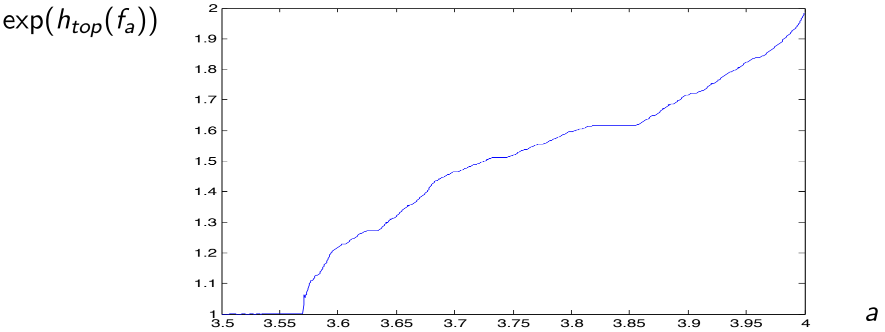

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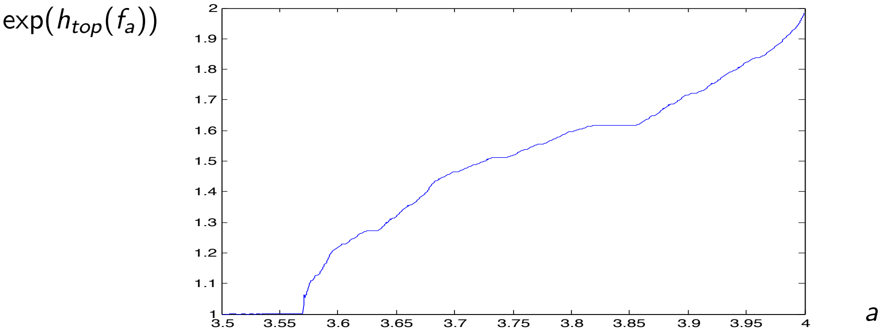

For $f_a(x) = ax(1-x)$, the entropy map $a \mapsto h_{top}(f_a)$ is

- Continuous - but what is the modulus of continuity?
- Monotone - but not strictly.

- Entropy is constant on every interval of hyperbolicity (where f_a has a stable periodic orbit) and every successive interval of period doubling cascade.

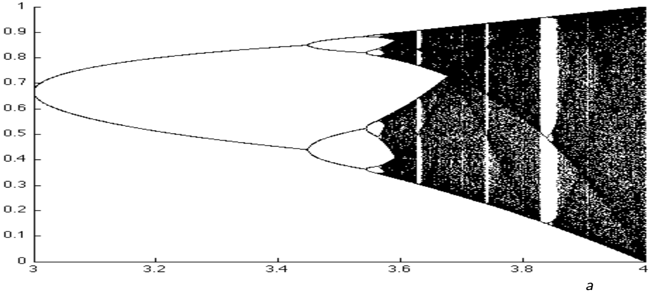

Figure: Bifurcation diagram for $f_a(x) = ax(1-x)$

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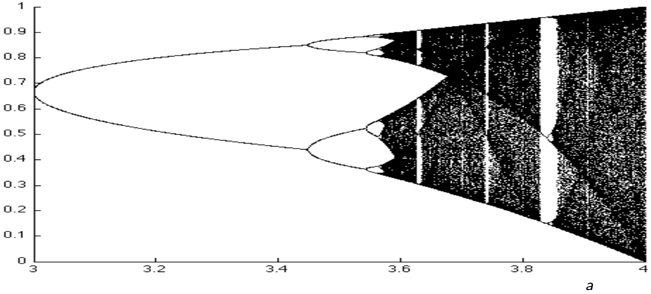

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- Stronger than monotonicity: there are only period doublings, no period halfings.

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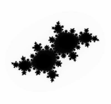
- However, the measure of non-hyperbolic parameters is positive, see Jakobson (1981), Benedicks & Carleson (1984).

Step 1: For polynomials $f_c(z) = z^2 + c$, the **filled-in Julia set** is



$$\mathcal{K}_c = \{z \in \mathbb{C} : f_c^n(z) \not\to \infty\} \subset \text{dynamical space}$$

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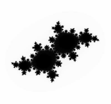

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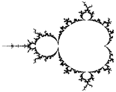

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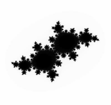

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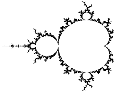

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$$\phi : \bar{\mathbb{C}} \setminus \mathcal{M} \to \bar{\mathbb{C}} \setminus \mathbb{D}, \quad \phi'(\infty) = 1,$$

and for $c \in \partial\mathcal{M}$:

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Proof of monotonicity - Douady & Hubbard

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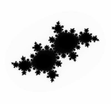

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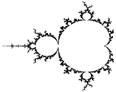

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The map ϕ_c **conjugates** f_c on $\bar{\mathbb{C}} \setminus \mathcal{K}_c$ to $z \mapsto z^2$ on $\bar{\mathbb{C}} \setminus \mathbb{D}$.

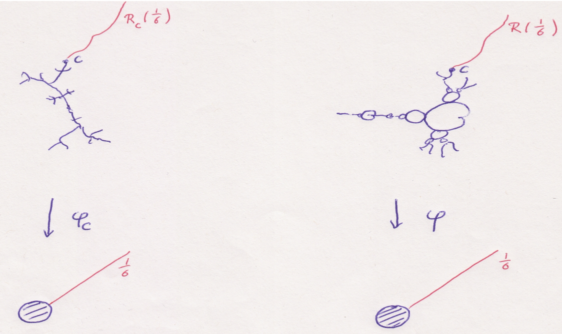

Figure: Riemann map ϕ and ϕ_c and external rays for angle $\frac{1}{6}$ for the filled-in Julia set and the Mandelbrot set.

Step 2: For each $\theta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, define **parameter rays**:

$$R(\theta) = \phi^{-1}(\{re^{2\pi i\theta} : r > 1\})$$

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- $R_c(\theta)$ lands at c as well, but here $c = f_c(0)$ is the critical value!

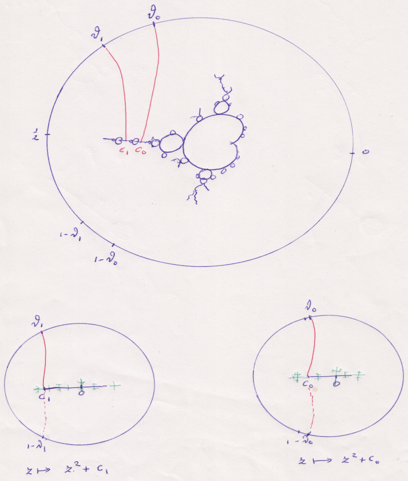

Figure: External rays θ_0 and θ_1 and corresponding rays for the Mandelbrot set and filled-in Julia sets.

Proof of monotonicity - continued

Step 3: Take $0 < \theta_0 < \theta_1 < \frac{1}{2}$ such that $R(\theta_0)$ and $R(\theta_1)$ land at **real** Misiurewicz-Thurston parameters c_0 and c_1.

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$$\Gamma_i = \{\gamma \neq 0 : g^n(\gamma) \in (-\theta_i, \theta_i) \text{ for all } n \geq 0\}$$

This is because f_{c_i} on the real core is a 2-to-1 factor of g on Γ_i.

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This is because f_{c_i} on the real core is a 2-to-1 factor of g on Γ_i. But $\theta_0 < \theta_1$, so $\Gamma_0 \subset \Gamma_1$ and

$$h_{top}(g|\Gamma_0) \leq h_{top}(g|\Gamma_1)$$

Finite-to-one factor maps preserve entropy, so

$$h_{top}(f_{c_0}|\text{real core}) \leq h_{top}(f_{c_1}|\text{real core}) \qquad \square$$

The **Hubbard tree** T of the filled-in Julia set is the equivalent of the core interval $[c, c^2 + c]$ of a **real** unimodal map f_c. It is (a schematic version of) the connected hull of $\text{orb}_f(0)$ within the filled-in Julia set $\mathcal{K}_c$

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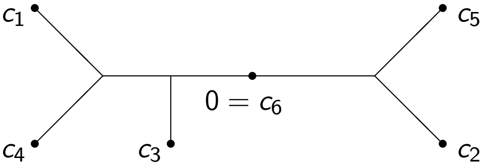

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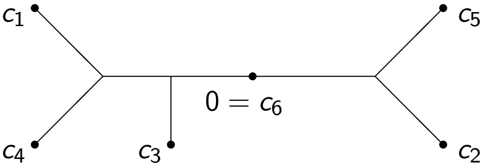

Whereas $h_{top}(f_c|\mathcal{K}_c) = \log 2$ for every $c \in \mathbb{C}$, the **core entropy** $h_{top}(f_c|T)$ can be strictly smaller than $\log 2$. In fact,

$$h_{top}(f_c|T) < \log 2 \qquad \text{unless } c = -2.$$

Monotonicity along antennae

Having the notion of Hubbard trees and core entropy, we can extend the Douady-Hubbard proof to other parts of $\mathcal{M}$:

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Full proof given in the PhD thesis of Tao Li (2007).

Largely unnoticed, fully symbolic, proof in the PhD thesis of Chris Penrose (1994).

Rules for Hubbard trees

1: $f : T \to T$ is continuous and surjective;

2: f is a local homeomorphism onto its image at every point $z \in T$, except at a unique critical point 0, where it is 2-to-1.

3: The set of **marked points** is

$$V = \{\text{endpoints}\} \cup \{\text{branchpoints}\} \cup \{c_k = f^k(0) : k \geq 0\}$$

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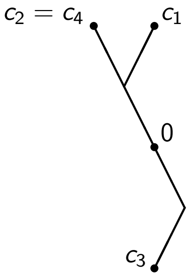

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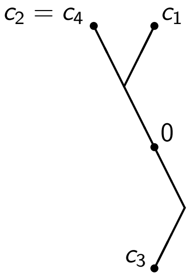
- The critical value c_1 is always an endpoint of T. So, 0 has at most two arms in T, and we can construct symbolic dynamics on two symbols.



Rules for Hubbard trees

1: $f : T \to T$ is continuous and surjective;

2: f is a local homeomorphism onto its image at every point $z \in T$, except at a unique critical point 0, where it is 2-to-1.

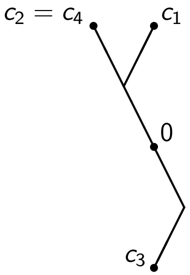
3: The set of **marked points** is

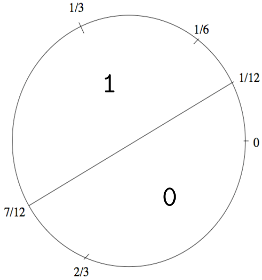
$$V = \{\text{endpoints}\} \cup \{\text{branchpoints}\} \cup \{c_k = f^k(0) : k \geq 0\}$$

For each $v \neq w \in V$ there is n such that $0 \in f^n(\text{arc}[v, w])$.

From this we can derive:

- The critical value c_1 is always an endpoint of T. So, 0 has at most two arms in T, and we can construct symbolic dynamics on two symbols.
- The symbolic itinerary ν of c_1 is called the **kneading invariant.**

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0
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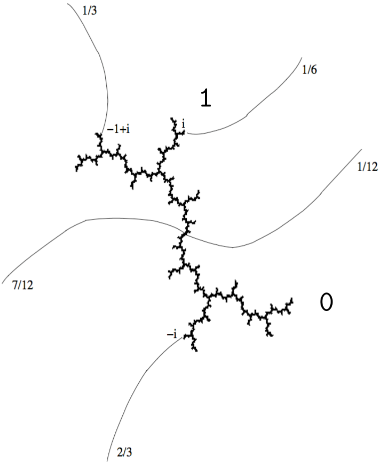
1
0


1/3
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i
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Biaccessible points

- For $z \in \mathcal{K}_c$,

$$\textbf{valency} \;\; = \#\{ \; \textit{arms of z in } \mathcal{K}_c \} = \#\{ \; \textit{rays landing at z} \}$$

Points of valency ≥ 2 are called biaccessible.

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Hence, if

$$A = \{\text{biacc. points in } T\} \qquad B = \{\text{biacc. points in } \mathcal{K}_c\}$$

then

$$B = \cup_n f^{-n}(A) \quad \text{and} \quad \dim_H(A) = \dim_H(B).$$

Here $\dim_H$ stands for Hausdorff dimension.

The ρ-function

There is an algorithm, based on itineraries only, to compute the valency of z. It depends on the ρ_e-**function**.

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Parameter space: The valency of $c \in \mathcal{M}$ equals the number of disjoint ρ_ν-orbits in $\mathbb{N}$.

Dimension Estimates

Using this characterization, one can estimate the Hausdorff dimension of biaccessible itineraries (dynamical space) or kneading sequences (parameter space) in $\{0,1\}^{\mathbb{N}}$.

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Technical Lemma: (From symbolics to external angles)

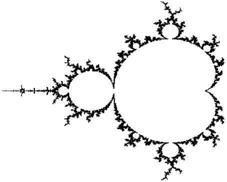
Dynamical space: The map $\gamma \mapsto e(\gamma)$ preserves Hausdorff dimension (fairly easy).

Parameter space: The map $\theta \mapsto \nu(\theta)$ preserves Hausdorff dimension (trickier to prove).

Estimates (abridged):

Dynamical space: The biaccessible angles of $\mathcal{K}_c$ for $c \in \partial\mathcal{M}$ has Hausdorff dimension

- < 1 iff $c \neq -2$;
- $= 0$ iff c is (infinitely) renormalizable via direct bifurcations from the Mandelbrot set.



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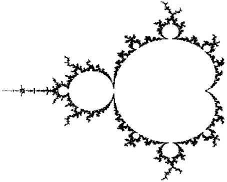

Parameter space: Except near $c = -2$, t The biaccessible angles of $\mathcal{M}$ have Hausdorff dimension

- < 1 iff not in a neighborhood of $\theta = \frac{1}{2}$;
- $= 0$ iff c is (infinitely) renormalizable via direct bifurcations from the Mandelbrot set.

Recall $h_{top}(f_c|J_c) = \log 2 \geq h_{top}(f_c|T_c) =$ core entropy

Theorem

Let $A_\theta = \{$biacc. dynamical angles landing on $T_\theta\} \subset \mathbb{S}^1$.
Then the core entropy is

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Proof: If T_θ is compact, use the Variational Principle which leads to the dimension formula for the angle doubling map g:

$$\dim_H(A_\theta) = \frac{\text{entropy}}{\text{Lyapunov exponent}} = \frac{h_{top}(g|A_\theta)}{g'}$$

It is trickier if T_θ is non-compact, as you need to estimate $\sigma(A)$ for infinite transition matrices. □

Hölder Monotonicity of Core Entropy

Theorem

The map

$$\theta \mapsto h_{top}(f|T_\theta)$$

is Hölder continuous with exponent $\alpha(\theta) = \dim_H(A_\theta)$.

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$$|h_{top}(f|T_\theta) - h_{top}(f|T_{\theta'})| \leq \frac{C}{-\log \max\{\alpha(\theta), \alpha(\theta')\}}$$

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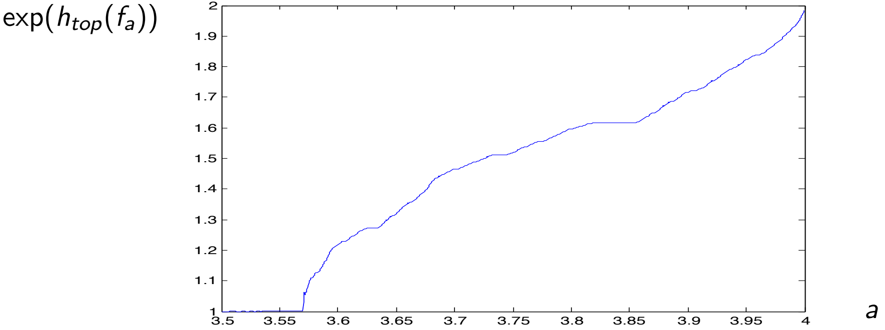
Question: What is the modulus of continuity of

$$\partial\mathcal{M} \ni c \mapsto h_{top}(f_c|T_c)?$$

$h_{top}(f_a)$ for quadratic family $f_a(x) = ax(1-x)$

This brings us back to an earlier picture. How smooth is this curve?



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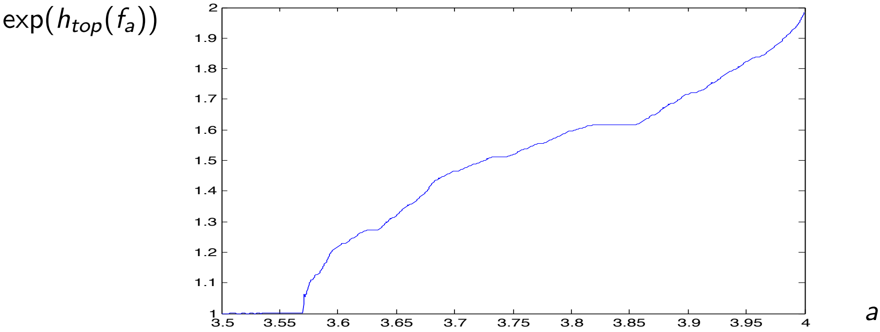
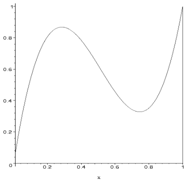


Known: Not absolutely continuous, and not Hölder at the Feigenbaum parameter (the last zero of the graph).

What about entropy for multimodal maps, i.e., maps with several critical points? Especially for the families of cubic, quartic, quintic, ... polynomials.



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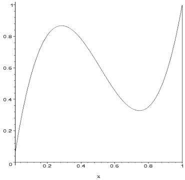

In their seminal paper 1977 preprint

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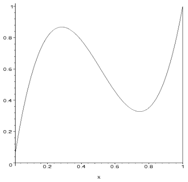

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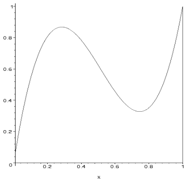

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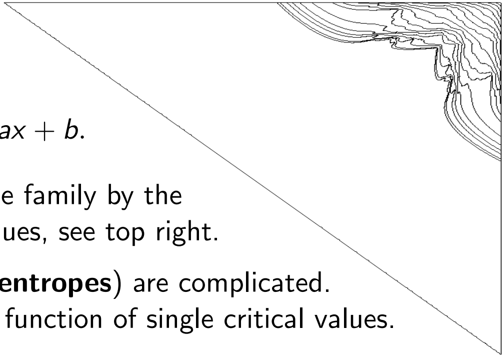
What about monotonicity?

Note that for families of degree $d + 1$ polynomials, parameter space is d-dimensional, and monotonicity means:

Isentropes, i.e., level sets of entropy, are ***connected****.*

Entropy in the cubic family

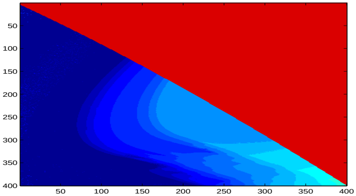


The general cubic family

$$x \mapsto x^3 - ax + b.$$

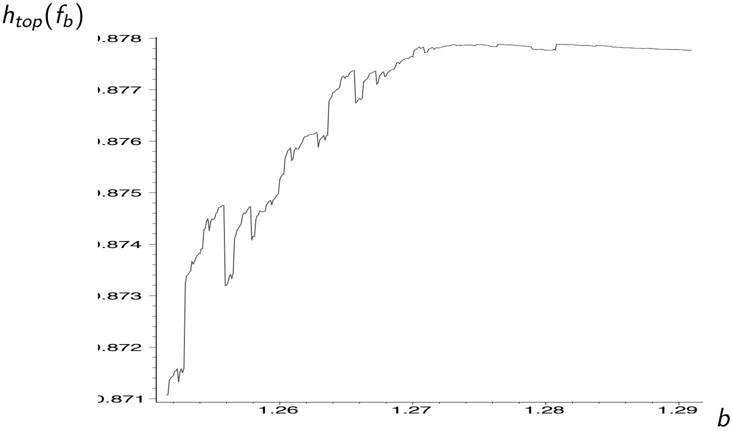
One can also parametrize the family by the height of the two critical values, see top right.

Level sets of the entropy (**isentropes**) are complicated. Entropy is not monotone as function of single critical values.



The cubic family
$$x \mapsto x^3 - ax + b.$$
Isentropes in blue colour:

We can prove in the case $d \geq 3$ that the entropy is not monotone on slices in parameter space. Below, the second critical value in the cubic map $x \mapsto x^3 - ax + b$ is fixed, the first, i.e., b, varies.



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Theorem (Milnor & Tresser (2000))

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Ingredients in the proof are:

- Denseness of hyperbolicity.
- Bones, i.e., set in parameter space where one critical point is periodic.
- Planar geometry (so fails for degree ≥ 4).
- The space of stunted saw-tooth maps as parameter space.

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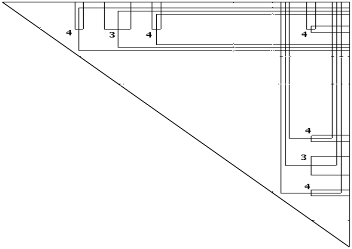
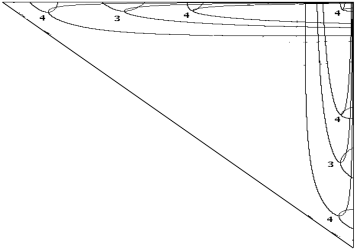
All these proofs use complex analysis!

Milnor and Tresser
analyse bifurcation curves,
see figures on the right.
They use planar topology
to show 'bones' are connected.





Bones: Milnor & Tresser's analysis of cubic parameter space

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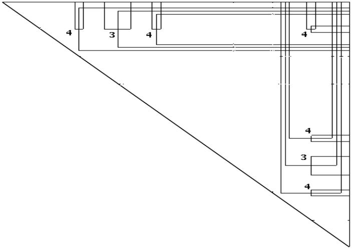
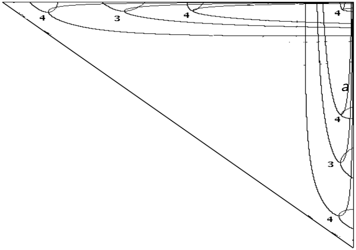
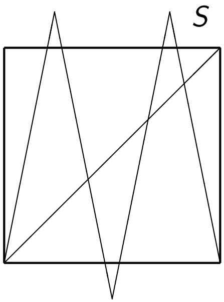




- Start with a piecewise linear saw-tooth map $S : [0,1] \to \mathbb{R}$ of $d+1$ laps. The critical values lie outside the interval!



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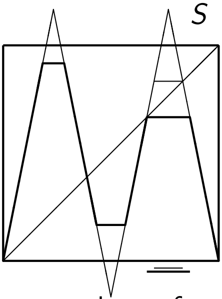

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Two stunted sawtooth maps, with different third plateaus.

- "Stunt" them at the preferred heights within $[0,1]$.

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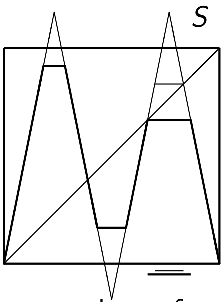

The saw-tooth map S

Two stunted sawtooth maps, with different third plateaus.

- "Stunt" them at the preferred heights within $[0,1]$.
- The result is a stunted saw-tooth map, with plateaus instead of critical points.

Let $\mathcal{S}^d$ be this space of stunted sawtooth maps. It will be used as parameter space.

What is good about the space $\mathcal{S}^d$?

- The saw-tooth map contains all itineraries in $\{0, \dots, d\}^{\mathbb{N}}$, hence $\mathcal{S}^d$ contains a map for every d-tuple of kneading sequences.
  (Kneading sequence ν_i is the itinerary of i-th critical value.)

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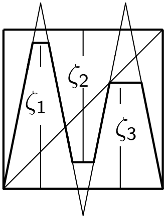

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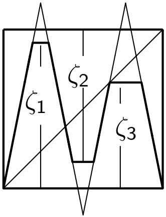

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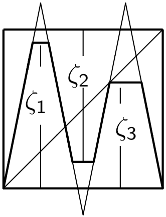

- $T \mapsto h_{top}(T)$ is monotone increasing in each parameter ζ_i.
- Using this, it is easy to show that **isentropes are connected (and even contractible)** for $\mathcal{S}^d$.

Let P^d be the space of degree $d + 1$ polynomials $f : [0, 1] \to [0, 1]$ such that

- f has d distinct critical points, all lying in $[0, 1]$.
- $f(0) = 0$ and $f(1) \in \{0, 1\}$.

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All isentropes L_s of P^d are connected.

This doesn't mean that isentropes are simple sets. We know that:

- For many value of entropy s, L_s is not locally connected.
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Question (Milnor): Are the isentropes contractible?

Question (Thurston): Is there a dense set of $s \in [0, \log d]$ such that hyperbolic maps are dense in L_s?

What is bad about the space $\mathcal{S}^d$?

To every $f \in P^d$, assign a stunted sawtooth map $\Psi(f) \in \mathcal{S}$, by taking the one with the same kneading invariants as f.

$$\Psi : P^d \to \mathcal{S} \text{ is well-defined and preserves entropy}$$

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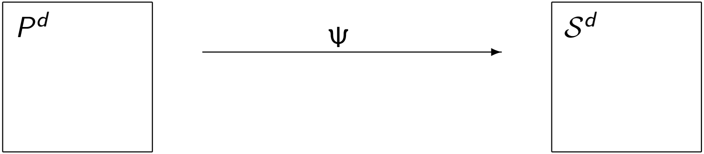

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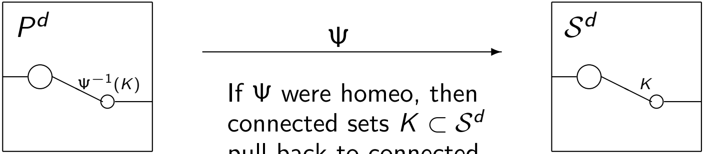

If Ψ were homeo, then connected sets $K \subset \mathcal{S}^d$ pull back to connected sets $\Psi^{-1}(K) \subset P^d$

However,

What is bad about the space $\mathcal{S}^d$?

To every $f \in P^d$, assign a stunted sawtooth map $\Psi(f) \in \mathcal{S}$, by taking the one with the same kneading invariants as f.

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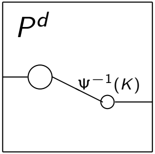

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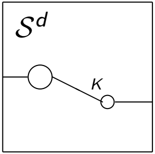

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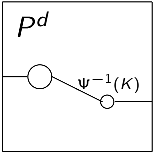

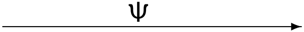

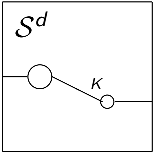

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- Ψ is really not surjective!

"Cells" in higher dimensional parameter space.

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  1. they are conjugate away from the basins of periodic attractors;
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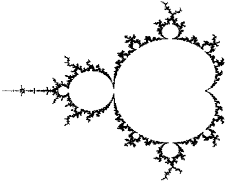
  The cell (**partial hyperbolic deformation space**) of $f \in P^d$ are all maps partially conjugate to it.
- These cells are indeed topological cells of the same dimension as number of critical points attracted to periodic attractors.
- When complexified, they are the higher-dimensional analog of **hyperbolic components** in the Mandelbrot set.



Cells in P^d are glued together via the following generic bifurcations

- sn saddle-node (creation of one-sided attractor, which then becomes becomes an attracting + repelling pair)
- pf pitchfork or reverse pitchfork (a two-sided attractor, which becomes repelling and spins off a pair of attracting orbits)
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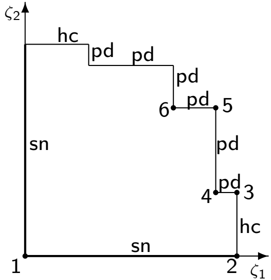
In $\mathcal{S}^d$, a cell is any set of $T \in \mathcal{S}^d$ for which

$$\{x \in [0,1] : \exists n \geq 0,\ \ T^n(x) \in (\cup Z_i)^\circ\}$$

remains unchanged.

Their bifurcations follow the same pattern.



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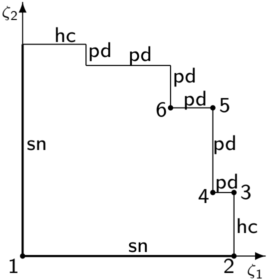

We overcome the continuity/injectivity/surjectivity problem by **(in a way)** quotienting out over the cells.

Serious non-surjectivity of Ψ due to wandering pairs

- An interval $J \subset [0, 1]$ is **wandering** if $f^n|J$ is monotone for all $n \geq 0$, but J is not attracted to a periodic orbit.
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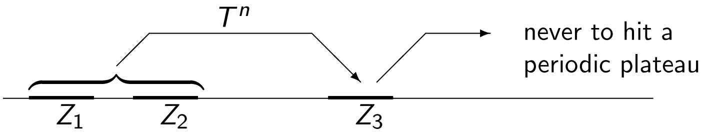

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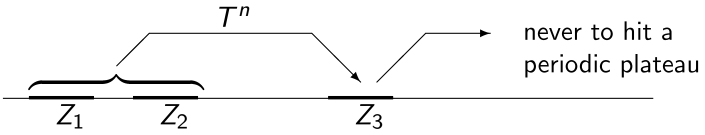

- A **pair** of adjacent intervals Z_1 and Z_2 is **wandering** if the above picture applies. The interval $[Z_1, Z_2]$ is eventually mapped to a point, but becomes never periodic. Hence, such a pair takes the role of a wandering interval.

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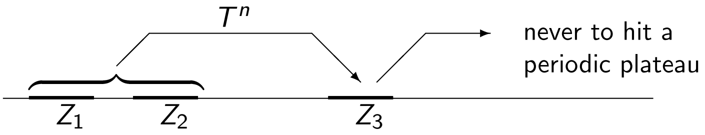

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- This is a serious obstacle for Ψ to be (even almost) surjective.
- Note that wandering pairs require at least three plateaus: $d \geq 3$. Milnor & Tresser didn't have to deal with this.

- As we cannot allow wandering intervals, let us define

$$\mathcal{S}_*^d = \{T \in \mathcal{S}^d : \nexists \text{ degenerate pair of plateaus}\}$$

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Theorem

The isentropes in $\mathcal{S}_*^d$ are connected and even contractible.

Ψ is almost injective, almost surjective and almost continuous

Proposition (Surjectivity)

For each $T \in \mathcal{S}^d_*$ *there exists* $f \in P^d$ *so that* $T \in cell(\Psi(f))$.

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If $f_1, f_2 \in P^d$ *and* $cell(\Psi(f_1)) \cap cell(\Psi(f_2)) \neq \emptyset$ *then* $\overline{cell(f_1)} \cap \overline{cell(f_2)} \neq \emptyset$.

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Proposition (Continuity)

Suppose $f_n \in P^d$ converges to $f \in P^d$. Then any limit of $\Psi(f_n)$ is contained in $cell(\Psi(f))$.

The upshot

Theorem

If K is closed and connected then

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Question (Milnor): Are isentropes contractible?

Probably yes, but this is work in progress.