Thermodynamic formalism for dissipative interval maps

Henk Bruin (University of Surrey)

joint work with

Mike Todd (St Andrews)

Basic thermodynamic formalism

Let $f : X \to X$ be continuous, and $\phi_t : X \to \mathbb{R}$ be a parametrised families of potentials.

The (variational) **pressure** is defined as

$$P(t) = \sup_{\mu} \left\{ \underbrace{h_\mu(f)}_{\text{entropy}} + \underbrace{\int \phi_t \, d\mu}_{\text{energy}} \right\},$$

where the sup is taken over all f-invariant probability measures μ.

An **equilibrium state** μ_t is a measure that assume the pressure.

Usually $\phi_t = t \cdot \phi$. Then at $t = 0$, we are maximising entropy, while for $t \to \infty$, we are minimising potential energy.

Values of t where $P(t)$ is not real analytic are called **phase transitions**. They indicate a qualitative (and abrupt) change in equilibrium state.

"Geometric" potential $\phi_t = -t \log |f'|$

For interval maps, an important class of potentials to choose is $\phi_t = -t \log |f'|$, which ties thermodynamic formalism to Lebesgue measure.

- Energy becomes the **Lyapunov exponent**:

$$\lambda(\mu) = \int \log |f'| \, d\mu.$$

- A result by Ledrappier [Le] says that:

  if $h_\mu(f) > 0$, then
  μ is equilibrium state for $t = 1$ if and only if μ is absolutely continuuous w.r.t. Lebesgue.

  Together with the Ruelle inequality $h_\mu \leqslant \lambda(\mu)$, this implies that $P(t) = 0$ for $t = 1$.

Lebesgue ergodic properties

Recall that

- f is **Lebesgue ergodic** if $f^{-1}(B) = B$ implies $Leb(B) \in \{0, 1\}$.
- f is **Lebesgue conservative** if $Leb(B) > 0$ implies that $f^n(B) \cap B \neq \emptyset$ for some $n > 0$.
- f is **Lebesgue dissipative** if not conservative.
- f is **Lebesgue totally dissipative** if there is no invariant set of positive measure on which it is conservative.
- If smooth unimodal map f is totally dissipative, then it has an attractor A, *i.e.,*
  - $f(A) \subset A$
  - The **basin** $\{x \in [0, 1] : \omega(x) = A\}$ has positive Lebesgue measure.
  - There is no smaller set with these two properties.

  Moreover, $A = \omega(c)$ and $Leb(A) = 0$. It can be one of the following:
  - a stable periodic orbit.
  - a **solenoidal attractor**, namely if f is infinitely renormalizable. (E.g. the Feigenbaum-Coullet-Tresser map).
  - a **wild attractor**: in this case, the basin has full measure, but is of first Baire category.

Interval dynamics - Fibonacci maps

Let f be a smooth unimodal map. For our purposes, it suffices to look at the family:

$$f = f_{a,\ell} : [0,1] \to [0,1], \qquad x \mapsto a(1 - |2x-1|^{\ell}).$$

with critical point $c = \frac{1}{2}$ and critical order $\ell > 0$.

The iterate n is a **cutting time** if the image of the central branch of f^n contains c. We denote cutting times by the strictly increasing sequence

$$1 = S_0 < S_1 < S_2 < \dots$$

We call f a **Fibonacci map** if its **cutting times** are the Fibonacci numbers.

For each $\ell > 0$, there is at least one (and if $\ell = 2k$ unique)

$$a = a(\ell) \text{ such that } f_{a,\ell} \text{ is Fibonacci.}$$

From now on let $f_\ell = f_{a(\ell),\ell}$ be a family of Fibonacci maps parametrised by its critical order.

Ergodic properties for smooth Fibonacci maps

The following properties are known for f_ℓ:

$$
\begin{cases}
\ell \leqslant 2 & f_\ell \text{ has an acip which is super-polynomially} \\
 & \quad \text{mixing, [LM, BLS],} \\
2 < \ell < 2 + \varepsilon & f_\ell \text{ has an acip which is polynomially mixing with} \\
 & \quad \text{exponent tending to infinity as } \ell \to 2, \text{ [KN, RS],} \\
\ell_0 < \ell < \ell_1 & f_\ell \text{ has a conservative } \sigma\text{-finite acim,} \\
\ell_1 < \ell & f_\ell \text{ has a wild attractor [BKNS], with dissipative} \\
 & \quad \sigma\text{-finite acim, [Ma].}
\end{cases}
$$

Linear versions of the induced map:

A linearised version of the induced map will be called F_λ, where λ denoted th exponential rate at which the distances $|z_k - c|$ decrease.

It is a two-to-one cover of a countably piecewise interval map $T_\lambda : (0,1] \to (0,1]$ defined in Stratmann & Vogt [SV] as follows:

For $n \geqslant 1$, let $V_n := (\lambda^n, \lambda^{n-1}]$ and define

$$T_\lambda(x) := \begin{cases} \frac{x-\lambda}{1-\lambda} & \text{if } x \in V_1, \\ \frac{x-\lambda^n}{\lambda(1-\lambda)} & \text{if } x \in V_n, \quad n \geqslant 2. \end{cases}$$

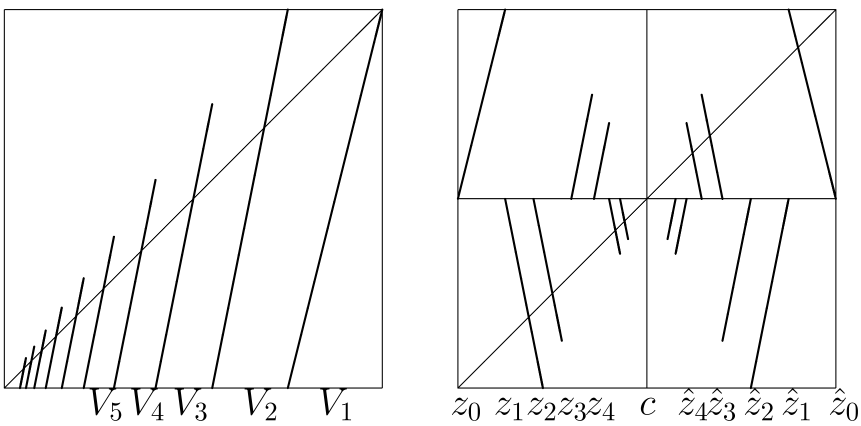

FIGURE 1. The maps $T_\lambda : [0,1] \to [0,1]$ and $F_\lambda : [z_0, \hat{z}_0] \to [z_0, \hat{z}_0]$.

Theorem A. *For each* $\lambda \in (0,1)$*, there is a countably piecewise linear unimodal map* f_λ *(with* $|W_k| = |\hat{W}_k| = \frac{1-\lambda}{2} \cdot \lambda^k$*) such that the induced map* F_λ *is has affine branches.*

Moreover:

a) The critical order $\ell = 3 + \frac{2\log(1-\lambda)}{\log \lambda}$.

b) If $\lambda \in (\frac{1}{2}, 1)$, i.e., $\ell > 5$*, then* f_λ *has a wild attractor.*

c) If $\lambda \in \left[\frac{2}{3+\sqrt{5}}, \frac{1}{2}\right]$, i.e., $4 \leqslant \ell \leqslant 5$*, then* f_λ *has no wild attractor, but an infinite* σ*-finite acim.*

d) If $\lambda \in (0, \frac{2}{3+\sqrt{5}})$, i.e., $\ell \in (3,4)$*, then* f_λ *has an acip.*

Theorem B. *The countably piecewise linear Fibonacci map f_λ, $\lambda \in (0,1)$, with potential ϕ_t has the following thermodynamical properties.*

a) The conformal and variation pressure coincide:

$$P_{\text{Conf}}(\phi_t) = P(\phi_t);$$

b) For $t < t_1$, there exists a unique equilibrium state ν_t for (I, f_λ, ϕ_t); this is absolutely continuous w.r.t. the appropriate conformal measure n_t.

c) For $t > t_1$, the unique equilibrium state for (I, f_λ, ϕ_t) is ν_ω, the measure supported on the critical omega-limit set $\omega(c)$. For $t = t_1$, ν_ω is an equilibrium state, and if $\lambda \in (0, \frac{2}{3+\sqrt{5}})$ then so is the acip, denoted ν_{t_1};

d) The map $t \mapsto P(\phi_t)$ is real analytic on $(-\infty, t_1)$. Furthermore $P(\phi_t) > 0$ for $t < t_1$ and $P(\phi_t) \equiv 0$ for $t \geqslant t_1$, so there is a phase transition at $t = t_1$.

Theorem C. *The pressure function $P(\phi_t)$ of the countably piecewise linear Fibonacci map f_λ, $\lambda \in (0,1)$, with potential ϕ_t has the following shape:*

a) On a left neighbourhood of t_1, there exist $\tau_0 = \tau_0(\lambda), \tau_0' = \tau_0'(\lambda) > 0$ such that

$$P(\phi_t) > \begin{cases} \tau_0 e^{-\pi \frac{\Gamma}{\sqrt{t_1 - t}}} & \text{if } t < t_1 \leqslant 1 \text{ and } \lambda \geqslant \frac{1}{2}; \\ \tau_0'(1-t)^{\frac{\log \gamma_+}{\log R}} & \text{if } t < 1 \text{ and } \frac{2}{3+\sqrt{5}} \leqslant \lambda < \frac{1}{2}, \end{cases}$$

where $R = \frac{\left(1+\sqrt{1-4\lambda^t(1-\lambda)^t}\right)^2}{4\lambda^t(1-\lambda)^t}$ and $\lim_{t\to 1} \log R \sim 2(1-2\lambda)$ for $\lambda \sim \frac{1}{2}$.

b) On a left neighbourhood of t_1, there exist $\tau_1 = \tau_1(\lambda), \tau_1' = \tau_1'(\lambda) > 0$ such that

$$P(\phi_t) < \begin{cases} \tau_1 e^{-\frac{5}{6} \frac{\Gamma}{\sqrt{t_1 - t}}} & \text{if } t < t_1 \leqslant 1 \text{ and } \lambda \geqslant \frac{1}{2}; \\ \tau_1'(1-t)^{\frac{\lambda \log \gamma_+}{2t(1-2\lambda)}} & \text{if } t < 1 \text{ and } \frac{2}{3+\sqrt{5}} \leqslant \lambda < \frac{1}{2}. \end{cases}$$

*c) If $\lambda \in (0, \frac{2}{3+\sqrt{5}})$, then $\lim_{s\uparrow t_1} \frac{d}{ds} P(\phi_s) < 0$; otherwise (*i.e., *if $\lambda \in [\frac{2}{3+\sqrt{5}}, 1)$), $\lim_{s\uparrow t_1} P(\phi_s) = 0$.*

Dimension results:

Let

$$\text{Bas}_\lambda = \{x \in I : f_\lambda^n(x) \to \omega(c) \text{ as } n \to \infty\}$$

be the **basin** of $\omega(c)$. The **hyperbolic dimension** is the supremum of Hausdorff dimensions of hyperbolic sets Λ, *i.e.,* Λ is f_λ-invariant, compact but bounded away from c.

Theorem D.

$$\dim_{hyp}(f_\lambda) = \dim_H(Bas_{1-\lambda}) = \begin{cases} 1 & \text{if } \lambda \leqslant \frac{1}{2}; \\ -\frac{\log 4}{\log[\lambda(1-\lambda)]} & \text{if } \lambda \geqslant \frac{1}{2}, \end{cases}$$

is the first zero of the pressure function.

Conformal measure and pressure:

Definition 1. *A measure m on $[0,1]$ is called* ϕ-conformal *if for any measurable set $A \subset [0,1]$ on which $f : A \to g(A)$ is a bijection,*

$$m(f(A)) = \int_A e^{-\phi}\, dm.$$

For $\phi = -t \log |f'|$, this reduces to

$$m(f(A)) = \int_A |f'|^t\, dm.$$

Definition 2. *For a dynamical system $g : X \to X$ and a potential $\phi : X \to [-\infty, \infty]$, the* conformal pressure *for (X, g, ϕ) is*

$$P_{\text{Conf}}(\phi) := \inf \{ p \in \mathbb{R} : \exists \ a \ (\phi - p)\text{-}conformal\ measure \}.$$

Inducing and potential shifts:

If $F = f^\tau$ is an induced map, the potential ϕ_t induces to

$$\Phi_t(x) = \sum_{j=0}^{\tau(x)-1} \phi_t \circ f^j(x).$$

For $\phi_t = -t \log |f'|$, the chain rule gives $\Phi_t = -t \log |F'|$.

Note that a potential shift of p for f induces to a **non-constant** potential shift for F:

$$\Phi_t = -t \log |F'| - \tau p.$$

Finding conformal measures:

To find a p-conformal measure for T_λ (**CONSTANT** potential shift), we need to solve (with $w_k^t = m_t(V_k)$)

$$\sum_k w_k^t = 1 \quad \text{subject to } w_k^t \geqslant 0 \text{ for all } k,$$

where the w_k^t satisfy

$$\begin{aligned} w_1^t &= (1-\lambda)^t e^{-p} \\ w_2^t &= \lambda^t (1-\lambda)^t e^{-p} \\ w_3^t &= \lambda^t (1-\lambda)^t e^{-p}(1-w_1^t) \\ \vdots \quad & \quad \vdots \qquad \vdots \\ w_j^t &= \lambda^t (1-\lambda)^t e^{-p} \left(1 - \sum_{k<j-1} w_k^t\right). \end{aligned}$$

To find a p-conformal measure for T_λ (**NON-CONSTANT** potential shift), we need to solve (with $\tilde{w}_k^t = m_t(V_k)$)

$$\sum_k w_k^t = 1 \quad \text{subject to } w_k^t \geqslant 0 \text{ for all } k,$$

where the $\tilde{w}_k^t$ satisfy

$$\begin{aligned} \tilde{w}_1^t &= (1-\lambda)^t e^{-pS_0} \\ \tilde{w}_2^t &= \lambda^t (1-\lambda)^t e^{-pS_1} \\ \tilde{w}_3^t &= \lambda^t (1-\lambda)^t e^{-pS_2}(1-\tilde{w}_1^t) \\ \vdots \quad & \quad \vdots \qquad \vdots \\ \tilde{w}_j^t &= \lambda^t (1-\lambda)^t e^{-pS_{j-1}} \left(1 - \sum_{k<j-1} \tilde{w}_k^t\right). \end{aligned}$$

References

[AL] A. Avila, M. Lyubich, *Hausdorff dimension and conformal measures of Feigenbaum Julia sets,* J. Amer. Math. Soc. **21** (2008) 305–363.

[BFKP] O. Bandtlow, J. Fiala, P. Kleban, T. Prellberg, *Asymptotics of the Farey fraction spin chain free energy at the critical point*, Journ. Stat. Phys. **138** (2010), 447–464.

[BLL] A. Baraviera, R. Leplaideur, A. Lopes, *The potential point of view for renormalization,* Stoch. Dyn. (2011)

[BI] L. Barreira, G. Iommi, *Multifractal analysis and phase transitions for hyperbolic and parabolic horseshoes,* Israel J. Math. **181** (2011) 347–379.

[BM] M. Benedicks, M. Misiurewicz, *Absolutely continuous invariant measures for maps with flat tops,* Inst. Hautes Études Sci. Publ. Math. **69** (1989) 203–213.

[Bi] P. Billingsley, *Probability and measure,* Third edition, Wiley 1995.

[Bo] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms,* Springer Lect. Notes in Math. 470 (1975).

[B1] H. Bruin, *Invariant measures of interval maps,* Ph.D. thesis, Delft. (1994).

[B2] H. Bruin, *Topological conditions for the existence of absorbing Cantor sets,* Trans. Amer. Math. Soc. **350** (1998) 2229–2263.

[BH] H. Bruin, J. Hawkins, *Exactness and maximal automorphic factors of unimodal interval maps,* Ergodic Theory Dynam. Systems **21** (2001) 1009–1034.

[BK] H. Bruin, G. Keller, *Equilibrium states for S–unimodal maps,* Ergodic Theory Dynam. Systems **18** (1998) 765–789.

[BKNS] H. Bruin, G. Keller, T. Nowicki, S. van Strien, *Wild Cantor attractors exist,* Ann. of Math. **143** (1996) 97–130. (document)

[BLS] H. Bruin, S. Luzzatto, S. van Strien, *Decay of correlations in one-dimensional dynamics,* Ann. Sci. Ec. Norm. Sup. **36** (2003) 621–646. (document)

[BT1] H. Bruin, M. Todd, *Equilibrium states for interval maps: the potential $-t\log|Df|$,* Ann. Sci. Éc. Norm. Sup. **42** (2009), 559–600.

[BT2] H. Bruin, M. Todd, *Equilibrium states for interval maps: potentials with* $\sup\phi - \inf\phi < h_{top}(f)$, Commun. Math. Phys. **283** (2008) 579–611.

[BT3] H. Bruin, M. Todd, *Transience and thermodynamic formalism for infinitely branched interval maps,* To appear in J. Lon. Math. Soc.

[DK2] K. Dajani, C. Kraaikamp, *Ergodic Theory of Numbers,* Carus Math. Monogr., vol. 29, Mathematical Association of America, 2002.

[D] N. Dobbs, *Renormalisation-induced phase transitions for unimodal maps,* Commun. Math. Phys. **286** (2009) 377–387.

[GW] P. Gaspard, X. J. Wang, *Sporadicity: between periodic and chaotic dynamical behaviours,* Proc. Nat. Acad. Sci. USA **85** (1988) 4591–4595.

[Gu1] B. M. Gurevič, *Topological entropy for denumerable Markov chains,* Dokl. Akad. Nauk SSSR **10** (1969) 911-915.

[Gu2] B. M. Gurevič, *Shift entropy and Markov measures in the path space of a denumerable graph,* Dokl. Akad. Nauk SSSR **11** (1970) 744-747.

[H1] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy,* Israel J. Math. **34** (1979) 213-237.

[H2] F. Hofbauer, *The topological entropy of the transformation $x \mapsto ax(1-x)$,* Monatsh. Math. **90** (1980) 117–141.

[HK] F. Hofbauer, G. Keller, *Some remarks about recent results about S-unimodal maps,* Ann. Inst. Henri Poincaré, Physique Théorique **53** (1990) 413-425.

[KN] G. Keller, T. Nowicki, *Spectral theory, zeta functions and the distribution of periodic points for Collet–Eckmann maps,* Comm. Math. Phys. **149** (1992) 31–69. (document)

[IT1] G. Iommi, M. Todd, *Natural equilibrium states for multimodal maps,* Comm. Math. Phys. **300** (2010) 65–94.

[IT2] G. Iommi, M. Todd, *Transience in dynamical systems,* Preprint (arXiv:1009.2772).

[Le] F. Ledrappier, *Some properties of absolutely continuous invariant measures on an interval,* Ergodic Theory Dynam. Systems **1** (1981) 77–93. (document)

[LS] G. Levin, G. Świątek, *Dynamics and universality of unimodal mappings with infinite criticality,* Comm. Math. Phys. **258** (2005), 103–133.

[Lo] A. O. Lopes, *The zeta function, nondifferentiability of pressure, and the critical exponent of transition,* Adv. Math. **101** (1993) 133–165.

[Lu] J. Lüroth, *Ueber eine eindeutige Entwickelung von Zahlen in eine unendliche Reihe,* Math. Ann. **21** (1883) 411–423.

[LM] M. Lyubich, J. Milnor, *The unimodal Fibonacci map,* J. Amer. Math. Soc. **6** (1993) 425–457. (document)

[Ma] M. Martens, *The existence of σ-finite measures, applications to real one-dimensional dynamics,* Stony Brook Preprint, 1992–1. (document)

[MaS] N. Makarov, S. Smirnov, *On thermodynamics of rational maps, II. Non-recurrent maps,* J. London Math. Soc. **67** (2003), 417–432.

[MoS] C. G. Moreira, D. Smania, *Metric stability for random walks (with applications in renormalization theory),* Preprint, 2005 (updated 2009).

[MU] R. Mauldin, M. Urbański, *Dimensions and measures in infinite iterated function systems,* Proc. London Math. Soc. (3) **73** (1996) 105–154.

[NS] T. Nowicki, S. van Strien, *Invariant measures exist under a summability condition for unimodal maps,* Invent. Math. **105** (1991) 123–136.

[PS] Y. Pesin, S. Senti, *Equilibrium measures for maps with inducing schemes,* J. Mod. Dyn. **2** (2008) 1–31.

[PFK] T. Prellberg, J. Fiala, P. Kleban, *Cluster approximations for the Farey fraction spin chain*, Journ. Stat. Phys. **123** (2006), 455–471.

[RS] J. Rivera-Letelier, W. Shen, Personal communications and *Statistical properties of one-dimensional maps under weak hyperbolicity assumptions,* Preprint 2011, http://arxiv.org/abs/1004.0230 (document)

[S1] O. Sarig, *Thermodynamic formalism for null recurrent potentials,* Israel J. Math. **121** (2001) 285–311.

[S2] O. Sarig, *Thermodynamic formalism for countable Markov shifts,* Ergodic Theory Dynam. Systems **19** (1999) 1565–1593.

[S3] O. Sarig, *Lecture Notes on Thermodynamic Formalism for Topological Markov Shifts,* (http://www.wisdom.weizmann.ac.il/∼sarigo/TDFnotes.pdf).

[SV] B. Stratmann, R. Vogt, *Fractal dimension of dissipative sets,* Nonlinearity **10** (1997) 565–577. (document)

[Z] R. Zweimüller, *S-unimodal Misiurewicz maps with flat critical points,* Fund. Math. **181** (2004) 1–25.