Interval Translation Maps with Weakly Mixing Attractors

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based on a joint work with Silvia Radinger

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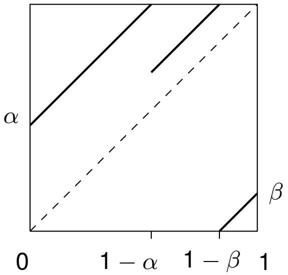
Renormalization of Interval Translation Maps

ITM introduced by Bruin & Troubetzkoy in 2003

$$T_{\alpha,\beta}(x) = \begin{cases} x+\alpha, & x \in [0, 1-\alpha), \\ x+\beta, & x \in [1-\alpha, 1-\beta), \\ x-1+\beta, & x \in [1-\beta, 1] \end{cases}$$

on the parameter space $U = \{(\alpha, \beta) : 0 < \beta \leq \alpha \leq 1\}$.



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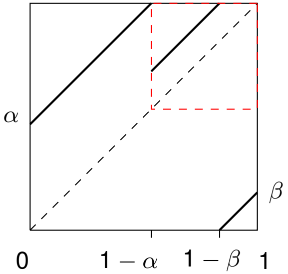

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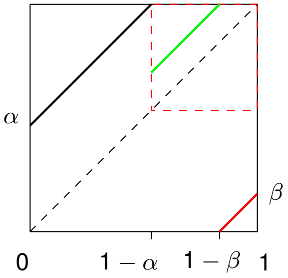

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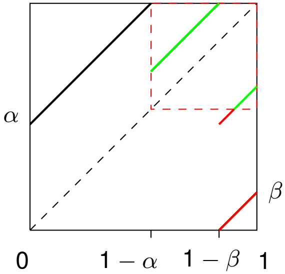

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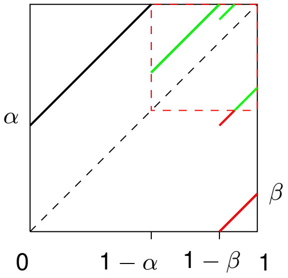

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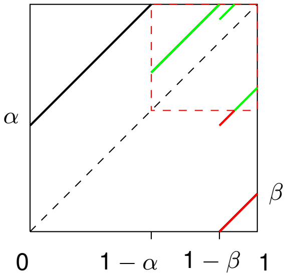
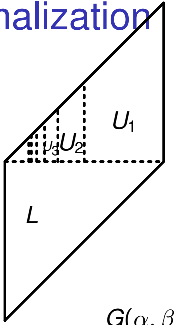
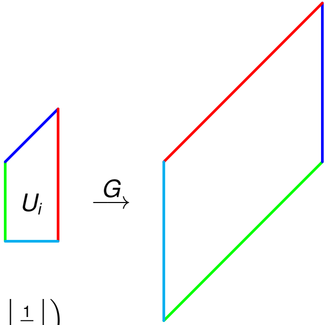


Renormalization transforms $T_{\alpha,\beta}$ into $T_{\alpha',\beta'}$ with:

$$(\alpha', \beta') = G(\alpha, \beta) = \left(\frac{\beta}{\alpha} \, , \; \frac{\beta - 1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor \right).$$

Renormalization





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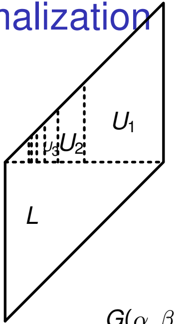
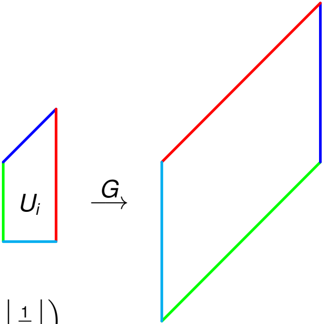




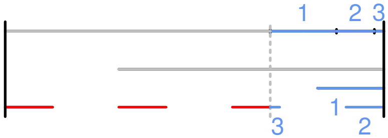
$$G(\alpha, \beta) = \left(\frac{\beta}{\alpha} \, , \, \frac{\beta - 1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor \right)$$

Types of parameters

- **Finite Type:** $G^n(\alpha, \eta) \notin U^\circ$ for some $n \geq 1$. Then $T_{\alpha,\beta}$ reduces to an interval exchange transformation.
- **Infinite Type:** $G^n(\alpha, \eta) \in U$ for all $n \geq 1$. Then $\Omega := \bigcap_{n \geq 0} \overline{T^n_{\alpha,\beta}([0,1])}$ is a Cantor set with $T_{\alpha,\beta}$ a minimal endomorphism.

The set of parameters (α, β) with $T_{\alpha,\beta}$ is of infinite type has Lebesgue measure zero but positive Hausdorff dimension.

S-adic Subshift



Symbolically, one renormalization step is given by the substitution

$$\chi_k : \begin{cases} 1 \to 2 \\ 2 \to 31^k \\ 3 \to 31^{k-1} \end{cases} \qquad \text{for } k = \left\lfloor \frac{1}{\alpha} \right\rfloor \in \mathbb{N}$$

with unimodular incidence matrix

$$A_k = \begin{pmatrix} 0 & k & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \det(A_k) = -1.$$

We define a S-adic subshift based on a sequence of substitutions χ_{k_i}, $k_i \in \mathbb{N}$. The itinerary of the point 1 is

$$\rho = \lim_{i \to \infty} \chi_{k_1} \circ \chi_{k_2} \circ \chi_{k_3} \circ \cdots \circ \chi_{k_i}(3).$$

Subshift X is the closure of $\{\sigma^n(\rho)\}_{n \in \mathbb{N}}$ where σ is the left-shift.

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Subshift X is the closure of $\{\sigma^n(\rho)\}_{n \in \mathbb{N}}$ where σ is the left-shift.

Every *ITM of infinite type* in this family is uniquely characterised by a sequence $(k_i)_{i \in \mathbb{N}} \subset \mathbb{N}$ such that

$k_{2i} > 1$ for infinitely many $i \in \mathbb{N}$ and $k_{2j-1} > 1$ for infinitely many $j \in \mathbb{N}$.

Ergodic Properties

Proposition

The S-adic subshift (X, σ), based on substitutions $(\chi_{k_i})_{i \in \mathbb{N}}$ from an ITM of infinite type, is

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- minimal;
- uniquely ergodic if $\liminf_i k_i < \infty$;
- not uniquely ergodic if $\liminf_i k_{i+1}/k_i > 1$;
- linearly recurrent if $(k_i)_i$ and the blocks of 1s are bounded.

Weak Mixing

Definition

A system (X, T) is called *weakly mixing* if the Koopman operator

$$U_T(f) = f \circ T$$

has 1 as its only eigenvalue.

If an eigenfunction f is

- in L^2, then its eigenvalue is called *measurable*,
- continuous, then its eigenvalue is called *continuous*.

Eigenvalue Conditions

Conditions for (non)existence of eigenvalues go back to Host and Veech.

- Typical weak mixing for interval exchange transformations: Nogueira & Rudolph, Sinaĭ & Ulcigrai, Avila & Forni.
- Conditions for Bratteli-Vershik systems by Bressaud, Durand, Frank, Maass,... several papers.
- Can more or less be translated into behaviour of the unimodular incidence matrices A_{k_i}, interpreted as toral automorphisms: There is an eigenvalue

$$\vec{t} A_{k_1} A_{k_2} \cdots A_{k_n} \bmod 1 \to 0, \qquad \text{for some } \vec{t} = (t, t, t) \neq \vec{0}.$$

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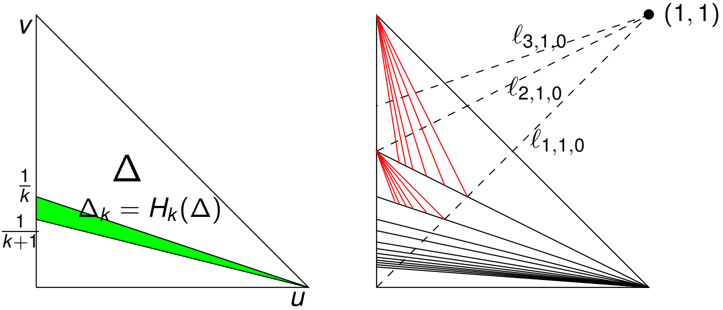
Lemma

The dynamics of $z \mapsto A_{k_1} A_{k_2} \cdots A_{k_n} z \bmod 1$ *has one stable and two unstable directions.*

Subtractive Algorithm

Determining whether $\vec{t}A_{k_1}A_{k_2}\cdots A_{k_n} \bmod 1 \to 0$ reduces to a problem for a "subtractive algorithm" on the simplex $\Delta = \{0 \le u \le u+v \le 1\}$:

$$H_k : (u,v) = \frac{1}{D_k}(v\ ,\ 1-v) \quad \text{for} \quad D_k = k(1-v)+1-u,$$



We have $\vec{t}A_{k_1}A_{k_2}\cdots A_{k_n} \bmod 1 \to 0$ "if and only if"

$(u,v) \in \Delta$ lies on a line $\ell_{p,q,r} = \{u(q-r) = v(p-r)+q-p\}$.

Subtractive Algorithm

Facts about the subtractive algorithm $H : \Delta \to \Delta$:

$$H(x,y) = H_k^{-1}(x,y) \mapsto \left(\frac{x+(k+1)y-1}{x+y}, \frac{x}{x+y} \right) \quad \text{if } (u,v) \in \Delta_k.$$

- H is *full Markov*: $H(\Delta_k) = \Delta$ for all k.
- the line $L = \{x+y=1\}$ consists of neutral period 2 points (and $(\frac{1}{2}, \frac{1}{2})$ is fixed).
- every rational point in Δ is eventually mapped into L (finite type case).
- elsewhere H is eventually expanding (in Hilbert metric): every infinite type $(k_i)_{i \geq 1}$ is realized by exactly one point $(x,y) \in \Delta^\circ$.

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Question: Does H preserve an (infinite) invariant measure $\ll Leb$?

Results (weak mixing)

Theorem

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Every infinite type sequence $(k_i)_{i\geq 1}$ such that the corresponding $(u, v) \notin \bigcup_{p,q,r} \ell_{p,q,r}$ and $\liminf_i k_i < \infty$ corresponds to a weakly mixing ITM.

Results (eigenvalues)

Recall that every infinite type sequence $(k_i)_{i\geq 1}$ corresponds to a unique $(u, v) \in \Delta$.

Theorem

Almost every (w.r.t. 1-dim. Lebesgue) infinite type sequence $(k_i)_{i\geq 1}$ such that $(u, v) \in \ell_{p,q,r}$ corresponds to an ITM with a continuous eigenvalue $e^{2\pi it}$ of the Koopman operator.

However, for every $p, q, r \in \mathbb{N}$ there exist parameters

$$(u, v) = (u(t), v(t)) \in \ell_{p,q,r}$$

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Question: Do there exists ITMs in this family where the eigenvalue is measurable but not continuous?

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Measurable Eigenvalues

Conditions for measurable eigenvalues are more difficult to compute:

Theorem (Durand, Frank, Maass in 2019)

There is a sequence of functions $\rho_n : V_{n+1} \to \mathbb{R}$ such that

$$g_n(x) := t\left(\tilde{S}_n(x) + \rho_n(w)\right) \bmod 1$$

converges for μ-a.e. $x \in X_{BV}$ as $n \to \infty$,

where $\tilde{S}_n(x) = \sum_{j=1}^{n} \langle \tilde{s}_j(x), h_j(v) \rangle$ is the minimal number of steps to the base of a tower.

Theorem

If $\liminf_n k_n < \infty$ and $\vec{t} \notin W^s$, then the corresponding ITM is weakly mixing.

It is an open question if there are ITMs with measurable and non-continuous eigenvalues.

Results for Continuous Eigenvalues

Theorem (Host in 1986)

For a primitive substitution system a sufficient condition to have an eigenvalue $e^{2\pi it}$ for some $t \in (0,1)$ is

$$\sum_{n=1}^{\infty} \|\vec{t}A^n\| < \infty, \qquad \vec{t} = (t,t,t),$$

where $\|x\|$ is the distance of a vector to the nearest integer lattice point.

This condition was later expanded to hold for linearly recurrent S-adic shifts and their continuous eigenvalues:

$$\sum_{n=1}^{\infty} \|\vec{t}\tilde{A}_1 \cdots \tilde{A}_n\| < \infty, \qquad \vec{t} = (t,t,t),$$