Interval Translation Maps with Weakly Mixing Attractors of Interval translation Maps

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based on a joint work with Serge Troubetzkoy and with Silvia Radinger

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Dynamical Systems

The simplest form of a **dynamical system** is the iteration of a map $T : X \to X$. That is, we look at **orbits**:

$$\text{orb}(x) = \{x, T(x), T \circ T(x), \dots, T^n(x) := \underbrace{T \circ \cdots \circ T(x)}_{n \text{ times}}, \dots\}.$$

Sometimes orbits are simple, like fixed points $T(x) = x$ or periodic points $T^p(x) = x$ for period $p \in \mathbb{N}$, or asymptotic to e.g. fixed points: $\lim_{n\to\infty} T^n(x) = y = T(y)$.

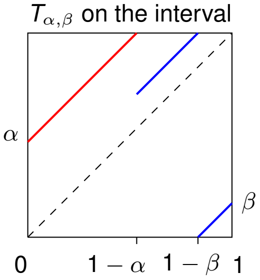
Most of the time orbits are complicated and erratic (chaotic). How to understand all (or at least most) orbits?

Interval Translation Maps

Our dynamical system will be an **interval translation map** (ITM) introduced by Bruin & Troubetzkoy in 2003:

$$T_{\alpha,\beta}(x) = \begin{cases} x+\alpha, & x \in [0, 1-\alpha), \\ x+\beta, & x \in [1-\alpha, 1-\beta), \\ x-1+\beta, & x \in [1-\beta, 1] \end{cases}$$

on the parameter space $U = \{(\alpha, \beta) : 0 < \beta \le \alpha \le 1\}$.



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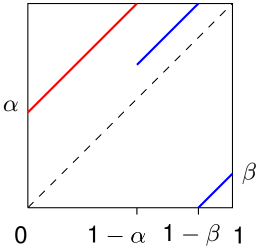
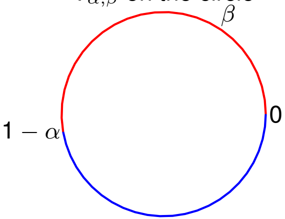
$T_{\alpha,\beta}$ on the interval



$T_{\alpha,\beta}$ on the circle



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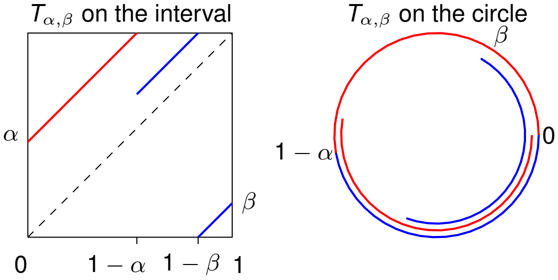
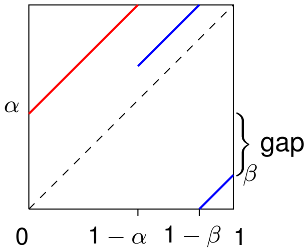


Interval Translation Maps

Note that $T_{\alpha,\beta}$ is usually **not** continuous, **not** one-to-one and **not** onto: $T([0,1]) \subsetneq [0,1]$.



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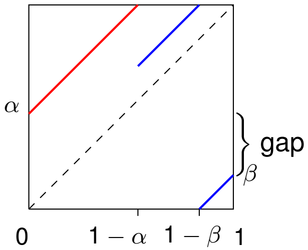


Set $I_0 = [0,1]$ and $I_n = \overline{T(I_{n-1})}$ for $n \geq 1$. Then

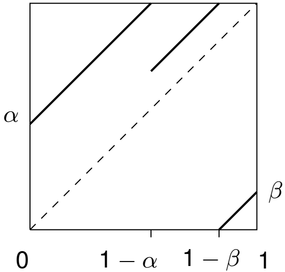
$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_\infty := \bigcap_{n \geq 0} I_n.$$

We call the set I_∞ the **attractor** of $T_{\alpha,\beta}$.

What kind of set is I_∞ and what is the dynamics on it?

Renormalization of Interval Translation Maps

We look at the **first return map** of $T_{\alpha,\beta}$ to the subinterval $[1-\alpha, 1]$.



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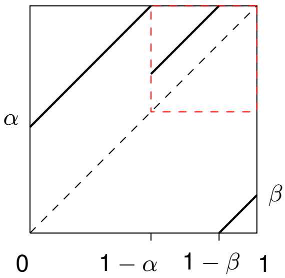

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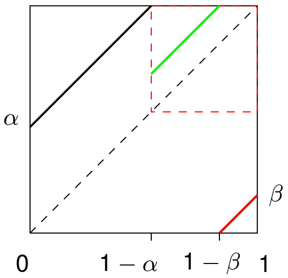

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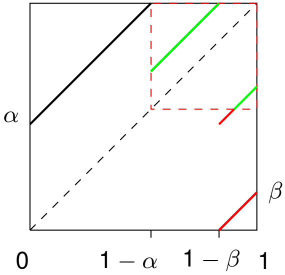

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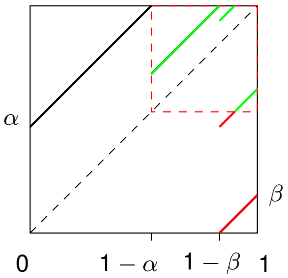

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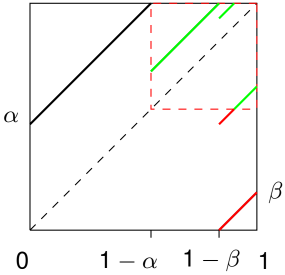
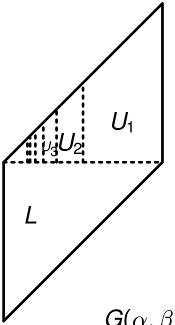
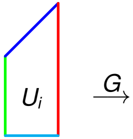


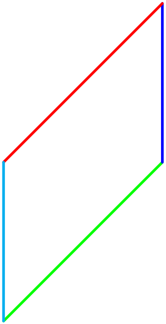
Renormalization transforms $T_{\alpha,\beta}$ into $T_{\alpha',\beta'}$ with:

$$(\alpha', \beta') = G(\alpha, \beta) = \left(\frac{\beta}{\alpha}\ ,\ \frac{\beta-1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor\right).$$

Renormalization







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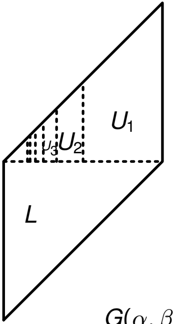

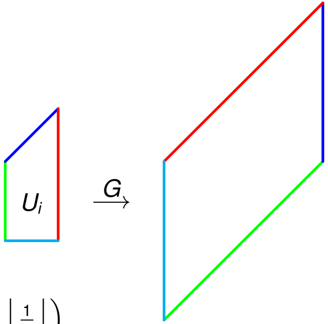

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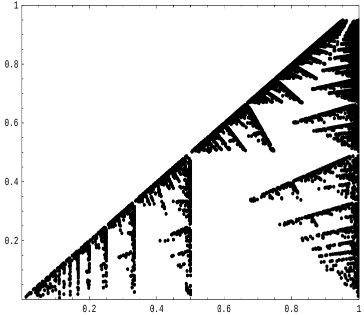
Types of parameters

- **Finite Type:** $G^n(\alpha, \beta) \notin U^\circ$ for some $n \geq 1$. Then I_∞ is a finite union of intervals and $T_{\alpha,\beta}$ reduces to a circle rotation.
- **Infinite Type:** $G^n(\alpha, \beta) \in U$ for all $n \geq 1$. Then I_∞ is a Cantor set.

Renormalization

Approximation of the set Ω of parameters (α, β) with $T_{\alpha,\beta}$ of infinite type $(10,000$ pixels).



The set Ω has Lebesgue measure zero but positive Hausdorff dimension.

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Every renormalization step gives an integer $k = \lfloor \frac{1}{\alpha} \rfloor$.

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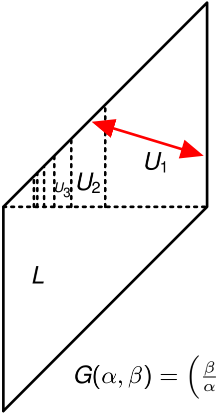

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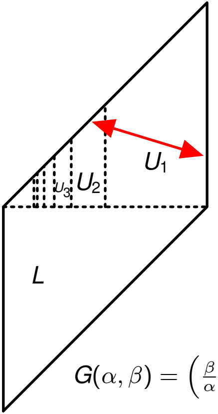

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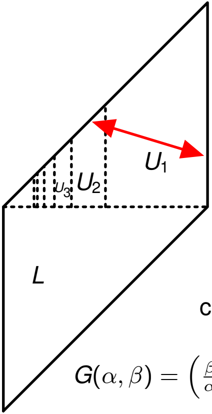

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Such parameters have $k_i = 1$ for every odd or every even i. This is the only restriction

Every sequence $(k_i)_{igeq1}$ satisfying:

$k_{2i} = 1$ and $k_{2i-1} = 1$ for only finitely many i

corresponds to a unique parameter of infinite type

and vice versa.

$$G(\alpha, \beta) = \left(\frac{\beta}{\alpha} , \frac{\beta - 1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor \right)$$

Invariant measures

Orbits can be described statistically by means of invariant measures:

Definition: A measure μ on a space X is a σ-additive function

$$\mu : \{\text{Borel sets}\} \to [0, 1]$$

such that $\mu(\emptyset) = 0$, $\mu(X) = 1$. μ is called **T-invariant** if

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Birkhoff's Ergodic Theorem

Let T be a transformation of a compact metric space X and μ an ergodic T-invariant measure. Then for every $f : X \to \mathbb{R}$ continuous,

$$\underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j(x)}_{\text{time average}} \quad = \quad \underbrace{\int_X f \, d\mu}_{\text{space average}}$$

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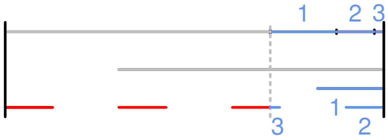
for all x except for a set of μ-measure zero. If there is only one T-invariant measure (T is **uniquely ergodic**), then this holds for **all** $x \in X$ and the convergence is uniform (Oxtoby's Theorem).



Symbolically, one renormalization step is given by the substitution

$$\chi_k : \begin{cases} 1 \to 2 \\ 2 \to 31^k \\ 3 \to 31^{k-1} \end{cases} \qquad \text{for } k = \left\lfloor \frac{1}{\alpha} \right\rfloor \in \mathbb{N}$$

with unimodular incidence matrix

$$A_k = \begin{pmatrix} 0 & k & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \det(A_k) = -1,$$

This matrix indicates how many letters a there are in $\chi_k(b)$.

Recap

Every *ITM of infinite type* in this family is uniquely characterised by a sequence $(k_i)_{i \in \mathbb{N}} \subset \mathbb{N}$ such that

$$k_{2i} = 1 \text{ and } k_{2i-1} > 1 \text{ for only finitely many } i \in \mathbb{N}.$$

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We define a (so-called **S-adic) subshift** based on the sequence of substitutions χ_{k_i}, $k_i \in \mathbb{N}$. The itinerary of the point 1 is

$$\rho = \lim_{i\to\infty} \chi_{k_1} \circ \chi_{k_2} \circ \chi_{k_3} \circ \cdots \circ \chi_{k_i}(3).$$

The (left-)shift σ removes the first symbols and moves the other symbols one lace to the left:

$$\rho = \rho_1\rho_2\rho_3\rho_4\ldots \qquad \sigma(\rho) = \rho_2\rho_3\rho_4\ldots$$

The subshift X is the closure of $\{\sigma^n(\rho)\}_{n\in\mathbb{N}}$ where σ.

Unique ergodicity

Birkhoff's ergodic theorem implies that each shift-invariant measure μ determines fixed "frequency" of letters $a \in \{1, 2, 3\}$:

$$v_a(x) = \lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n : x_j = a\}$$

and the same for frequencies of blocks. Let $\vec{e}_b$, $b = 1, 2, 3$, be the unit vectors in $\mathbb{R}^3$. Then

$$v_a(\rho) = \left(\lim_{n\to\infty} \frac{A_1 \cdot A_2 \cdots A_n \vec{e}_3}{\|A_1 \cdot A_2 \cdots A_n \vec{e}_3\|} \right)_a.$$

Lemma

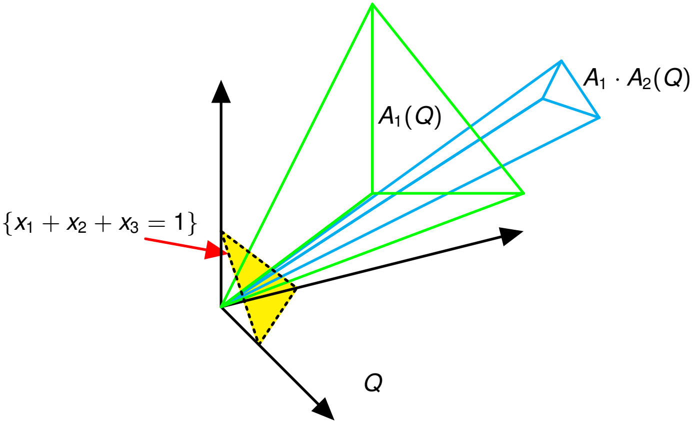
Let $Q = [0, \infty)^3$ be the positive octant.

The symbolic shift (Σ_ρ, σ) is uniquely ergodic if and only if

$$\bigcap_{n\geq 1} A_1 \cdot A_2 \cdots A_n(Q) \text{ is a single line } \ell.$$

The frequency vector $\vec{v}(\rho)$ is the intersection $\ell \cap \{x_1 + x_2 + x_3 = 1\}$.

The task is now to (find conditions to) ensure that the matrices A_k squeeze the positive octant to a single line.



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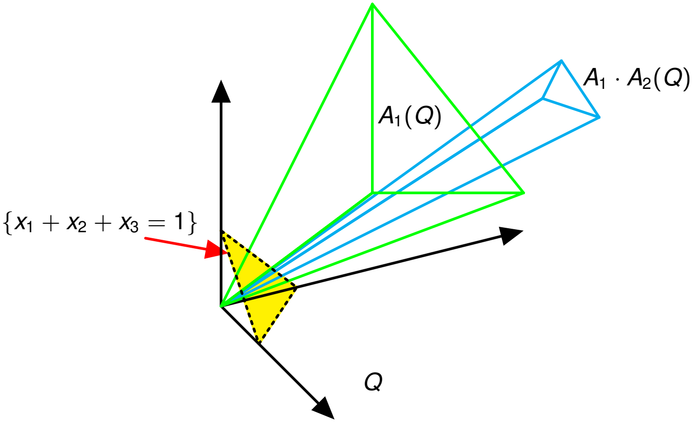

If all A_k were the positive and the same (or just bounded), then this would follow from the Perron-Frobenius Theorem.
But if the A_k increase too fast, then $\bigcap_{n \geq 1} A_1 \cdot A_2 \cdots A_n(Q)$ can be more than a line.

Unique ergodicity

We solve the problem using Hilbert semi-metric - in this metric the matrices are contractions, but the contraction factors $r_k < 1$ depend on A_k. Under certain condition $\prod_{k=1}^{\infty} r_k = 0$, and this assures that $\bigcap_{n\geq 1} A_1 \cdot A_2 \cdots A_n(Q)$ is a single line, and unique ergodicity follows.

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Theorem

Let $(k_i)_{i\geq 1}$ be the sequence corresponding to a parameter (α, β) of infinite type.

- If $\liminf_i k_i < \infty$ then $T_{\alpha,\beta}$ is uniquely ergodic.
- If $k_{i+1} \geq \lambda k_i$ for some $\lambda > 1$ and all i sufficiently large, then $T_{\alpha,\beta}$ is not uniquely ergodic.