Rarity of one-sided Bernoulli systems

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Definition 0.1. Let \( n \geq 2 \) and let \( \mathcal{A} = \{1, \ldots, n\} \) be the alphabet.
Let \( p = \{p_1, \ldots, p_n\}, p_k > 0 \) be a probability vector.
Let \( \Omega = \mathcal{A}^\mathbb{Z} \) or \( \mathcal{A}^\mathbb{N} \) and let \( \mathcal{D} \) be the \( \sigma \)-algebra generated by cylinder sets.
Let \( \rho \) be the product measure determined by \( p \).
Let \( \sigma \) is the left-shift.
Then \((\Omega, \mathcal{D}, \rho; \sigma)\) is the two-sided (respectively one-sided) Bernoulli shift.
**Definition 0.2.** An *isomorphism* $\psi$ between $(X_1, \mathcal{B}_1, \mu_1; T_1)$ and $(X_2, \mathcal{B}_2, \mu_2; T_2)$ is a measurable a.e.-bijection such that

\[ (X_1, \mathcal{B}_1, \mu_1) \xrightarrow{T_1} (X_1, \mathcal{B}_1, \mu_2) \]

\[ \downarrow \psi \hspace{1cm} \downarrow \psi \]

\[ (X_2, \mathcal{B}_2, \nu) \xrightarrow{T_2} (X_2, \mathcal{B}_2, \mu_2) \]

commutes.

More precisely
- There are $Y_1 \subset X_1$, $Y_2 \subset X_2$ of full measure such that $\psi : Y_1 \to Y_2$ is a bijection.

- $T_2 \circ \psi = \psi \circ T_1$ for all $x \in Y_1$.

- $\psi^{-1}B \in \mathcal{B}_1$ and $\mu_1(\psi^{-1}B) = \mu_2(B)$ for all $B \in \mathcal{B}_2$. 
The Two-sided Bernoulli Property.

Definition 0.3. A invertible measure preserving transformation \((X, \mathcal{B}, \mu; T)\) is Bernoulli if it is isomorphic to a two-sided Bernoulli shift.

For two-sided Bernoulli shifts, and hence, invertible measure preserving transformations, entropy is a complete invariant.

For one-sided Bernoulli shifts, entropy is an invariant, but not a complete invariant.
Noninvertible Bernoulli Properties.

Let \((X, \mathcal{B}, \mu; T)\) be a non-invertible measure preserving transformation. There are several ways of relating it to Bernoulli shifts.

(a) The natural extension is Bernoulli.

(b) \((X, \mathcal{B}, \mu; T)\) is weakly Bernoulli.

(c) \((X, \mathcal{B}, \mu; T)\) is one-sided Bernoulli, i.e., isomorphic to a one-sided Bernoulli shift.

The implications are as follows:

\[(c) \Rightarrow (b) \Rightarrow (a)\]

but the reverse implications are both false.
Definition 0.4. Let $(X, \mathcal{B}, \mu; T)$ be a measure preserving endomorphism. Let $\zeta = \{P_1, P_2, \cdots\}$ and $\eta = \{Q_1, Q_2, \cdots\}$ be partitions. The partition $\zeta$ is independent of $\eta$ if
\[
\sum_{i,j} |\mu(P_i \cap Q_j) - \mu(P_i)\mu(Q_j)| = 0
\]
and $\varepsilon-$independent of $\zeta$ if
\[
\sum_{i} \sum_{j} |\mu(P_i \cap Q_j) - \mu(P_i)\mu(Q_j)| \leq \varepsilon.
\]
A partition $\zeta$ is weak Bernoulli if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \geq 1$,
\[
\bigvee_{0}^{m} T^{-i}\zeta \text{ is } \varepsilon-\text{independent of } \bigvee_{N}^{N+m} T^{-i}\zeta,
\]
and $(X, \mathcal{B}, \mu; T)$ is weakly Bernoulli if it has a generating weak Bernoulli partition.
**Theorem 0.5** (Friedman & Ornstein). If \((X, \mathcal{B}, \mu; T)\) be an invertible measure preserving system, and \(\eta\) is a weak-Bernoulli partition such that

\[
\zeta_{-\infty}^{\infty} \equiv \bigvee_{i=-\infty}^{\infty} T^{-i}(\zeta)
\]

generates \(\mathcal{B}\), then \(T\) is isomorphic to a two-sided Bernoulli shift.

Therefore, if a measure preserving endomorphism is weakly Bernoulli, its natural extension is two-sided Bernoulli.

Among endomorphism shown to be weakly Bernoulli are:

- Toral endomorphisms (Adler & Smorodinski, 1972)

- Various interval maps with acips (Ledrappier, 1981)

- Equilibrium states for rational maps of \(\overline{\mathbb{C}}\) with supremum gap (Haydn, 2000)
\textbf{\textit{n-to-one Endomorphisms.}}

Due to Rohlin (1952) a \textbf{\textit{n-to-one}} (for \( n = 2, 3, \ldots, \aleph_0 \)) measure preserving endomorphism \((X, \mathcal{B}, \mu; T)\) has a proper factor \((Y, T^{-1}\mathcal{B}, \nu; T)\) with factor map \(\varphi\) such
\[
\begin{array}{cccc}
(X, \mathcal{B}, \mu) & \xrightarrow{T} & (X, \mathcal{B}, \mu) \\
\downarrow \varphi & & \downarrow \varphi \\
(Y, T^{-1}\mathcal{B}, \nu) & \xrightarrow{T} & (Y, T^{-1}\mathcal{B}, \nu)
\end{array}
\]
commutes, where \(\nu = \mu|_{T^{-1}\mathcal{B}}\).

Thus we can decompose
\[
\mu(B) = \int_Y \mu_y(B) d\nu(y)
\]
where for \(\nu\)-a.e. \(y \in Y\) \(\mu_y = \mu|_{T^{-1}x}\) is a measure that is nonsingular for \(T\), purely atomic (since \(T\) is at most countable-to-one), and its support is contained in the set of points \(\{T^{-1}x\}\) such that \([y] = [T^{-1}x]\).
The Index of a Point.

**Definition 0.6.** For a nonsingular endomorphism $T$, the *index function* (or index) $\text{ind}_T(x)$ is defined to be, $(\mu \mod 0)$, the cardinality of the support of $\mu_{[T^{-1}x]} = \mu_{[\varphi(x)]]}$ for $x \in X$.

If $(X, \mathcal{B}, \mu; T)$ is one-sided Bernoulli, then the index is constant $n$.

Moreover, the Jacobians

$$J(x) = \frac{d\mu \circ T}{d\mu}(x) \in$$

satisfy

$$\{J_{\mu T}(y)\}_{y \in \text{supp}(\mu_{[T^{-1}x]})} = \{1/p_1, 1/p_2, \ldots, 1/p_n\}.$$ for $\mu$-a.e. $x$.

*Figure 1.* The map $T(x) = |\min\{3x - 1, 2 - 3x\}|$ preserves an acip $\mu$ with $\frac{d\mu}{dm} = \frac{4}{3}$ on $[0, \frac{1}{2})$ and $\frac{d\mu}{dm} = \frac{2}{3}$ on $(\frac{1}{2}, 1]$. $T$ is bounded-to-one w.r.t. Lebesgue, but 2-to-1 w.r.t. Hausdorff measure supported on the middle thirds Cantor set.
Rohlin Partitions An bounded-to-one measure preserving endomorphisms $(X, \mathcal{B}, \mu; T)$ has a ordered partition $\zeta = \{A_1, A_2, A_3, \ldots \}$ satisfying:

1. $\mu(A_i) > 0$ for each $i$;

2. the restriction of $T$ to each $A_i$, which we will write as $T_i$, is one-to-one ($\mu \text{ mod } 0$);

3. each $A_i$ is of maximal measure in $X \setminus \bigcup_{j<i} A_j$ with respect to property 2;

4. $T_1$ is one-to-one and onto $X$ ($\mu \text{ mod } 0$) by numbering the atoms so that

$$\mu(T A_i) \geq \mu(T A_{i+1})$$

for $i \in \mathbb{N}$. 
Non-uniqueness of Rohlin Partitions.

- For the angle doubling map (preserving Lebesgue measure), any partition
  \[ \zeta_t = \{ A_0 = [0, t) \cup (t + \frac{1}{2}, 1], \quad A_1 = [t, t + \frac{1}{2}) \} \]
  is a Rohlin partition.

- \( \zeta_t \) generates \( B \) for all \( t \in (0, \frac{1}{2}) \) except \( t = \frac{1}{4} \).

- The coding map \( \pi_t \) is surjective but not injective. for all \( t \in (0, \frac{1}{2}) \).
  For \( t = 0 \), \( \pi_t \) is injective, but no point has code 111...).
  For the map \( T_{p,t} \) below, Lebesgue measure is one-sided \( \{p, 1-p\}\)-Bernoulli, except for \( t = \frac{1}{4} \).

\[ \text{Figure 2. The map } T_{p,t} \text{ is not one-sided Bernoulli for } t = \frac{1}{4} \text{ (left) but it is for e.g. } t = \frac{3}{20} \text{ (right).} \]
Theorem 0.7. Suppose $p \neq \frac{1}{2}$:

(1) Let $\sigma$ on $(\Omega, \rho)$ be the one-sided $\{p, 1-p\}$ Bernoulli shift. Then there exists no nontrivial nonsingular automorphism $\varphi : (\Omega, \rho) \to (\Omega, \rho)$ with $\varphi \circ \sigma = \sigma \circ \varphi (\mu \mod 0)$.

(2) If $T$ on $(X, \mathcal{B}, \mu)$ is a one-sided $\{p, 1-p\}$ Bernoulli endomorphism, then there is no nontrivial nonsingular commuting automorphism $\varphi : (X, \mu) \to (X, \mu)$.

Corollary 0.8 (Parry). Suppose $(X, \mathcal{B}, \mu; T)$ is a measure preserving 2-to-one endomorphism. If there exists a nontrivial nonsingular automorphism $\varphi$ commuting with $T$, then $T$ is not isomorphic to a one-sided $\{p, 1-p\}$ Bernoulli shift.

Figure 3. $T(x) = 2x + \varepsilon \sin 4\pi x$ preserves an acip $\mu$ but is not one-sided Bernoulli, because of its symmetry $x \mapsto 1-x$. 


A Livsič-like Result

**Theorem 0.9.** Let $T : I = [0, 1] \to I$ be a piecewise $C^2$ $n$-to-1 map and assume $T$ preserves a probability measure $\mu \sim m$.

Assume that the Radon-Nikodym derivative

$$g(x) = \frac{d\mu}{dm}$$

is continuous and bounded away from 0.

Then $T$ is one-sided Bernoulli on $(I, \mathcal{B}, m)$ if and only if $T$ is $C^1$-conjugate to a map $S : I \to I$ whose graph consists of $n$ linear pieces, with slopes $\pm \frac{1}{p_i}$ such that $h_\mu(T) = - \sum_{i=1}^{n} p_i \log p_i$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Commutative diagram to construct $\Psi = \psi \circ \pi^{-1}$.}
\end{figure}
Applications to Postcritically Finite Maps:

Any degree $n$ Chebyshev system is Bernoulli.

Let $T : \mathbb{C} \to \mathbb{C}$ be the rational map associated to the Newton algorithm for finding the roots of the equation $z^d - 1 = 0$:

$$T(z) = z - \frac{z^d - 1}{dz^{d-1}} = \frac{(d - 1)z^d + 1}{dz^{d-1}}.$$

Then $T$ preserves a measure $\mu \ll m_t$, where $t = \dim_H(\mathcal{J})$ and $m_t$ is $t$-conformal measure.

The dihedral group $\mathcal{G}$ generated by $z \mapsto e^{2\pi i/d}z$ and $z \mapsto \overline{z}$ is the group of symmetries of $\mathcal{J}$, which also transitively permutes the atoms of the Rohlin partition $\{A_1, \ldots, A_d\}$.

The system $(\mathcal{J}, \mathcal{B}, \mu; T)$ is not one-sided Bernoulli.
Selected references.

References