Hubbard Trees and Symbolic Dynamics for Quadratic Polynomials.

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Iteration of Quadratic Maps

$$f_c : z \mapsto z^2 + c, \qquad z, c \in \mathbb{C}.$$

Depending on the behaviour of the orbits

$$\text{orb}(z) := \{z, f_c(z), \underbrace{f_c^2(z)}_{f_c \circ f_c(z)}, f_c^3(z), \dots\},$$

we divide the complex plane into

- the **Fatou set**: regular behaviour.
  a neighbournood of z is
  - asymptoticially periodic,
  - goes of to ∞, or
  - behaves as an irrational rotation.
- the **Julia set**: chaotic behaviour.

Properties of the Julia Set

For polynomials $z \mapsto z^2 + c$, the Julia set is

$$\begin{aligned} J_c &= \overline{\{\text{repelling periodic points}\}} \\ &= \partial\{z \in \mathbb{C} : f_c^n(z) \to \infty\} \\ &= \partial\underbrace{\{z \in \mathbb{C} : \text{orb}(z) \text{ is bounded}\}}_{\text{filled-in Julia set } K_c} \end{aligned}$$

The Julia set is compact, fully invariant, *i.e.*,

$$f_c(J_c) = f_c^{-1}(J_c) = J_c$$

and self-similar: For every open U intersecting J_c there is n such that $f_c^n(U) \supset J_c$.

The Mandelbrot Set

The critical point of f_c is $z = 0$.

$$J_c \text{ is } \begin{cases} \text{connected} & \text{if orb}(0) \text{ is bounded,} \\ \text{a Cantor set} & \text{if } f_c^n(0) \to \infty. \end{cases}$$

The **Mandelbrot set** is

$$\mathcal{M} = \{c \in \mathbb{C} : J_c \text{ is connected}\}.$$

N.B.

$\mathcal{M} \subset$ parameter space.
$J_c \subset$ dynamical space.

Hubbard Trees

A **Hubbard tree** is a tree T with map $f : T \to T$ and a single **critical point** 0, such that:

1. $f : T \to T$ is continuous and surjective;

2. every $z \in T$ has at most two preimages;

3. at every $z \neq 0$, the map f is a local homeomorphism onto its image;

4. all endpoints of T are on the critical orbit;

5. the critical point is (pre)periodic, but not fixed;

6. (expansivity) if $x \neq y$ are branch points or points on the critical orbit, then there is an $n \geq 0$ such that $f^n([x, y]) \ni 0$.

Theorem 1. *Every prepriodic or $*$-periodic sequence has a Hubbard tree, but not all of them correspond to a quadratic polynomial.*

Symbolic Dynamics

Divide the Hubbard tree into

$$T = \underbrace{T_1}_{\ni f(0)} \cup \underbrace{T_*}_{= \{0\}} \cup T_0$$

The **itinerary** of z is

$$e(z) = e_0 e_1 e_2 \text{ with } e_k = a \text{ if } f^k(z) \in T_a.$$

The **kneading sequence** ν is the itin. of $f(0)$.

Reverse Question:

Given a sequence $\nu \in \{0,1\}^{\mathbb{N}}$

- Is it the kneading sequence of some HT?
- If so how to construct it?
- Combinatorial properties of the HT? (branchpoints, their itineraries, number of arms, relative positions)
- Is it the kneading sequence of a quadratic polynomial?

The ρ-function

Given $\nu, e \in \{0,1\}^{\mathbb{N}}$ define

$$\rho(n) = \min\{m > n : \nu_m \neq \nu_{m-n}\}$$

and

$$\rho(n) = \min\{m > n : e_m \neq \nu_{m-n}\}$$

Facts:

- $\text{orb}_\rho(1) =$ internal address.
- If ζ_k is a **closest precritical point (ccp)** with $f^k(\zeta_k) = c_1$, then the next ccp on $[f(0), \zeta_k]$ is $\zeta_{\rho(k)}$.
- The first ccp on $[f(0), f^{k+1}(0)]$ is $\zeta_{\rho(k)-k}$.
- If $e = e(z)$, then the number of arms of z in J_c equals the number of disjoint ρ_e-orbits.

Evil Branchpoints

A branchpoint is called **evil** if it is n-periodic but f^n doesn't permute its arm cyclically.

For the **characteristic** (*i.e.,* closest to $f(0)$) evil branchpoint, f^n fixes the arms towards 0 and permutes the other arms cyclically.

A kneading sequence $\nu \in \{0,1\}^{\mathbb{N}}$ **fails the admissibility condition for period** m if the following hold:

1. m is not in the internal address of ν;
2. if $k < m$ divides m, then $\rho(k) \leq m$;
3. $\rho(m) < \infty$ and

$$\rho(m) = qm + r, \quad r \in \{1, \ldots, m\},$$

then $m \in \text{orb}_\rho(r)$.

Branchpoints and Arms

Proposition 1. *If ν fails the admissibility condition for m, such that*

$$\rho(m) = qm + r, \quad r \in \{1, \dots, m\}$$

then the Hubbard tree contains an evil branch point z with

$$e(z) = \overline{\nu_1 \dots \nu_m} \quad \text{and} \quad q + 2 \ \text{arms.}$$

Proposition 2. *If $k \in \mathrm{orb}_[\rho(1)$ is such that*

$$\rho(k) = qk + r, \quad r \in \{1, \dots, k\}$$

then the Hubbard tree contains a tame branch point z with with $e(z) = \overline{\nu_1 \dots \nu_k}$ and

$$\left\{ \begin{array}{c} q+1 \\ q+2 \end{array} \right\} \text{ arms if } \left\{ \begin{array}{l} k \in \mathrm{orb}_\rho(r); \\ k \notin \mathrm{orb}_\rho(r). \end{array} \right.$$

Propositions 1 and 2. account for all periodic branch points.