The Ingram Conjecture

Henk Bruin
University of Surrey

joint with

Marcy Barge (Montana)

and

Sonja Štimac (Zagreb)

Tent maps and inverse limits:

Let $T_s : I \to I$, $x \mapsto \min\{sx, s(1-x)\}$ be the tent map with slope $s \in [0,1]$.

The **inverse limit space**

$$\varprojlim([0,1], T_s) = \{(x_k)_{k \le 0} : T_s(x_k) = x_{k+1} \forall k < 0\}$$

is the space of all backward orbits, equipped with product topology.

Theorem: (Ingram's Conjecture)
If $1 \le s < s' \le 2$, then $\varprojlim([0,1], T_s)$ and $\varprojlim([0,1], T_{s'})$ are non-homeomorphic.

Some History:

Tom Ingram posed the question in the early 1990s, specifically for the case that c is periodic.

Partial results for specific critical behaviour.

Barge & Diamond	three period 5 non-homeo
	$\exists$ complete algebraic inv.
Swanson & Volkmer	all period ≤ 15 non-homeo
Bruin	$\frac{\log s}{\log s'}$ irrational $\Rightarrow$ non-homeo
Kailhofer	all periodic cases
Block, Jakimovik, Kailhofer & Keesling	non-homeo
Štimac	all (pre)periodic non-homeo
Raines & Štimac	all non-recurrent non-homeo
Barge & Diamond	uncountable many
Brucks & Bruin	non-homeo recurrent cases

Notation:

$c = \frac{1}{2}$ is critical point. $\qquad c_n = c_n(s) = T_s^n(c)$

$I = [0,1]$ and $[c_2, c_1]$ is the **core** of T_s.

The shift homeomorphism is

$$\sigma(x) = (\dots, x_{-3}, x_{-2}, x_{-1}, x_0, T_s(x_0)).$$

The projection on coordinate $-q$ is

$$\pi_q : \varprojlim([0,1], T_s) \to I, \qquad \pi_q(x) = x_{-q}$$

commutes as $T \circ \pi_q = \pi_q \circ \sigma$.

The **zero-composant**

$\mathfrak{C} = \{x \in \varprojlim([0,1], T_s) : x_{-k} < c \,\forall\, k \text{ suff. large}\}$

is a ray disjoint from but converging to the **core** of the inverse limit space $\varprojlim([c_2, c_1], T_s)$.

Main Result:

If $s \in (\sqrt{2}, 2]$, then $\varprojlim([c_2, c_1], T_s)$ is indecomposable.

Theorem: If $\sqrt{2} < s \leq 2$, then every self-homeomorphism

$$h : \varprojlim([0,1], T_s) \to \varprojlim([0,1], T_s)$$

is **isotopic** to a power of the shift.

Remark: We started proving that h is pseudo-isotopic to σ^R, *i.e.,* h permutes the composants of $\varprojlim([c_2, c_1], T_s)$ in the same way as σ^R.

This makes the Ingram Conjecture easier to prove as well.

Chains:

Unimodal inverse limits are **chainable**, i.e., for every $\varepsilon > 0$ there is a chain $C = \{\ell_j\}_{j=1}^N$ such that

- $\varprojlim([0,1], T_s) \subset \cup_j \ell_j$;
- ℓ_j are open;
- $\ell_i \cap \ell_j \neq \emptyset$ if and only if $|i-j| \leq 1$;
- $\mathrm{mesh}(C) = \max_j \mathrm{diam}(\ell_j) < \varepsilon$.

We will use q-**chains** constructed as follows:

- $I \subset \cup\{I_j^q\}$, where I_j^q are intervals of length $< \varepsilon s^{-q}/2$;
- $\ell_j^q = \pi_q^{-1}(I_j^q)$;
- The resulting chain $\mathcal{C}_q = \{\ell_j^q\}$.

More notation:

A point $x \in \mathfrak{C}$ is a p-**point** if $x_{-(p+j)} = c$ for some $j \geq 0$. The largest such $j = L_p(x)$ is the p-**level** of x.

The set of p-points E_p can be ordered according to arc-length $\bar{d}$ along $\mathfrak{C}$:

$$x \preceq y \text{ if } \bar{d}(\alpha, x) \leq \bar{d}(\alpha, y)$$

where α is the end-point of $\mathfrak{C}$.

The **folding pattern** is the list of p-levels of p-points in an arc $A \subset \mathfrak{C}$:

$$FP(A) = L_p(x_1) L_(p_2) \dots L_p(x_N)$$

if $x_1 \preceq x_2 \preceq \cdots \preceq x_N$ are the p-points of A.

s_k is the k-th **salient** p-point if

- $L_p(s_k) = k$;
- $L_p(x) < k$ for all $\alpha \prec x \prec s_k$.

p-link-symmetry:

Definition: Given an arc $A \subset \mathfrak{C}$ and a p-chain $\mathcal{C}_p$, let $\ell^0, \ell^1, \dots, \ell^N$ be the links successively visited by A. Then A is

- **symmetric** if the folding pattern $FP(A)$ is a palindrome and $\pi_p(\partial A)$ is a single point;
- p-**link-symmetric** if $\ell^j = \ell^{N-j}$ for $0 \le j \le N$;
- **maximal** p-**link-symmetric** if there is no p-**link-symmetric** arc $A' \supset A$ passing though more links.

A symmetric arc has a well-defined midpoint.

If $h(\mathcal{C}_q)$ refines $\mathcal{C}_p$, then q-link-symmetric arcs map to p-link-symmetric arcs.

Main Lemmas on p-link-symmetric Arcs:

Assume that c has an infinite orbit.

Let $\kappa := \min\{i \geq 3 : c_i > c\}$.

Lemma 1: The maximal p-link-symmetric arc A_i centred at s_i, $i \geq \kappa$, contains κ salient points, namely

$$s_{i-\kappa+2}, \dots, s_i, s_{i+1}$$

and $s_{i-\kappa+2}$ is interior to this arc.

Lemma 2: If a p-link-symmetric arc J is **not** centred at a salient point y, say $s_{i-1} \prec y \prec s_i$, then J contains at most one salient point, and $J \subset A_i$.

How to use p-link Symmetry

Folding points.

A point is called **folding point** if it has no neighbourhood in the core $\varprojlim([c_2, c_1], T_s)$ homeomorphic with a Cantor set of arcs.

Characterisations of folding points:

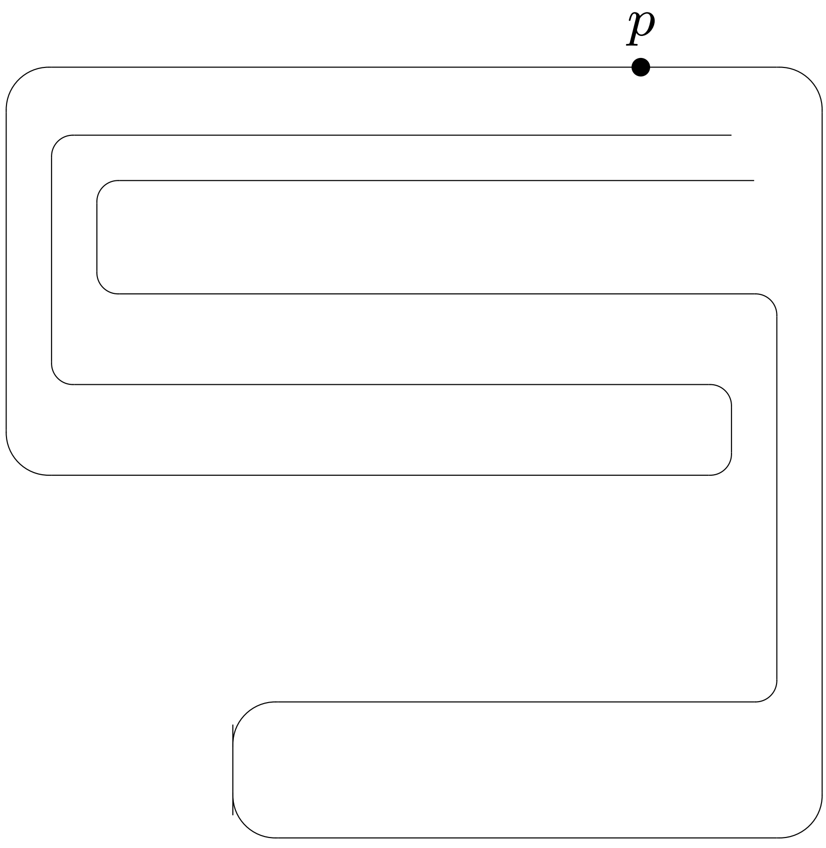
- x is folding point iff $x_k \in \omega(c)$ for all k.

- x is folding point iff there is a sequence of p-points $x^k \to x$ such that $L(x^k) \to \infty$.

- x is folding point iff there is a sequence of salient points $s_{i_k} \to x$.

Theorem: If $s \in (\sqrt{2}, 2]$ and $h : \varprojlim([c_2, c_1], T_s) \to \varprojlim([c_2, c_1], T_s)$ is a homeomorphism, then there is R such that for every folding point x: $h(x) = \sigma^R(x)$

A typical example of this is when T has a periodic critical point of period 3. The below picture illustrates the resulting inverse limit space; p is the fixed point of $\widehat{f}$.



This representation has a single infinite Wada channel.