On unimodal inverse limit spaces

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partially based on joint work with

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Unimodal maps

Unimodal maps are maps of the interval with a single critical point, and increasing/decreasing at the left/right of the critical point. The best known examples are quadratic (logistic) maps and tent-maps

$$T_a : x \mapsto 1 - a|x|, \qquad Q_a : x \mapsto 1 - ax^2.$$

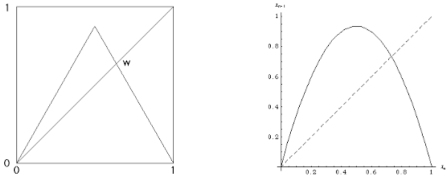

Figure: A tent map and a quadratic map

Lozi and Hénon

A simple way to T_a and Q_a them invertible is by introducing a second coordinate, and thicken the map:

$$T_a : x \mapsto 1 - a|x|, \qquad L_{a,b} : (x, y) \mapsto 1 - a|x| + by, x),$$

$$Q_a : x \mapsto 1 - ax^2, \qquad H_{a,b} : (x, y) \mapsto (1 - ax^2 + by, x),$$

and obtain the Lozi map and the Hénon map.

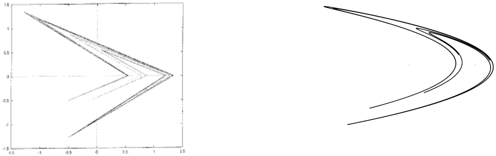

Figure: The Lozi and Hénon attractor

The Lozi-attractor (resp. Hénon-attractor) obtained as $\cap_{n \geq 0} L^n_{a,b}(U)$ for some well-chosen, forward invariant open disk U.

Definitions

Let $f = T_a$ or Q_a. The critical point is $c := 1/2$. Write $c_k := f^k(c)$. The closed f-invariant interval $[c_2, c_1]$ is called the core.

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The inverse limit space $\varprojlim([0,1], f)$ is the collection of all backward orbits

$$\{x = (\dots, x_{-2}, x_{-1}, x_0) : f(x_{-i-1}) = x_{-i} \in [0,1] \text{ for all } i \in \mathbb{N}_0\},$$

equipped with metric $d(x,y) = \sum_{i \le 0} 2^i |x_i - y_i|$. The map f is called the bonding map of $X := \varprojlim([0,1], f)$.

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Define the induced, or shift homeomorphism on $\varprojlim([0,1],f)$ as

$$\sigma(x) := \sigma_f(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1}, x_0, f(x_0)).$$

Let $\pi_i : \varprojlim([0,1],f) \to [0,1]$, $\pi_i(x) = x_{-i}$ be the i-th projection map.

Examples

Figure: The $\sin\frac{1}{x}$ continuum and the Knaster continuum.

Simple examples of such unimodal inverse limit spaces are the $\sin\frac{1}{x}$-continuum and the Knaster continuum (bucket handle).

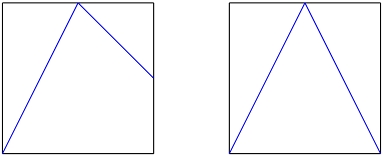
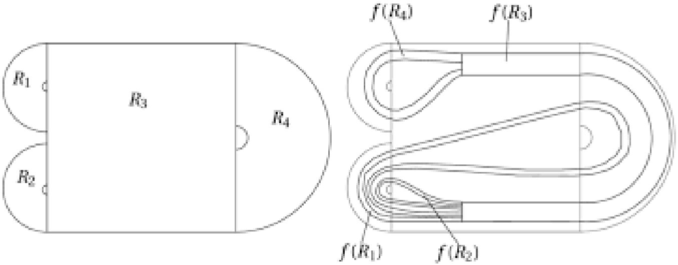


Figure: Maps with $\sin\frac{1}{x}$ continuum and the Knaster continuum as ILs

In 1967, Bob Williams [W] proved that hyperbolic one-dimensional attractors can be represented as inverse limits of maps on branched manifolds and that every point has a neighbourhood homeomorphic to the product of a Cantor set and an open arc.

A standard example of this is the Plykin attractor.



Inverse limits and Hénon attractors

The similarity between a Hénon attractors and the Knaster continuum may suggest that inverse limit spaces are homeomorphic to Hénon attractors in some generality, but in fact, the generality is very limited.

Theorem (Barge & Holte [BH])

If a is such that 0 is a periodic for $Q_a(x) = 1 - ax^2$, then for $|b|$ sufficiently small, then the attractor of $H_{a,b}$ and the inverse limit space of Q_a are homeomorphic.

Barge [B1] showed that under fairly general assumptions, Hénon attractors (and homoclinic tangle emerging from a homoclinic bifurcations) are **not** homeomorphic to unimodal inverse limit spaces, not even if you allow varying bonding maps.

Embedding ILs in the plane

All unimodal inverse limit spaces are chainable, and all chainable continua can be embedded in the plane, i.e., there is a continuous injection $h : X \to \mathbb{R}^2$ (called embedding) such that $h(X)$ and X are homeomorphic.

Embedding ILs in the plane

Definition

A point $a \in X \subset \mathbb{R}^2$ is accessible if there exists an arc $A = [x, y] \subset \mathbb{R}^2$ such that $a = x$ and $A \cap X = \{a\}$.

Unimodal inverse limit spaces can therefore be embedded in the plane, but in general there are many (in fact uncountably non-homotopic) ways to do so.

There are two standard ways that yield an embedding very much like the Lozi-attractor (or Hénon-attractor) with $b > 0$ (orientation reversing, making the composant $\mathfrak{R}$ of the fixed point $p = (\dots, r, r, r)$ accessible, see [B2]) and $b < 0$ (orientation preserving, making the zero-composant accessible, see [BD]) respectively.

Embedding ILs in the plane

The result of Anušić et al. gives an idea how much variety there is in embeddings:

Theorem ([ABC1])

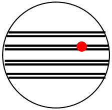
For every point a in th core UIL X' there exists an embedding of X in the plane such that a is accessible.

As a corollary, there are uncountably many nonequivalent embeddings of X' in the plane.

Folding points

A point $x \in \varprojlim\{I, T\}$ is

- "laminar" if it has a neighborhood $\simeq$ Cantor set of (open) arcs.



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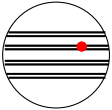
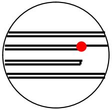

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  - endpoint ($x \in \mathcal{E}$) if $x \in$ subcontinua A, B implies $A \subset B$ or $B \subset A$.

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  - endpoint ($x \in \mathcal{E}$) if
    $x \in$ subcontinua A, B implies $A \subset B$ or $B \subset A$.
  - non-end folding point ($x \in \mathcal{F} \setminus \mathcal{E}$) if $x \in A^\circ$ for some arc.

Endpoints

Further classification of the end-points $\mathcal{E}$:

- Flat end-points $\mathcal{E}_F$: x is the end-point of a non-degenerate **basic arc**:

$$A(x) = \text{largest continuum } A \ni x \text{ such that } \pi_0 : X \to [0,1] \text{ is bijective.}$$

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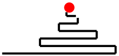

- Spiral end-points $\mathcal{E}_S$: x is the end-point of an arc but not of a non-degenerate **basic arc**.
- Nasty (= solitary) end-points $\mathcal{E}_N$: x is not contained in any arc.

General results

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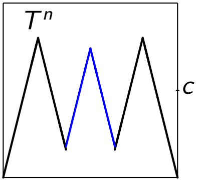
Proposition [AABC]: For $Y = \mathcal{E}_F, \mathcal{E}_S, \mathcal{E}_N$ or $\mathcal{F} \setminus \mathcal{E}$ holds:

If $Y \neq \emptyset$, then Y is dense in $\mathcal{F}$.

Cutting times and the kneading map

Definition: An iterate n is called a **cutting time** if the image of the central branch of T^n contains c.



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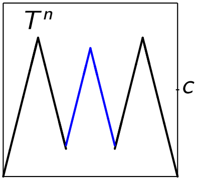

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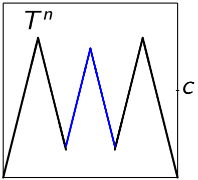

Denote the cutting times as $1 = S_0 < S_2 < S_3 < S_4 < \dots$

Lemma: The difference of two consecutive cutting times is a cutting time.

Therefore define the **kneading map** $Q : \mathbb{N} \to \mathbb{N}_0$:

$$S_k - S_{k-1} = S_{Q(k)}.$$

Properties of $\mathcal{E}$

Definition (Blokh & Lyubich)

The critical point c is reluctantly recurrent if there is $\varepsilon > 0$ and an arbitrary long (but finite!) backward orbit $\bar{y} = (y_{-j}, \dots, y_{-1}, y)$ in $\omega(c)$ such that the ε-neighbourhood of $y \in I$ has monotone pull-back along $\bar{y}$. Otherwise, c is persistently recurrent.

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Theorem ($\mathcal{F} = \mathcal{E}$)

All folding points are end-points iff c is persistently recurrent.

Flat end-points $\mathcal{E}_F$

Definition: The map T is longbranched (i.e.,
$\exists \delta > 0 \ \forall n \geq 0 \ \forall J$ branch domain of $T^n : |T^n(J)| \geq \delta$).

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Proposition: If T is long-branched and c is recurrent, then $\mathcal{E} = \mathcal{E}_F$.

Question: Give a necessary and sufficient condition for $\mathcal{E} = \mathcal{E}_F$.

Spiral end-points $\mathcal{E}_S$

Lemma ($\mathcal{E} = \mathcal{E}_S$)

If $Q(k) \to \infty$ *and*

$$Q(k+1) > Q(Q(k)+1)+1 \qquad \text{for all sufficiently large } k,$$

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Lemma: If $Q(k) \not\to \infty$, then $\mathcal{F} \neq \mathcal{E}_S$.

Conjecture: $Q(k) \to \infty$ implies (and is equivalent to?) $\mathcal{E} = \mathcal{E}_S$.

Nasty end-points $\mathcal{E}_N$

Known: If f is an infinitely renormalizable unimodal (e.g. quadratic) map, then $\mathcal{F} = \mathcal{E} = \mathcal{E}_N$, but there are no infinitely renormalizable tent maps.

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Theorem (after Barge, Brucks & Diamond 1996)

If $\{c_{S_k} : k \in \mathbb{N}\}$ is dense in $[c_2, c_1]$, then every neighbourhood in $\varprojlim\{I, T\}$ contains a subcontinua homeomorphic to $\varprojlim\{I, \tilde{T}\}$ for every tent map $\tilde{T}$.

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Corollary: If $\{f^{S_k}(c) : k \in \mathbb{N}\}$ is dense in $[c_2, c_1]$, then

$\mathcal{E}_N$ is dense in $\varprojlim\{I, T\}$,

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Arc-components and composants

Definition

Let X be a continuum and $x \in X$. The arc-component $A(x)$ of x is the union of points y such that there is an arc in X connecting x and y. The composant $C(x)$ of a point x is the union of all proper subcontinua of X.

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For example, if $X = [0,1]$ then $A(0) = [0,1]$ but $C(0) = [0,1)$ (it doesn't contain 1 because $[0,1]$ is not a proper subcontinuum of X). Also $A(\frac{1}{2}) = C(\frac{1}{2}) = [0,1]$ because $[0,1] = [0,\frac{1}{2}] \cup [\frac{1}{2},1]$.

If an indecomposable continuum has uncountably many composants, which may or may not be arc-components.

Asymptotic arc-components

Two arc-components A and $\tilde{A}$ are asymptotic if there are parametrizations

$$\varphi, \tilde{\varphi} : \mathbb{R} \to A, \tilde{A} \qquad \text{such that} \qquad \lim_{t\to\infty} d(\varphi(t), \tilde{\varphi}(t)) = 0.$$

The trivial case when $A = \tilde{A}$ is excluded, but A is self-asymptotic if there is a parametrization φ such that

$$\lim_{t\to\infty} d(\varphi(t), \tilde{\varphi}(-t)) = 0.$$

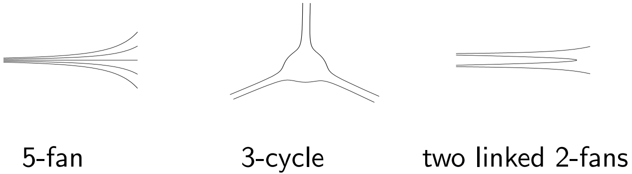

Figure: Configurations of asymptotic arc-components.

Asymptotic arc-components

Figure 6 give the UIL of a tent map with $f^3(c) = c$, for which the fixed composant $\mathfrak{R}$ is self-asymptotic. There is a single infinite Wada channel for which the entire shore is equal to $\mathfrak{R}$.

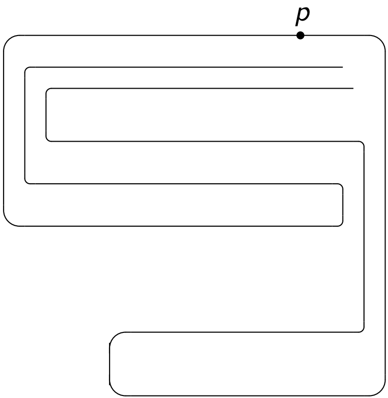

Figure: This representation has a single infinite Wada channel.

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The proof relies on substitution tilings and the fact that these spaces act as 2-to-1 coverings of inverse limit spaces. In fact, if the period is N, then there are at least $N-1$ and at most $2(N-1)$ "halves" of arc-components asymptotic to some other "halves" of an arc-components. □

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Conjecture

The upper bound is in fact $2(N-2)$. Given any two "halves" of arc-components H and H', H is asymptotic to or coincides with $\sigma^n(H')$ for some $n \in \mathbb{Z}$.

Asymptotic arc-components

If c is non-recurrent, then there are no asymptotic arc-components, see [B3], but:

Question

What is the situation with asymptotic arc-components when c is non-periodic but recurrent?

Asymptotic arc-components

	ν	type	periodic tail(s)
1	101	1-cycle	1
2	1001	3-fan	101
3	10001	4-fan	1001
4	10010	3-cycle	101
5	10111	three 2-fans	101110
6	100001	5-fan	10001
7	100010	4-cycle	1001
8	100111	four 2-fans	10010011
9	101110	two linked 3-fans	10, 1
10	1000001	6-fan	100001
11	1000010	5-cycle	10001
12	1000111	five 2-fans	1000100011
13	1000100	four 2-fans (l.i.p.)	10, 1001
14	1001101	four 2-fans	10011010
15	1001110	five 2-fans	10010, 10111
16	1001011	five 2-fans	1001011011

Asymptotic arc-components

17	1011010	5-cycle	10111
18	1011111	five 2-fans	1011111110
19	10000001	7-fan	1000001
20	10000010	6-cycle	100001
21	10000111	six 2-fans	100001110000
22	10000100	five 2-fans	10001, 10010
23	10001101	five 2-fans	1000110100
24	10001110	six 2-fans	100010, 100111
25	10001011	six 2-fans	100010110011
26	10011010	six 2-fans (l.i.p.)	101, 100111
27	10011111	six 2-fans	100111110110
28	10011100	five 2-fans	10010, 10111
29	10010101	five 2-fans	1001010111
30	10010110	six 2-fans (l.i.p.)	100, 101110
31	10110111	three 3-cycles	101101110
32	10111110	two linked 4-fans	101110, 1

(l.i.p = linked in pairs.)

Arc-components

Bandt [B] for the Knaster continuum that every two arc-components not containing the endpoint are homeomorphic.

Question

Given two arc-components without endpoints, are they homeomorphic? In particular, can a self-asymptotic arc-component be homeomorphic to a non-self-asymptotic arc-component?

In contrast, Fokkink (in his thesis and in [F]) showed that among all matchbox manifolds (i.e., continua that locally look like Cantor set of open arcs) there are uncountably many non-homeomorphic arc-components.

Aarts' Question

Question

Are two lines with irrational slopes wrapping for ever around the torus be homeomorphic as spaces?

This question is due to Aarts almost half a century ago, but beyond the fact that if the slopes θ and θ' have continued fraction expansion with the same tail then the lines are indeed homeomorphic, nothing is known.

Thank you for your attention.

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