Li-Yorke chaos for maps on the interval.

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Li-Yorke, distal and asymptotic pairs

Let $f : X \to X$ be a continuous map on a metric space (X, d).

We call the pair (x, y):

Distal: if $\liminf_n d(f^n(x), f^n(y)) > 0$.

Asymptotic: if $\lim_n d(f^n(x), f^n(y)) = 0$.

Li-Yorke: if

$$\liminf_n d(f^n(x), f^n(y)) = 0$$

and

$$\limsup_n d(f^n(x), f^n(y)) > 0.$$

A set B is **scrambled** if every pair $x \neq y \in B$ is Li-Yorke, and f is **Li-Yorke chaotic** if there is an uncountable scrambled set.

Note: $h_{top}(f) > 0$ implies that (X, f) is Li-Yorke chaotic (Blanchard et al.)

Multimodal maps.

In this talk the interval map $f : I \to I$ will

- be C^2 or C^3 multimodal (i.e., finite set of critical points: $f'(c) = 0$);

- have non-flat critical points c:

$$f(x) = f(c) + O(|x - c|^{\ell_c})$$

  for $x \approx c$, and critical order $\ell_c \in (1, \infty)$.

- Sometimes we will assume that f is topologically mixing, i.e., every iterate f^n has a dense orbit.

C^1 maps gives different results

The are C^1 interval maps that

- have a scrambled set of positive Lebesgue measure, or

- have a scrambled set of full outer measure (but the scrambled set is non-measurable).

These C^1 results are due to Smital '84.

Jímenez-López '91 proved that no C^1 can have a measurable scrambled set of full Lebesgue measure.

What is an Attractor?

For interval maps, the following measure-theoretic definition was introduced by Milnor '85.

The omega-limit set

$$\omega(x) = \cap_n \overline{\cup_{m \geq n} f^m(x)}$$

is the set of limit points of an orbit.

The basin $\text{Bas}_{\mathcal{A}} = \{x : \omega(x) \subset \mathcal{A}\}$.

Let λ be Lebesgue measure. $\mathcal{A}$ is an attractor (a la Milnor) if

$$\lambda(\textbf{Bas}_{\mathcal{A}}) > 0$$

and if $\mathcal{A}' \subset \mathcal{A}$, $\mathcal{A}' \neq \mathcal{A}$, then $\lambda(\textbf{Bas}_{\mathcal{A}'}) = 0$.

Remark: An attractor is closed and forward invariant.

Classification of attractors of multimodal maps.

Theorem 1 *If $f : I \to I$ is a non-flat C^2 multimodal map, then it has $\leq \#$Crit attractors, which are of the following types:*

1. *$\mathcal{A}$ is an attracting periodic orbit.*

2. *$\mathcal{A}$ is a collection of intervals $\{I_j\}_{j=0}^{N-1}$ permuted cyclically. (Topological mixing $\Rightarrow$ $N = 1$.)*

3. *$\mathcal{A}$ is a minimal Cantor set consisting of the infinite intersection of cykels of intervals as above.*
   (**solenoidal attractor** - *infinitely renormalizable, Bas$_{\mathcal{A}}$ is of 2^{nd} Baire category.)*

4. *$\mathcal{A}$ is a minimal Cantor set, but not of the above type.*
   (**wild attractor** - *not Lyapunov stable, Bas$_{\mathcal{A}}$ is of 1^{st} Baire category.)*

Some references.

The classification of attractors was proved more or less independently by Blokh & Lyubich '91, Keller '90 and by Martens '90.

Milnor '85 posed the question whether wild attractors really exist. This was proved by Bruin, Keller, Nowicki & van Strien ('96) for the Fibonacci unimodal map with very large critical order ℓ. Bruin ('98) extended this to more general combinatorial types.

For the quadratic family, Lyubich ('94) proved that there are no wild attractors (neither for $\ell \leq 2 + \varepsilon$, Keller & Nowicki '95).

These result are the starting point of our 'Li-Yorke chaos classification'.

When attractor $\mathcal{A} = I$.

Theorem 2 *If $\mathcal{A} = I$, and $\lambda(B) > 0$. Then*

- *there are $x \neq y \in B$ such that $f^n(x) = f^n(y)$ for some $n \geq 0$.*
  Hence, there is no scrambled set of positive Lebesgue measure.

- *there are $x \neq y \in B$ such that (x, y) is* **not** *asymptotic. Hence, there is no asymptotic set of positive Lebesgue measure.*

- *there are $n > m \geq 0$ such that $\lambda(f^n(B) \cap f^m(B)) > 0$: there are no* **strongly wandering** *sets (Blokh & Misiurewicz).*

- *For all $x \in I$ there is a set C_x of full measure such that for all $y \in C_x$*

$$\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0, \text{ and}$$

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| \geq diam(I)/2.$$

so f is **Li-Yorke sensitive**.

Dense orbits for $f \times f : I^2 \to I^2$.

Proposition 3 *Let f be a non-flat C^3 unimodal map, then the following are equivalent:*

1. *f has an invariant prob. measure $\ll \lambda$;*

2. $\liminf_n \lambda(f^n(A)) > 0$ *for every* $A \subset I$, $\lambda(A) > 0$.

3. $\lim_n \lambda(f^n(A)) = 1$ *for every* $A \subset I$, $\lambda(A) > 0$.

In this case two-dimensional λ_2-a.e. (x, y) has a dense orbit. (cf. μ weak mixing)

Conjecture 4 *There exists a top. mixing $f \in C^\infty(I)$ such that λ_2-a.e. (x, y) is Li-Yorke, but does not have a dense orbit. In fact, λ_2 is dissipative, whereas λ is conservative.*

When attractor $\mathcal{A}$ is a solenoidal Cantor set.

A point x is **approximately periodic** if $f^n \not\to$ periodic orbit, but for every $\varepsilon > 0$, there is a periodic point p such that

$$|f^j(x) - f^j(p)| < \varepsilon$$

for all j sufficiently large.

Lemma 5 (Barrio-Blaya, Jiménez-López) $(\omega(x), f)$ *is conjugate to some* (p_i)*-adic adding machine if and only if* x *is approximately periodic.*

Theorem 6 *If* $\mathcal{A}$ *is solenoidal Cantor attractor, then*

$$\lambda_2(\textit{Distal pairs}) = \lambda_2(\textit{Bas}_{\mathcal{A}} \times \textit{Bas}_{\mathcal{A}})$$

and there are no Li-Yorke pairs in $\textit{Bas}_{\mathcal{A}}$.

When $\mathcal{A}$ is wild.

Theorem 7 *Let $\mathcal{A}$ be a wild attractor (with positive drift and basin of full measure), then we have these possibilities:*

(a) λ_2(*Distal pairs*) $= 1$ *and every point in the basin is approximately periodic.*
(Strange adding machine!)

(b) λ_2(*Distal pairs*) $= 1$, *but no point in the basin is approximately periodic.*

(c) λ_2(*LY pairs*) $= 1$.

(d) λ_2(*Distal pairs*) > 0 *and*
λ_2(*LY pairs*) > 0.

Each of these cases occurs. In cases (b)-(d) there is $\varepsilon > 0$ such that $Bas_{\mathcal{A}}$ contains uncountable ε-scrambled sets, and f is Li-Yorke sensitive on $Bas_{\mathcal{A}}$.

Cutting Times for Unimodals

The n-th iterate of f has **central branch**

$$f^n : J \to f^n(J) =: D_n \subset f^n(J),$$

where J is a maximal interval adjacent to c on which f^n is monotone.

The number n is a **cutting time** if $c \in f^n(J)$.

Cutting times are denoted as

$$1 = S_0 < S_1 < S_2 < \dots$$

Enumeration Scales

Given the integer sequence

$$\{S_k\}_{k\geq 0} \text{ with } S_0 = 1 \text{ and } S_k \leq 2S_{k-1}$$

let

$$\langle n\rangle \in \{0,1\}^{\mathbb{N}\cup\{0\}} \text{ such that } \sum_{k\geq 0} \langle n\rangle_k S_k = n$$

be the **greedy** representation of $\mathbb{N}$.

Let g be 'add one and carry':

$$g : \langle \mathbb{N}\cup\{0\}\rangle \to \langle \mathbb{N}\rangle, \quad g(\langle n\rangle) = \langle n+1\rangle.$$

The extension to the closure

$$g : E := \overline{\langle \mathbb{N}\cup\{0\}\rangle} \to E$$

is well-defined and continuous, provided there is $\alpha_k \to \infty$ such that

$$S_k = \sum_{j\geq \alpha_k} S_j.$$

Factors of Enumeration Scales

Let (E, g) be the enum. scale based on $\{S_k\}_{k\geq 0}$.

Let

$$h(x) = x - \text{round}(x) \in [-\frac{1}{2}, \frac{1}{2})$$

be the signed distance to the nearest integer.

Assume that there is $\rho \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$\sum_k k|h(\rho S_k)| < \infty. \qquad (1)$$

Then

$$\pi_\rho : E \to \mathbb{S}^1, \quad \pi(e) = \sum_k e_k h(\rho S_k) \pmod 1$$

is well-defined, onto and satisfies

$$\pi_\rho \circ g = R_\rho \circ \pi_\rho.$$

Idea of Proof part (b).

Assume that (1) holds.

Assume that f has a wild attractor with positive drift.

Define

$$b_n(x) = \max_j \{j : f^j(Z_j(x)) = D_n\}$$

where $Z_j(x)$ is order j cylinder containing x, and

$$\pi_n(x) = -\sum_k h(\rho\langle b_n(x) - n\rangle_k S_k) \pmod{1}.$$

Then $\{\pi_n(x)\}_n$ is Cauchy sequence in $\mathbb{S}^1$, defined λ-a.e. x in the basin of $\omega(c)$.

For the limit $\pi = \lim_n \pi_n$:

$$\pi \circ f = R_\rho \circ \pi_\rho, \qquad \lambda - \text{-a.e.}$$

Existence wild attractors.

Wild attractors exist for unimodal (i.e. one critical point c) maps with specific **Fibonacci-like** combinatorics and sufficiently large critical order ℓ_c.

Idea of proof:

- Construct pairs of symmetric intervals U_k converging to c.

- Use **induced map**

$$F|_{U_k} = f^{S_k}|_{U_k}$$

and consider its dynamics as random walk:

$$\chi_n(x) = k \text{ if } F^n(x) \in U_k.$$

Existence of wild attractors: idea of proof continued:

- If combinatorics are good and ℓ_c is large, then this random walk has **positive drift:**

$$\mathbb{E}(\chi_{n+1} - k | \chi_n = k) \geq \eta > 0$$

w.r.t. λ, uniformly in n and k.

- Then $\chi_n \to \infty$ λ-a.s., so $F^n(x) \to c$ λ-a.e. and $f^j(x) \to \omega(c)$ λ-a.e.