Li-Yorke chaos from the Lebesgue measure point of view.

Henk Bruin (University of Vienna)

on papers joint with

Victor Jiménez-López (University of Murcia)
Piotr Oprocha (AGH Krakow)

Budapest, June 2019

Throughout we consider maps $f : I \to I$ of the interval I.

Definitions: A pair of distinct points (x, y) is called:

- **distal** if $\liminf_{n\to\infty} |f^n(y) - f^n(x)| > 0$;
- **asymptotic** if $\lim_{n\to\infty} |f^n(y) - f^n(x)| = 0$;
- **Li-Yorke** if

$$0 = \liminf_{n\to\infty} |f^n(y) - f^n(x)| < \limsup_{n\to\infty} |f^n(y) - f^n(x)|.$$

  i.e., neither asymptotic nor distal.
- **proximal** if $\liminf_{n\to\infty} |f^n(y) - f^n(x)| = 0$.

Throughout we consider maps $f : I \to I$ of the interval I.

Definitions: A pair of distinct points (x, y) is called:

- **distal** if $\liminf_{n\to\infty} |f^n(y) - f^n(x)| > 0$;
- **asymptotic** if $\lim_{n\to\infty} |f^n(y) - f^n(x)| = 0$;
- **Li-Yorke** if

  $$0 = \liminf_{n\to\infty} |f^n(y) - f^n(x)| < \limsup_{n\to\infty} |f^n(y) - f^n(x)|.$$

  i.e., neither asymptotic nor distal.
- **proximal** if $\liminf_{n\to\infty} |f^n(y) - f^n(x)| = 0$.
- A set $S \subset I$ is called scrambled if every two distinct $x, y \in S$ form a Li-Yorke pair.
- The map f is called Li-Yorke chaotic if it has an uncountable scrambled set.

Some results:

The notion comes from the 1975 Li-Yorke paper: Period 3 implies chaos.

Some results:

The notion comes from the 1975 Li-Yorke paper: Period 3 implies chaos.

Kuchta & Smítal: If $f : I \to I$ has a Li-Yorke pair then it has a scrambled Cantor set: One Li-Yorke pair implies Li-Yorke chaos. (Extended by Ruette & Snoha to graph maps.)

Some results:

The notion comes from the 1975 Li-Yorke paper: Period 3 implies chaos.

Kuchta & Smítal: If $f : I \to I$ has a Li-Yorke pair then it has a scrambled Cantor set: One Li-Yorke pair implies Li-Yorke chaos. (Extended by Ruette & Snoha to graph maps.)

Blanchard et al.: Positive entropy implies Li-Yorke chaos.

Two points to raise:

1. Are scrambled sets are large in the sense of Lebesgue measure?

Continuous interval maps can have scrambled sets of $\text{Leb}(S) > 0$, but

Theorem: For C^3 interval maps with non-flat critical points (i.e., $D^\ell f(c) \neq 0$ for some $\ell \geq 1$), every measurable scrambled set has $\text{Leb}(S) = 0$.

Still unknown if f has flat critical points.

Two points to raise:

1. Are scrambled sets are large in the sense of Lebesgue measure?

Continuous interval maps can have scrambled sets of $\mathrm{Leb}(S) > 0$, but

Theorem: For C^3 interval maps with non-flat critical points (i.e., $D^\ell f(c) \neq 0$ for some $\ell \geq 1$), every measurable scrambled set has $\mathrm{Leb}(S) = 0$.

Still unknown if f has flat critical points.

What about the 2-dim. Lebesgue measure of LY-pairs $(x, y) \in I^2$?

2. What about Li-Yorke tuples instead of Li-Yorke pairs?

2. What about Li-Yorke tuples instead of Li-Yorke pairs?

Definitions: A d-tuple $\underline{x} = (x_1, \dots, x_d)$ is called:

- **asymptotic** if $\lim_n \max_{i,j} |f^n(x_i) - f^n(x_j)| = 0$;
- **proximal** if $\liminf_n \max_{i,j} |f.(x_i) - f^n(x_j)| = 0$;
- δ-**separated** if $\limsup_n \min_{i \neq j} |f^n(x_i) - f^n(x_j)) > \delta$;
  if $\underline{x}$ is δ-separated for some $\delta > 0$, we call it **separated**;
- **Li-Yorke** if it proximal and separated, that is:

$$\begin{cases} \liminf_n \max_{i,j} |f^n(x_i) - f^n(x_j)| = 0, \\ \limsup_n \min_{i \neq j} |f^n(x_i) - f^n(x_j)| > 0. \end{cases}$$

- δ-**Li-Yorke** for some $\delta > 0$, if $\underline{x}$ is LY and:

$$\limsup_n \min_{i \neq j} |f^n(x_i) - f^n(x_j)| > \delta.$$

Definition: A set A is a measure-theoretic attractor if

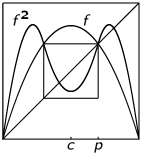
- it is closed and forward invariant;
- the basin $\mathrm{Bas}(A) = \{x : \omega(x) \subset A\}$ has positive Lebesgue measure;
- it is minimal w.r.t. these two properties.

Definition: A set A is a measure-theoretic attractor if

- it is closed and forward invariant;
- the basin $\text{Bas}(A) = \{x : \omega(x) \subset A\}$ has positive Lebesgue measure;
- it is minimal w.r.t. these two properties.

Classification measure-theoretic attractors: if f is C^3 multimodal with negative Schwarzian derivative, then there are four types of such attractors:

- A (one or two-sided) attracting periodic orbit;
- A finite union of intervals, cyclically permuted by f;
- A Lyapunov stable Cantor set (infinite renormalization);



A renormalizable map

- A wild attractor - not Lyapunov stable, $Bas(A)$ is a meager set of positive measure.

Theorem: Let $f : I \to I$ be a C^3 topologically mixing multimodal map with non-flat critical points, and having no Cantor attractors. Then the Cartesian product $(I^2, \mathrm{Leb}_2, f \times f)$ is ergodic and for every $x \in I$ there is a full measure set $H_x \subset I$ such that

$$\liminf_{n\to\infty} |f^n(y) - f^n(x)| = 0, \quad \limsup_{n\to\infty} |f^n(y) - f^n(x)| \geq \delta,$$

for every $y \in H_x$ and $\delta = \frac{1}{2}\mathrm{diam}(I)$.

In particular, the set of δ-Li-Yorke pairs has full measure in I^2 for and f is Li-Yorke sensitive.

Theorem: If f is a C^3 multimodal map with non-flat critical point, but with a wild attractor, then one of the following occurs:

(a) Lebesgue a.e. pair of points in $\text{Bas}(A)$ is distal and no point in $\text{Bas}(A)$ is approximately periodic;

(b) Lebesgue a.e. pair of points in $\text{Bas}(A)$ is Li-Yorke;

(c) Both the sets of distal pairs and of Li-Yorke pairs have positive Lebesgue measure in $\text{Bas}(A) \times \text{Bas}(A)$.

There are examples of polynomial unimodal maps of all above types so that additionally $\text{Bas}(A)$ contains ε-scrambled sets for a fixed $\varepsilon > 0$ and f is Li-Yorke sensitive on $\text{Bas}(A)$.

Recap

2. What about Li-Yorke tuples instead of Li-Yorke pairs?

Recap

2. What about Li-Yorke tuples instead of Li-Yorke pairs?

More specifically:

Does the set of Li-Yorke d-tuples have positive d-dim. Lebesgue measure?

Can it happen that the Li-Yorke d-tuples have positive d-dim. Lebesgue measure but the Li-Yorke $d+1$-tuples not?

Definition: Let $f : I \to I$ be a non-singular map. Lebesgue measure is called

- lim sup **full** if $\limsup_n \mathrm{Leb}(f^n(A)) = 1$ whenever $\mathrm{Leb}(A) > 0$;
- **full** if $\lim_n \mathrm{Leb}(T^n(A)) = 1$ whenever $\mathrm{Leb}(A) > 0$.

Definition: Let $f : I \to I$ be a non-singular map. Lebesgue measure is called

- lim sup **full** if $\limsup_n \text{Leb}(f^n(A)) = 1$ whenever $\text{Leb}(A) > 0$;
- **full** if $\lim_n \text{Leb}(T^n(A)) = 1$ whenever $\text{Leb}(A) > 0$.

Denote by Leb_d the d-dimensional Lebesgue measure, and

$$\text{LY}_d^\delta = \{\underline{x} = (x_1, \dots, x_d) : \underline{x} \text{ is } \delta - \text{Li-Yorke tuple}\}$$

Theorem:

1. If Leb is lim sup full then $\text{Leb}_2(LY_2^\delta) = 1$ for every $\delta < \text{diam}(I)/2$,
2. If Leb is full then $\text{Leb}_d(LY_d^\delta) = 1$ for every $\delta < \text{diam}(I)/2(d-1)$.

Definition: Let $f : I \to I$ be a non-singular map. Lebesgue measure is called

- lim sup **full** if $\limsup_n \mathrm{Leb}(f^n(A)) = 1$ whenever $\mathrm{Leb}(A) > 0$;
- **full** if $\lim_n \mathrm{Leb}(T^n(A)) = 1$ whenever $\mathrm{Leb}(A) > 0$.

Denote by Leb_d the d-dimensional Lebesgue measure, and

$$\mathrm{LY}_d^{\delta} = \{\underline{x} = (x_1, \ldots, x_d) : \underline{x} \text{ is } \delta - \text{Li-Yorke tuple}\}$$

Theorem:

1. If Leb is lim sup full then $\mathrm{Leb}_2(LY_2^{\delta}) = 1$ for every $\delta < \mathrm{diam}(I)/2$,
2. If Leb is full then $\mathrm{Leb}_d(LY_d^{\delta}) = 1$ for every $\delta < \mathrm{diam}(I)/2(d-1)$.

The analogon holds for every continuous map on connected space with a non-sigular measure.

The Pomeau-Manneville map T_α is defined as

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

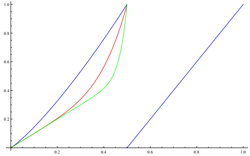


Figure: Graph of $T_{\frac{1}{2}}$, T_4 and T_{15}.

Theorem: For the Pomeau-Manneville map T_α, we have

1. Leb_2-a.e. pair is $\frac{1}{3}$-Li-Yorke,
2. and for $d \geq 3$:
   - if $\alpha \leq \frac{d-1}{d-2}$ and $\delta < 1/2(d-2)$, then Leb_d-a.e. d-tuple is δ-Li-Yorke;
   - if $\alpha > \frac{d-1}{d-2}$ and $\varepsilon > 0$, then Leb_d-a.e. d-tuple is not ε-separated; in particular Leb_d-a.e. d-tuple is not Li-Yorke.

Theorem: For the Pomeau-Manneville map T_α, we have

1. Leb_2-a.e. pair is $\frac{1}{3}$-Li-Yorke,
2. and for $d \geq 3$:
   - if $\alpha \leq \frac{d-1}{d-2}$ and $\delta < 1/2(d-2)$, then Leb_d-a.e. d-tuple is δ-Li-Yorke;
   - if $\alpha > \frac{d-1}{d-2}$ and $\varepsilon > 0$, then Leb_d-a.e. d-tuple is not ε-separated; in particular Leb_d-a.e. d-tuple is not Li-Yorke.

Remark: No matter how large α, a typical d-tuple is never asymptotic.

Theorem: For the Pomeau-Manneville map T_α, we have

1. Leb_2-a.e. pair is $\frac{1}{3}$-Li-Yorke,
2. and for $d \geq 3$:
   - if $\alpha \leq \frac{d-1}{d-2}$ and $\delta < 1/2(d-2)$, then Leb_d-a.e. d-tuple is δ-Li-Yorke;
   - if $\alpha > \frac{d-1}{d-2}$ and $\varepsilon > 0$, then Leb_d-a.e. d-tuple is not ε-separated; in particular Leb_d-a.e. d-tuple is not Li-Yorke.

Remark: No matter how large α, a typical d-tuple is never asymptotic.

For $\alpha > 2$, the product system is Leb_2-dissipative, whence orbits of typical pairs are not dense in $[0,1]^2$, however still Leb_2-a.e. pair is LY. (This addresses a question by Bruin & Jiménez-López.)

Some Questions:

1. What is the situation with Pomeau-Manneville maps with two or more neutral fixed point (of the same exponent α)?

Some Questions:

1. What is the situation with Pomeau-Manneville maps with two or more neutral fixed point (of the same exponent α)?

2. What is the Lebesgue typical situation with LY d-tuples for smooth multimodal maps; I mean without exploiting neutral periodic orbits?

- F. Blanchard, E. Glasner, S. Kolyada, A. Maass, *On Li-Yorke pairs,* J. Reine Angew. Math. **547** (2002), 51–68.
- H. Bruin, Víctor Jiménez-López, *On the Lebesgue measure of Li-Yorke pairs for interval maps,* Comm. Math. Phys. **299** (2010) 523–560.
- H. Bruin, P. Oprocha, *On "observable" Li-Yorke tuples for interval maps,* Nonlinearity **28** (2015), 1675-1694.
- M. Kuchta and J. Smítal, *Two-point scrambled set implies chaos,* World Sci. Publ., Teaneck, NJ, (1989), 427–430.
- T. Li, J. A. Yorke, *Period three implies chaos,* Amer. Math. Monthly, **82** (1975) 985-992.
- S. Ruette and L. Snoha, *For graph maps, one scrambled pair implies Li-Yorke chaos,* Proceedings AMS, **142**, (2014), 2087–2100, 2014.