Dynamics of Selmer's continued fraction algorithm

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The Euclidean Algorithm

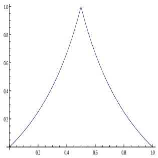
An example from very old Greeks:

Let $x < y$ be positive real numbers.

The Euclidean algorithm to approximate $\frac{x}{y}$ by rationals goes by iterating:

$$(x, y) \to \begin{cases} (x, y - x) & \text{if } x < y - x, \\ (y - x, x) & \text{if } x > y - x. \end{cases}$$

If we scale the largest coordinate to 1, we get the Farey map:



$$f(x) = \begin{cases} \frac{x}{1-x} & \text{if } x < \frac{1}{2}, \\ \frac{1-x}{x} & \text{if } x > \frac{1}{2}. \end{cases}$$

The Gauß map

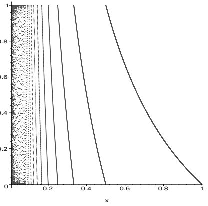
To speed up this algorithm, define

$$\tau(x) = 1 + \min\{n \geq 0 : f^n(x) \in (\frac{1}{2}, 1]\}.$$

The induced map $G = f^\tau$ is the **Gauß map**: $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$

It produces the standard continued fraction of x by $x_n = G^n(x)$, $a_{n+1} = \lfloor \frac{1}{x_n} \rfloor$:

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$



The Gauß map

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However ν does **not** pull back to an f-invariant **probability** measure.

Instead, the Farey map preserves the infinite density

$$d\mu = \frac{1}{x} \, dx$$

The Lebesgue statistical properties of the Farey map are nevertheless very well understood, *e.g.* Thaler.

Algorithms in higher dimension.

Let $\vec{x} = (x_1, \ldots, x_d)$ be a d-tuple of positive reals. and π a permutation on $\{1, \ldots, d\}$.
Any subtractive algorithm can be composed of basic maps

$$T_\pi(\vec{x}) = \pi \circ (x_1, \ldots, x_{d-1}, x_d - x_1)$$

and then iterated

$$T^n(\vec{x}) = T_{\pi_n} \circ T_{\pi_{n-1}} \circ \cdots \circ T_{\pi_1},$$

where the permutations may depend on the argument $\vec{x}$, (for example, to `sort` in increasing order).

Algorithms in higher dimension.

$$T^n(\vec{x}) = T_{\pi_n} \circ T_{\pi_{n-1}} \circ \cdots \circ T_{\pi_1},$$

You can scale to unit size (say $\max x_j = 1$) at any moment:

$$f^n(\vec{x}) = \frac{1}{\max \hat{x}_j}\, \hat{x} \quad \text{for } \hat{x} = T^n(\vec{x}).$$

Thus f acts on

$$\Delta_d = \{\vec{x} = (x_1, \dots, x_{d-1}) : 0 \le x_i \le 1\}.$$

NB: The boundary $x_1 \equiv 0$ of Δ_d consists of neutral fixed points.

Selmer's Algorithm and Selmer's Generalised Algorithm

Let $a \in \mathbb{N}$ and define:

$$T(\vec{x}) = \mathtt{sort}(x_1, \ldots, x_a, x_{a+1} - x_1)$$

Here `sort` means: rearrange in increasing order.

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That means: typically. If there are rational relations between the coordinates, e.g. $x_{a+1} = x_1$, then x_1 can become zero in finitely many steps, and $\vec{x}$ won't change anymore.

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$$T(x_1, \ldots x_3) = \textbf{sort}(x_1, x_2 - x_1, x_3 - x_1),$$

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Therefore, if at some iterate $\eta > 0$, then $x_3^\infty = \eta > 0$.

In particular, Lebesgue measure is not ergodic.

We call $\{\vec{x} \in \mathbb{R}^3_+ : x_1 + x_2 < x_3\}$ the trapping region.

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For Selmer's Generalised Algorithm we have the following answer:
Trapping Theorem The r-th coordinate of $\vec{x}^{\infty} := \lim_{n\to\infty} T^n(\vec{x})$ is zero

$$\begin{cases} \text{almost surely} & \text{if } r \leq a+1, \\ \text{with probability strictly between 0 and 1} & \text{if } a+1 < r \leq \min\{a+b, 2a\}. \end{cases}$$

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For $r > 2a$ there is no Markov partition. Numerical experiments suggest that the r-th coordinate is positive for Lebesgue-a.e. $\vec{x}$.

Trapping regions

Recall Selmer's generalised algorithm

$$\begin{cases} T(\vec{x}) = \texttt{sort}(x_1, \ldots, x_a, x_{a+1} - x_1, \ldots, x_d - x_1), \\ f(\vec{x}) = \frac{1}{\hat{x}_d}\ \vec{x} \qquad \hat{x} = T(\vec{x}). \end{cases}$$

for $a, b \in \mathbb{N}$, $d = a + b$.

The **r-th Trapping Region** is

$$\mathcal{T}_r = \{\vec{x} \in \Delta_d : \frac{1}{r-a} \sum_{j \le r} x_j < x_r\}.$$

If $\vec{x} \in \mathcal{T}_r$, then $x_1, \ldots, x_{r-1}$ combined are too small to pull x_r to zero.

Rational Approximations

The map T is piecewise linear; each iterate T^k is given by an integer matrix A_k that depends on $\vec{x}$. Its inverse

$$A_k^{-1} = \begin{pmatrix} p_{1,k} & p_{1,k-1} & \cdots & \cdots & p_{1,k-d+1} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ p_{d-1,k} & p_{d-1,k-1} & \cdots & \cdots & p_{d-1,k-d+1} \\ q_k & q_{k-1} & \cdots & \cdots & q_{k-d+1} \end{pmatrix}.$$

is also integer and has non-negative entries.

In projective space, the columns of A_k^{-1} approximate $\vec{x}$, **provided $T^k(\vec{x}) \to \vec{0}$.**

Rational Approximations

Hence,

$$\left(\frac{p_{1,k-j}}{q_{k-j}}, \ldots, \frac{p_{d-1,k-j}}{q_{k-j}}\right) \qquad \text{for each } 0 \le j < d,$$

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To see this: Each column of A_k^{-1} is orthogonal to all rows of A_k, except one. Each column therefore spans the orthogonal complement of $d-1$ rows. If $\lim_{k\to\infty} A_k\vec{x} = \vec{0}$, then $\vec{x}$ is nearly orthogonal to all rows of A_k.

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Therefore, in projective space, $\vec{x}$ is close to the column vectors of A_k^{-1}. The quality of the approximation depends on the rate of convergence of $T^k(\vec{x}) \to \vec{0}$; if $\lim_k T^k(\vec{x}) \neq \vec{0}$, then T gives no approximations at all.

Qualifty of Approximations

Dirichlet's Theorem states that every vector $\vec{x}$ has infinitely many rational approximations $\vec{w}$ of denominator $q = q(\vec{w})$ such that

$$\|\vec{w} - \vec{x}\| \leq q^{-(1+1/(d-1))}. \tag{1}$$

(NB: The norm is taken after dividing by the largest coordinate!)

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The standard continued fraction algorithm in dimension $d - 1 = 1$ achieves this: It finds the best approximants, with $|w - x| \leq q^{-2}$.

In higher dimension, there is no known subtractive algorithm that finds all best approximants, or even achieves infinitely many approximants satisfying (1).

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The **best approximation exponent** is

$$\eta(\vec{x}) = \limsup_{k \to \infty} \sup_{0 \le i < d} \eta(\vec{w}_{k,i}, \vec{x})$$

The **uniform approximation exponent** is

$$\eta^*(\vec{x}) = \inf_k \frac{\min_{0 \le i < d} - \log \|\vec{w}_{k,i} - \vec{x}\|}{\max_{0 \le i < d} \log q_{k-i}}.$$

Qualifty of Approximations

These are chosen such that we can conclude

$$\|\vec{w}_{k,i} - \vec{x}\| \leq \begin{cases} q_{k-i}^{-\eta(\vec{x})} & \text{infinitely often} \\ q_{k-i}^{-\eta^*(\vec{x})} & \text{for all } k, i. \end{cases}$$

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Thus Dirichlet's Theorem states that

$$\eta(\vec{x}) \geq 1 + 1/(d-1)$$

for every $\vec{x}$, provided the algorithm finds infinitely many of the best approximations.

Qualifty of Approximations

Main Theorem: For Selmer's Generalised Algorithm with $a \geq \max\{2, b\}$, Lebesgue-a.e. vector $\vec{x} \in \mathcal{V}_d$ satisfies

$$\eta(\vec{x}) = \eta^*(\vec{x}) = 1 - \frac{\lambda_2}{\lambda_1} > 1,$$

where $\lambda_1 > 0 > \lambda_2$ are the largest two typical Lyapunov exponents of the cocycle A_k^{-1}.

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Remark: If all negative Lyapunov exponents are equal, then

$$1 - \frac{\lambda_2}{\lambda_1} = 1 + 1/(d-1).$$

Finding an algorithm with this equality of Lyapunov exponents is extremely unlikely.

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Remarks on the Proof:

The Main Theorem follows from the work of Lagarias '93, based on Oseledec' Theorem on Lyapunov exponents of matrix-valued cocycles (here A_k^{-1}).

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- Challenge: Estimate λ_1 and λ_2;
- Challenge: What about non-typical $\vec{x}$?

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