Piecewise Contractions are Asymptotically Periodic

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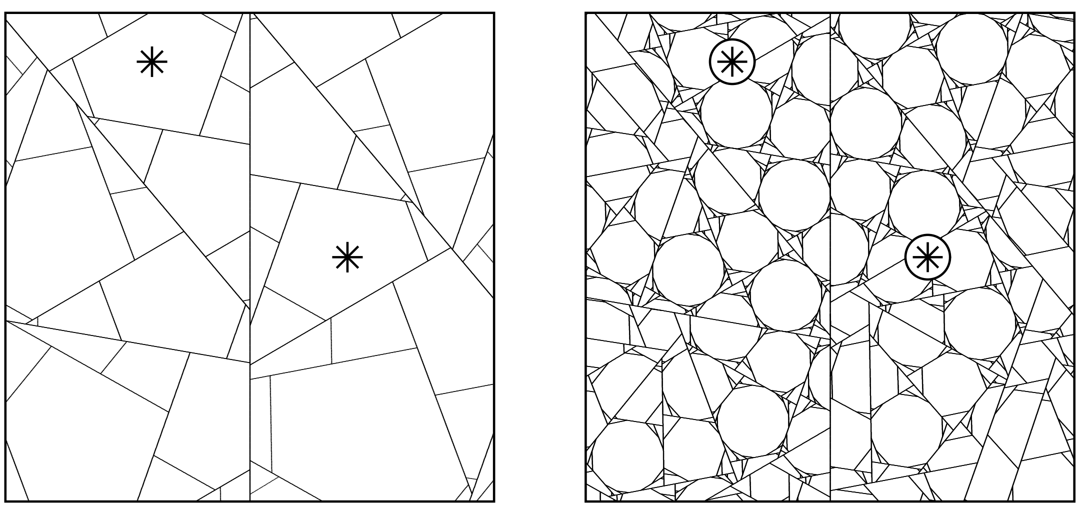
Piecewise Isometries on the Plane

Partition $\mathbb{C}$ into $\{X_k\}_{k=1}^{K}$. Fix $|\lambda_k| = 1$.

$$G(z) := G_k(z) = \lambda_k z + (1 - \lambda_k) w_k \qquad \text{if } z \in X_k.$$

Example: A Goetz map:

$$X_1 = \{Re\ z < 0\} \quad w_1 = 1 \qquad \lambda_1 = e^{0.7i}$$
$$X_2 = \{Re\ z > 0\} \quad w_2 = -1 + 2i \quad \lambda_2 = e^{0.7i}$$



Contraction (left) and Isometry (right).

Piecewise Isometries: Take $|\lambda_k| < 1$.

Motivation

Electronic Circuits with Dissipation.

- A **Buck converter** leads to the Goetz map.

- A Σ-Δ **modulator** (digital circuit).

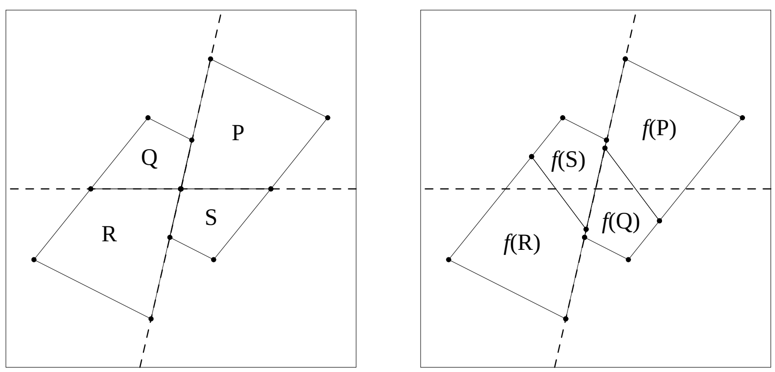
  The bandpass Σ-Δ modulator is a one-bit analogue to digital converter with desirable noise characteristics. The voltage at time interval n is described by a delay equation:

  $$x_{n+1} = F(x_n, x_{n-1}).$$

  The Z-transform description in the zero-input and unity gain case leads to a piece-wise isometry f_s (contraction if there is dissipation) of the plane (cf. Feely and Fitzgerald):

$$\begin{pmatrix} w_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2\cos\theta \end{pmatrix} \begin{pmatrix} w_n \\ t_n \end{pmatrix} + \begin{pmatrix} 0 \\ \text{sign } w_n - 2\cos\theta \text{ sign} t_n. \end{pmatrix}$$

The invariant set for this PWI is as follows:



Left: the invariant set $M = P \cup Q \cup R \cup S$ on which the mapping f_s is invertible. Right: the effect of f_s on each of P, Q, R and S leaves their union M unchanged.

Main Result

There exists R such that $B_R = \{|z| \leq R\}$ is forward invariant and attracts every orbit.

$$S = \cup_k \partial X_k \cap B_R$$

is a finite set of (rectifiable) arcs.

The **exceptional set** $\mathcal{E}$ is

$$\mathcal{E}_n = \cup_{0 \leq i \leq n} G^{-i}(S) \cap B_R$$

and

$$\mathcal{E} = \overline{\cup_{n \geq 0} \mathcal{E}_n}.$$

Theorem 1. *For all $\lambda \in \mathbb{D}^K$ and Lebesgue a.e. $w \in \mathbb{C}^K$, the following hold:*

- *There exists a finite N such that $\mathcal{E} = \mathcal{E}_N$.*
- *There is a finite number of attracting periodic orbits,*
- *Every point is attracted to one of them.*

Ideas of Proof

Step 1: I_n are itineraries of length n.

$\#I_n$ increases subexponentially

(zero entropy).

Step 2: The 'multivalued' omega-limit set

$$\widetilde{\omega}(z) = \cap_m \overline{\cup_{n \geq m} \widetilde{G^n}(z)}.$$

for

$$\widetilde{G_n}(z) = \bigcup_{e_0 \dots e_{n-1} \in I_n} \{G_{e_{n-1}} \circ \cdots \circ G_{e_0}(z)\}.$$

is the same for every $z \in B_R$. It is covered by

$\#I_n$ discs of radius $\lambda_{\max}^n$.

Hence $\dim_H(\widetilde{\omega}(z) = 0$.

Ideas of Proof (continued)

Step 3: $\dim_H(S) = 1$ so for typical $w \in \mathbb{C}^K$,

$$S \cap \widetilde{\omega}(z) = \emptyset.$$

(Use Lebesgue Density Theorem to show that

$$B_\varepsilon(\widetilde{G}_n(0)) \cap S = \emptyset$$

for n large and most w.)

Lemma 2. *If $\widetilde{\omega}(0) \cap S = \emptyset$, then there exists $N \in \mathbb{N}$ such that $\mathcal{E}_N = \mathcal{E}$.*

Step 4: Each component Y of $B_R \setminus \mathcal{E}_N$ intersects at most one periodic orbit, which attracts Y.

O. Feely and D. Fitzgerald, Non-ideal and chaotic behaviour in bandpass sigma-delta modulators, *Proceedings of NDES 1996*, Sevilla, Spain, pp. 399–404 (1996)

O. Feely, Nonlinear dynamics of bandpass sigma-delta modulation, *Proceedings of NDES*, Dublin, pp 33–36 (1995)