Markov extensions, inducing and Lyapunov exponents of measures

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Let (X, f) be a dynamical system, $Y \subset X$ and f_Y is the return map to Y. If μ_Y is an f_Y-invariant probability measure, then

- f preserves a σ-finite measure

$$\mu(A) = \sum_k \sum_{i=0}^{k-1} \mu_Y(f^{-i}(A) \cap \{\tau_Y = k\})$$

  μ is σ-finite because $\mu(f^k(\{\tau_Y \geq k\})) < 1$ for all $k \in \mathbb{N}$.

- $\int_Y \tau_Y \; d\mu_Y < \infty$ if and only if μ is finite. Indeed:

$$\begin{aligned} \mu(X) &= \sum_k \sum_{i=0}^{k-1} \mu_Y(f^{-i}(X) \cap \{\tau_Y = k\}) \\ &= \sum_k k\mu_Y(\{\tau_Y = k\}) = \int_Y \tau \; d\mu_Y \end{aligned}$$

If an interval $J \subset I$ is such that

1. $\mathrm{orb}(\partial J) \cap J^\circ = \emptyset$ (J is *nice*),

2. J contains only hyperbolic repelling periodic points,

3. $\overline{\mathrm{orb}(\mathrm{Crit})} \cap \overline{J} = \emptyset$

then the first return map to J is Markov map with onto branches.

Induced maps with "good" transfer times

A good transfer time $n = n(x)$ for an induced (jump) transformation on J (with extension $T \supset J$) is such that

1 $f^n(x) \in J$

2a there exists $J_x \supset x$ such that $f^n : J_x \to J$ is monotone onto (n is natural), or

2b there exists $T_x \supset J_x \supset x$ such that $f^n : T_x \to T$ is monotone onto (n is naturally extendible).

3 n is minimal with respect to properties 1-2.

Then $F(x) := f^{n(x)}$ is a natural (naturally extendible) induced map.

Note that naturally extendible reduces to natural if you take $T = J$.

The **Markov extension** (Hofbauer tower) is $(\hat{I}, \hat{f})$ where $\hat{I} = \sqcup_n D_n$ and $\pi \circ \hat{f} = f \circ \pi$.

If f is non-renormalisable and has no periodic attractors, then $(\hat{I}, \hat{f})$ is transitive.

Liftability of measures

An f-invariant probability measure μ is **liftable** if, given $\mu_1 = \mu \circ \pi|_{D_2}$, the sequence of Cesaro means

$$\widehat{\mu}_n := \frac{1}{n} \sum_{k=0}^{n-1} \widehat{\mu}_1 \circ f^{-k}$$

converges vaguely (along a subsequence) to a probability measure $\widehat{\mu}$.

Theorem 1 (Keller) *If μ is an f-invariant probability measure such that*

- *the entropy $h_\mu > 0$, or*
- *the Lyapunov exponent $\lambda(\mu) > 0$,*

then μ is liftable.

Remark: The second condition is actually equivalent to liftability, provided μ is non-atomic.

For intervals $J \subset T$ in I, let

$$\hat{T} = \sqcup \{\pi^{-1}(T) \cap D_n : \pi(D_n) \supset T\},$$

and

$$\hat{J} = \pi^{-1}(J) \cap \hat{T}.$$

Theorem 2 • *If (F, J, T) is a natural(ly extendible) induced map with transfer time $n(x)$, then $\hat{F} : \hat{J} \to \hat{J}$ defined by*

$$\hat{F}(\hat{x}) = \hat{f}^{n(\pi(\hat{x}))}(\hat{x})$$

is the first return map to $\hat{J}$.

- *Conversely, if $(\hat{F}, \hat{J})$ is a first return map, then $F : \hat{J} \to J$ defined by*

$$F(x) = \pi(\hat{F}(\pi^{-1} \cap \hat{J}))$$

is a natural(ly extendible) induced map on J.

Corollaries:

- If $\hat{J}_0$ is compactly contained in some D_k and the first return map $\hat{F}_0$ is defined for Leb-a.e. $x \in \hat{J}_0$, then any interval compactly contained in some D_k has an acip for its first return map.

  Sketch of Proof: $(\hat{J}_0, \hat{F}_0)$ has an acip $\hat{\mu}_0$ (Folklore Theorem). Pull it back to obtain a σ-finite acim on $\hat{I}$. By transitivity, $\frac{d\hat{\mu}}{dLeb} > 0$. Therefore the first return map $\hat{F}_1$ of any interval $\hat{J}_1$ is defined Leb-a.e. If $\hat{J}_1$ is compactly contained in some D_k, then $\hat{F}_1$ preserves an acip.

- If (I, f) has some naturally extendible induced map with an acip, then every naturally extendible induced map has an acip,

- If (I, f) has some natural induced map with acip and integrable transfer time, then every natural induced map has an acip with integrable transfer time.

Theorem 3 (Bruin & Luzzatto) *Let f and $\tilde{f}$ be conjugate ($f \circ \psi = \psi \circ \tilde{f}$) C^3 multimodal maps with nonflat critical points.*
If μ is a non-atomic f-invariant probability measure and $\tilde{\mu} = \mu \circ \psi$, then the signs of the Lyapunov exponents $\lambda(\mu)$ and $\lambda(\tilde{\mu})$ are the same.

Remark: By Ruelle's inequality $\lambda(\mu) \geq h_\mu$, and h_μ is preserved under conjugacy. So if $h_\mu > 0$, the result is immediate.

Remark: The fact that μ is non-atomic is important. Indeed, if $\mu = \delta_p$ for fixed point p, then one can alter the sign of $\lambda(\mu)$ from $+$ to 0 (from $-$ to 0) if p is repelling (attracting).

Proposition 4 *The sign of the lower pointwise Lyapunov exponent $\underline{\lambda}(x)$ is not preserves by conjugacy.*

This was shown by Przytycki, Rivera-Letelier and Smirnov in the bimodal setting. It holds for unimodal maps as well.

Conjecture: If $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)} \not\to$ an atomic measure, then the sign of the upper pointwise Lyapunov exponent $\overline{\lambda}(x)$ is preserved under conjugacy.

Remark: For unimodal maps: If

$$\inf\{\lambda(\mu) : \mu \text{ erg. } f\text{-inv. prob. meas.}\} > 0,$$

then f is Collet-Eckmann and $\overline{\lambda}(x) > 0$ for all x non-precritical.

Example: There is a quadratic map such that $\lambda(\mu) > 0$ for all ergodic f-invariant probability measures, but there is x such that $\lambda(x)$ exists and $= 0$.

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