Arc-ray subcontinua for unimodal inverse limit spaces.

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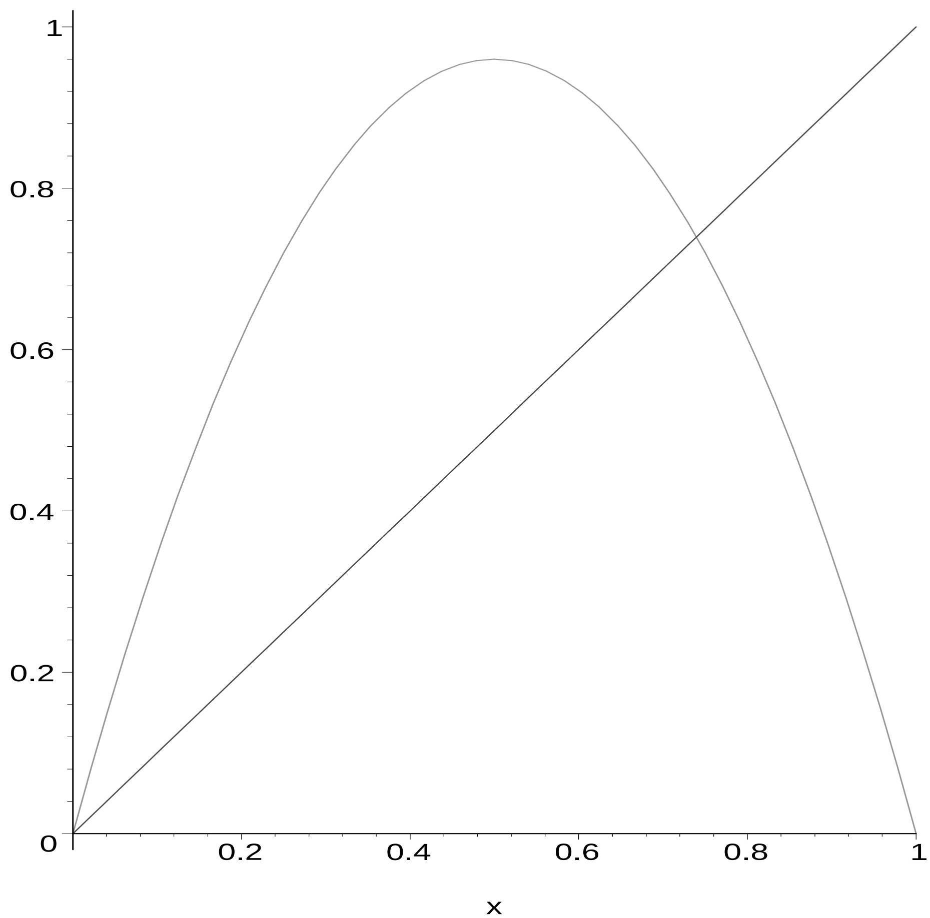
Definition: Let $f : I \to I$ be a unimodal interval map, so it has a single turning point c. This map is called **locally eventually onto (leo)** if for every non-degenerate interval $J \subset I$ there is n such that $f^n(J) \supset [f(c), f^2(c)]$.



Example of a unimodal map.

Definition The **inverse limit space** (I, f) of $f : I \to I$ is the space

$\{x = (x_1, x_2, x_3 \dots) : x_i \in I$ and $x_i = f(x_{i+1})\}$,

equipped with product topology.

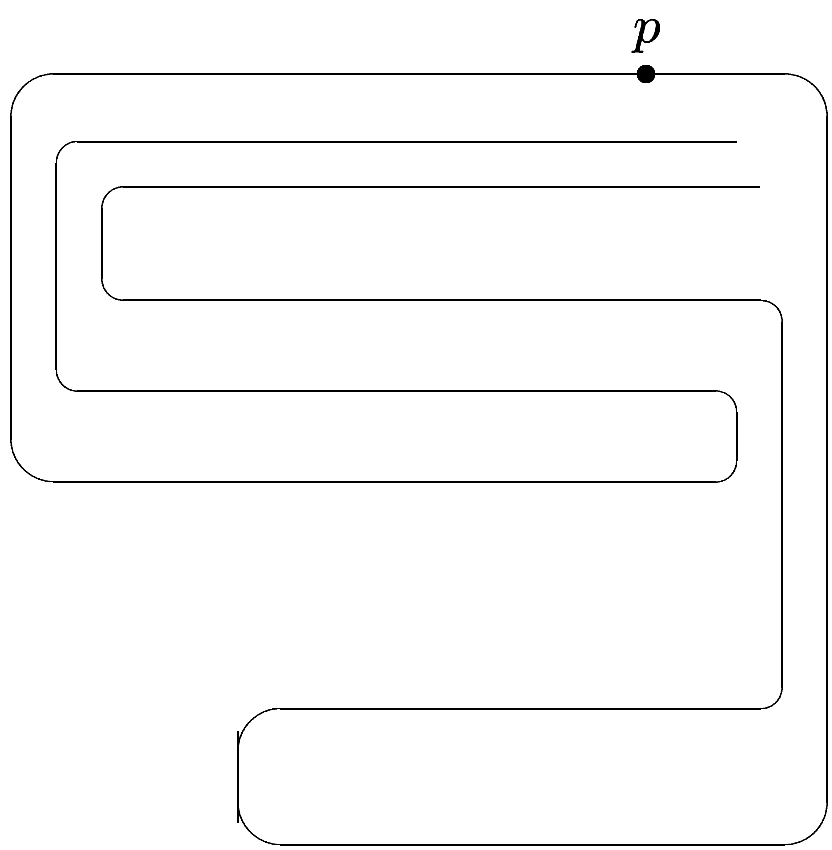
The **induced homeomorphism** is the map

$$\hat{f}(x_1, x_2, x_3, \dots) = (f(x_1), x_1, , x_2, x_3, \dots)$$

Let π_i denote the i-th projection:

$$\pi_i(x) = x_i$$

Examples of unimodal inverse limit spaces are the Knaster continuum ($f(x) = 4x(1 - x)$), or the one with c periodic of period 3. The latter has a single infinite Wada channel.



The classification of inverse limit spaces of unimodal maps with a finite critical orbit is in an advanced stage: If

$$f, \tilde{f} : I \to I$$

are non-conjugate unimodal maps with finite critical orbit, then (I, f) and $(I, \tilde{f})$ are non-homeomorphic (L. Kailhofer 2004).

The natural question is:

How to classify inverse limit spaces for maps with infinite critical orbit?

Definition: A **continuum** is a connected, compact, metric space. A **subcontinuum** H is (I, f) is a subset of (I, f) which is a continuum on its own right.

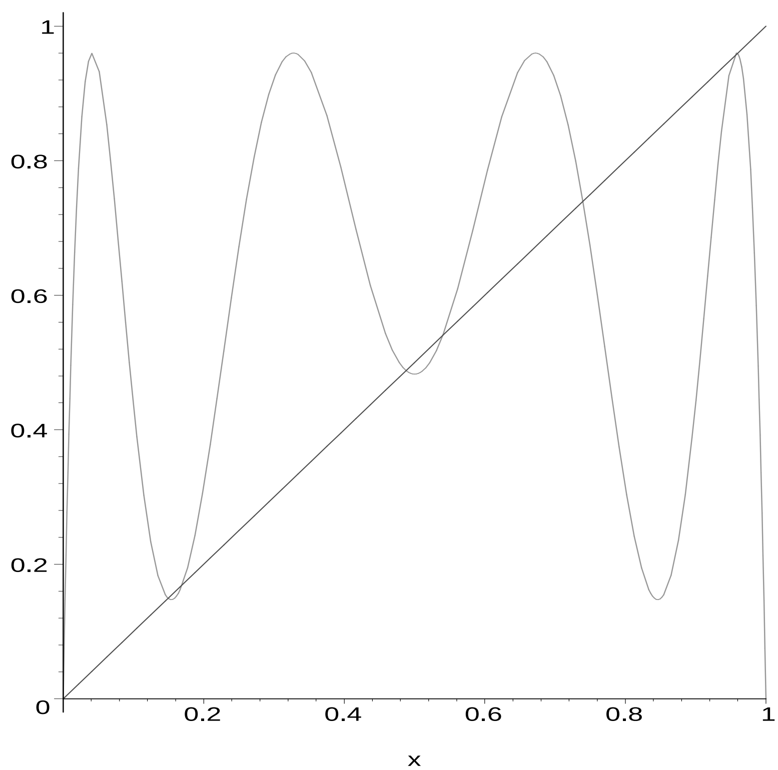
Lemma: If H is a subcontinuum, then $\pi_i H$ is an arc, (i.e. a homeomorphic copy of $[0, 1]$).

If $\pi_i H \ni c$ for only finitely many i, then H is an arc or a point.

Lemma: H is a proper subcontinuum if and only if $|\pi_i H| \to 0$ as $i \to \infty$.

Central branches and cutting times

A branch of f^n is a maximal interval J on which f^n is monotone. It is a **central branch** if $c \in \partial J$.



An integer $n > 0$ is a **cutting time** if $f^n(J) \ni c$ for a central branch of f^n.

Notation of cutting times:

$$S_0 < S_1 < S_2 < \dots$$

If f is leo and has a non-periodic critical point, then there are infinitely many cutting times, and there is a map, the **kneading map**

$$Q : \mathbf{N} \to \mathbf{N} \cup \{0\}$$

such that

$$S_k - S_{k-1} = S_{Q(k)}$$

The combinatorial class of f is completely determined by the cutting times (and hence by the kneading map).

A map f is called **long-branched** if there exists $\varepsilon > 0$ such that

$$|f^n(J)| > \varepsilon$$

for all $n \geq 1$ and all branches of f^n.

Assume f is leo. If c is periodic, then f is longbranched. Otherwise, f is long-branched if and only if the kneading map is bounded.

Remark: Long-branched maps **can** have a recurrent turning point, even if c is not periodic.

Definition: n is a **critical projection** of subcontinuum H if $\pi_n H \ni c$. Denote the critical projections by

$$n_1 < n_2 < n_3 < n_4 < \dots$$

Then

$$n_i - n_{i-1} = S_{k_i} \text{ is a cutting time.}$$

Write

$$\pi_{n_i} = M_i \cup L_i$$

where $\overline{M_i} \cap \overline{L_i} = c$ and $f^{S_{k_i}}(M_i) = \pi_{n_{i-1}} H$.

Proposition: Every subcontinuum H of (If) is a point or contains a dense ray. (A ray is a continuous copy of $\mathbf{R}$ or $[0, \infty)$.)

Corollary: The only proper subcontinua of the inverse limit space of a long-branched unimodal map are arcs and points.

Proof of Corollary Let $n_i > n_{i+1}$ be two critical projections. Then $f(c), f^{n_{i+1}-n_i+1}(c) \in \pi_{n_i-1}H$. By long-branchedness,

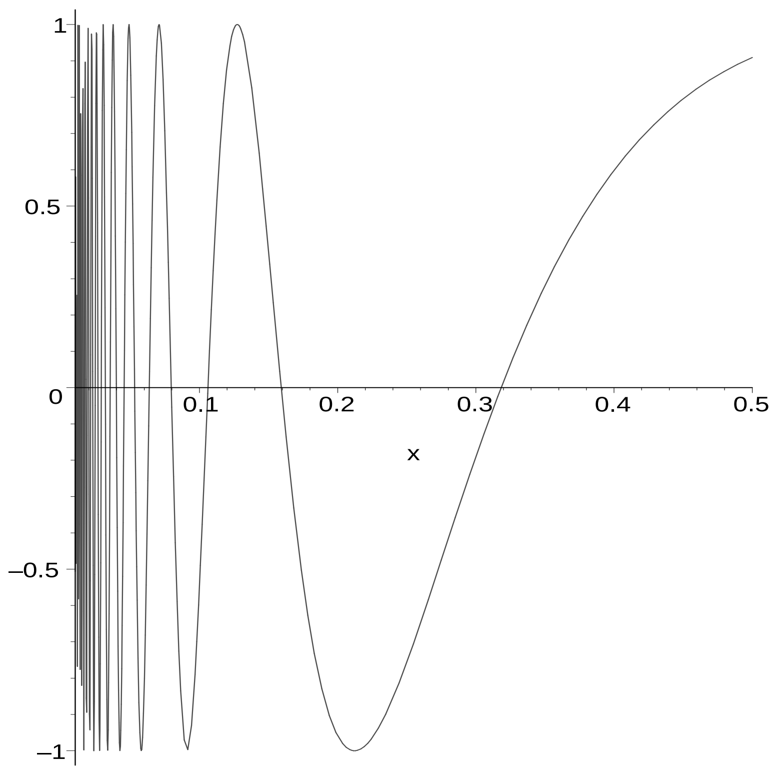
$$|f(c) - f^{n_{i+1}-n_i+1}(c)| > \varepsilon.$$

If there are only finitely many n_i, then H is a point or an arc. If there are infinitely many n_i, then $\lim_n |\pi_n H| \not\to 0$, so H is not a proper subcontinuum. $\square$

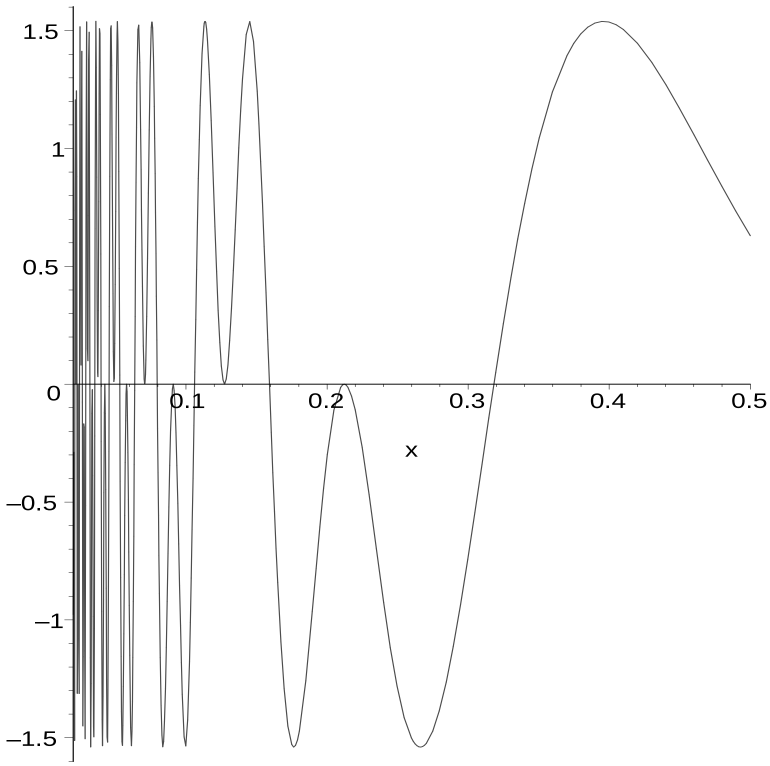
Definition: A continuum H is an **arc+ray** if

$$H = A \cup R_H$$

where A is an arc, and R_H a dense ray in H.



Example: The $\sin\frac{1}{x}$-continuum.



Example: The MW-continuum, generated as the closure of the graph of $\sin\frac{1}{x} + \sin\frac{3}{x}$.

Theorem Let H be a subcontinuum with critical projections $\{n_i\}$. If there exists i_0 such that for all $i_0 < j < i$,

$$f^{n_i - n_j}(L_{n_i}) \not\ni c$$

then H is a point, an arc or an arc+ray continuum, or two arc+ray continua glued together at their rays.

If additionally $Q(k) \to \infty$, then H is an arc or a point.

Barge, Brucks & Diamond (1996) have shown that subcontinua as complicated as unimodal inverse limits itself are abundant in inverse limit spaces of typical unimodal maps.

Barge & Diamond (1999) showed that the same is true in the closure of the unstable manifold at homoclinic tangency (modulo some non-resonance condition).

The construction of such subcontinua can be set in a combinatorial framework: it is in principal possible to read the subcontinua of (I, f) off from the kneading data.

Theorem: Let $\mathcal{F}$ be a finite or countable collection of unimodal maps, each of them having a periodic critical point. Then there exists a unimodal map g such that

- for each $f \in \mathcal{F}$ there exist a subcontinuum H of (I, g) such that $H \simeq (I, f)$.

- if H is a subcontinuum (I, g), then H is a point, an arc, an arc+ray subcontinuum or homeomorphic to (I, f) for some $f \in \mathcal{F}$.

Conjecture: If f and $\tilde{f}$ have eventually the same kneading map, then (I, f) and $(I, \tilde{f})$ have the same subcontinua.