σ-invariant measures for interval maps

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Setting:

Let $f : I \to I$ be a C^2 unimodal map, with critical point c and critical order $1 < \ell \leq \infty$. Let $c_n = f^n(c)$.

Assume that $c_2 < c_1$ and $c_2 \leq c_3$. Scale I such that $I = [c_2, c_1]$. Let $q \in [c, c_1]$ be the fixed point.

An acip μ is an invariant probability measure that is absolutely continuous w.r.t. Lebesgue.

An aciσ μ is an infinite σ-finite invariant measure that is absolutely continuous w.r.t. Lebesgue.

Attractors:

An attractor A is one of the following:

1. a periodic orbit.
2. a periodic interval (period 1 is possible; in this case $A = I$)
3. $\omega(c)$, when f is infinitely renormalizable.
4. $\omega(c)$ but not infinitely renormalizable: the wild attractor. $\text{orb}(x) \to \omega(c)$ for Lebesgue a.e. point, but a 2nd Baire category behaves as in case 2.

   Case 4. does not occur for $\ell = 2$ [**Lyub**], but does for ℓ sufficiently large.

For C^2 maps there are no wadering intervals H, i.e., $f^n|H$ is monotone for every n, but $f^n(H) \not\to \text{orb}(p)$.

Ergodicity and conservativity of Lebesgue measure:

Legesgue measure is

ergodic	except in case 1.
conservative (and exact)	if f is nonrenormalizable
dissipative	if f is finitely renormalizable
totally dissipative	if f is infinitely renormalizable or has a wild attractor

If f is (in)finitely renormalizable, then f is not exact. If f has a wild attractor, then f may be exact or not. (E.g. the Fibonacci map with wild attractor is not exact.)

If Lebesgue measure is totally dissipative, then there is no acip, but there is a dissipative aci σ.

Existence of acips (finite):

Historically, acips were found for

- $f(x) = 4x(1-x)$, [**NU**].
- if $c \notin \omega(c)$, [**Mis**], provided $\ell < \infty$.
- if $|Df^n(c_1)| \geq C\lambda^n$ $(\lambda > 1)$, [**CE**], provided $\ell < \infty$. (Already [**Jak**] for $|Df^n(c_1)| \geq C\lambda^{\sqrt{n}}$.)
- if $\sum_n |Df^n(c_1)|^{-1/\ell} < \infty$, [**NvS**].
- if $\liminf_n |Df^n(c_1)| \geq K = K(\ell)$, [**BSS03**] and $\frac{d\mu}{dx} \in L^p$ for $1 \leq \ell/(\ell-1)$, [**BRSS**].

Non-existence of acips:

The existence of acips is prevented if (there is)

(a) Cascade of Johnson boxes [**John**].

(b) Cascade of almost saddle nodes (almost tangencies), [**Br94**].

(c) $\ell =$ "sufficiently" ∞, [**BeMi**, **Thun**]

(d) $\ell \gg 2$ and Fibonacci-like combinatorics, [**BKNS**, **Br98a**].

The existence of an acip is not a topological property, even if $\ell \equiv 2$, [**Br98b**].

If there is no acip, there is still a range of possible physical measures, e.g. Dirac measure δ_q, [**HK90**].

Existence of aciσs:

- Constructions by Hofbauer & Keller, Bruin
- If $\omega(c)$ is nowhere dense, then there exists an acip or acia, σ with $\mu(J) < \infty$ for any compact interval in $I \setminus \omega(c)$.
- If $\omega(c) = I$ and $\mu(I) = \infty$, then $\mu(J) = \infty$ for every non-degenerate interval J.
- If there is a wild attractor, then there is a dissipative aci σ, [**Mar**, **BH01**]
- If f is infinitely renormalizable, then there is a dissipative aci σ, [**BH01**]
- $\exists f$, $\omega(c) = I$, $\exists$ aci σ. Sets of finite μ-measure are certain Cantor sets.
- If f is only C^1, then the absence of an aci σ, [**BH01**] and [**Quas**].

Fibonacci-like maps and aciσs:

For Fibonacci-like maps, $\omega(c)$ is a minimal Cantor set, with strong recurrence properties, and $f|\omega(c)$ is uniquely ergodic, [**BSS06**].

The proof of the existence of Cantor attractors for Fibonacci maps depends on an induced map similar to this:

$$F(x) = f^{S_{k-1}} \quad \text{for} \quad x \in (z_{k-1}, z_k) \cup (\hat{z}_k, \hat{z}_{k-1})$$

where z_k and $\hat{z}_k$ are the closest points to$c$ that are mapped to c by $f^{S_k}(z_k)$, and cutting times satisfy

$$S_0 = 1, \quad S_k = S_{k-1} + S_{Q(k)}.$$

To evade the effects of non-linearity of the induced map we create a countable piecewise linear map such that the induced map has linear branches.

Let kneading map Q, $\varepsilon_k = |z_k - z_{k-1}|$ and slopes $\kappa_k = |Df(|z_k - z_{k-1})|$ satisfy:

$$Q(k+1) > Q(Q^2(k)+1)$$

$$\kappa_0 := \frac{1}{2\varepsilon_0} \text{ and } \kappa_1 := \frac{1}{\varepsilon_1}\sum_{i\geq 1}\varepsilon_i = \frac{\frac{1}{2}-\varepsilon_0}{\varepsilon_1}.$$

$$\kappa_j := \begin{cases} \frac{s_j}{\kappa_0}\frac{\kappa_{j-1}}{s_{j-1}} & \text{if } Q(j-1)=0, \\ \frac{s_j \cdot \kappa_{j-1}}{s_{j-1}\cdot s_{Q(j-1)}\cdot s_{Q^2(j-1)+1}} & \text{if } Q(j-1)>0. \end{cases}$$

and using notation: $x^f := f(x)$

$$\frac{s_j}{\kappa_j}|c^f - z_j^f| = \frac{s_j}{\kappa_j}\sum_{i=j+1}^{\infty}\kappa_i\varepsilon_i \leq \varepsilon_{Q(j)},$$

$$\frac{s_j}{\kappa_j}|c^f - z_j^f| = \frac{s_j}{\kappa_j}\sum_{i=j+1}^{\infty}\kappa_i\varepsilon_i \leq \frac{\varepsilon_{Q^2(j)+1}}{s_{Q(j)}}$$

Proposition 1 *If f is a unimodal map satisfying the above, then the induced map F is linear on each set (z_{k-1}, z_k) and $(\hat{z}_k, \hat{z}_{k-1})$ and the slopes slopes satisfy*

$$s_j := |\, DF|_{(z_{k-1},z_k)}\,| = \frac{1}{\varepsilon_j}\sum_{i\geq Q(j)+1}\varepsilon_i.$$

Theorem 2 *Let Q be the Fibonacci kneading map: $Q(k) = \max(0, k-2)$. Let*

$$\lambda \in (0,1) \quad \text{and} \quad \varepsilon_j = \frac{1-\lambda}{2}\lambda^j.$$

Then the corresponding countably piecewise linear unimodal map f satisfies:

- *The critical order*

  $$\ell = 3 + \frac{2\log(1-\lambda)}{\log \lambda}$$

  .

- *If $\lambda \in (\frac{1}{2}, 1)$, i.e. $\ell > 5$, then f has a wild attractor*

- *If $\lambda \in (\frac{2}{3+\sqrt{5}}, \frac{1}{2})$, i.e. $4 < \ell < 5$, then f has no wild attractor, but an absolutely continuous infinite σ-finite invariant measure.*

- *If $\lambda \in (0, \frac{2}{3+\sqrt{5}})$, i.e. $\ell < 4$, then f has an absolutely continuous invariant probability measure.*

Write

$$\varphi_n = k \quad \text{if} \quad F^n(x) \in (z_{k-1}, z_k) \cup (\hat{z}_k, \hat{z}_{k-1}).$$

The **drift** at state k is $\mathbf{E}(\varphi_n - k \mid \varphi_{n-1} = k)$.

For the Fibonacci map, we compute:

$$\begin{aligned} \mathbf{E}(\varphi_n - k \mid \varphi_{n-1} = k) &= \frac{\sum_{i \geq k-1} i\varepsilon_i}{\sum_{i \geq k-1} \varepsilon_i} - k \\ &= \frac{\lambda}{(1-\lambda)} - 1. \end{aligned}$$

Proposition 3 *If the drift > 0 for all k sufficiently large, then f has a wild attractor.*
For the Fibonacci map, this happens for $\lambda > \frac{1}{2}$.

The transition matric of the induced system is

$$(p_{i,j})_{i,j} = (1-\lambda) \begin{pmatrix} 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \dots & \dots \\ 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \dots & \dots \\ 0 & 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \dots \\ 0 & 0 & 1 & \lambda & \lambda^2 & \lambda^3 & \dots \\ \vdots & \vdots & 0 & 1 & \lambda & \lambda^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It has a left eigenvalue v for eigenvalue 1 with

$$|v|_1 = 1 \quad \text{and} \quad v_n = \kappa_n \rho^n,$$

where

$$\rho \in (0,1) \quad \text{for} \quad \lambda < \frac{1}{2} \quad \text{and} \quad \frac{1}{n} \log \kappa_n \to 0.$$

Proposition 4 *If*

$$\sum_k S_k v_k < \infty$$

then f has a acip. Otherwise, f has an aciσ.

For the Fibonacci map, we get

$$\begin{cases} \lambda > \frac{1}{2} & \exists \text{ wild attractor, dissipative aci}\sigma. \\ \frac{2}{3+\sqrt{5}} < \lambda < \frac{1}{2} & \exists \text{ conservative aci}\sigma. \\ \lambda < \frac{2}{3+\sqrt{5}} & \exists \text{ acip} \end{cases}$$

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