Mixing for almost Anosov maps

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joint with

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Krynica, Poland, June 2018

Marek Zdun

Overlap with this talk and the work of Marek Zdun revolves around the question:

> *When is a diffeomorphism f* **embeddable** *in a flow of a vector field? That is, when is f equal to the time-1 map of a flow φ^t?*

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Globally on an entire manifold, this is seldom the case.

But local solutions are often useful too. In bifurcation theory and in this talk.

Statistical Laws

Let (X, f) be a chaotic dynamical systems, and $v : X \to \mathbb{R}$ an observable.

Due to the chaos, precise predictions of $V_n := v \circ f^n$ are impossible.

If there is a "good" f-invariant measure μ, one can hope to prove **statistical laws**:

Statistical Laws

- Central Limit Theorem (CLT):

$$\frac{\sum_{j=0}^{n-1}(V_j - \mathbb{E}(v))}{\sqrt{n}} \Rightarrow_d \mathcal{N}(0, \sigma^2)$$

provided $\mathbb{E}(|v|^2) < \infty$ and $\mathbb{E}(|v|) < \infty$.

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- Many further laws, with fancy names and acronyms: (LIL, WIP, ASIP, LLT, return time statistics, extremal value statistics,....)

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- Dynamical systems with intermittency (specifically: neutral fixed points) tend to exhibit polynomial mixing.
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  to **accelerated** the dynamics to a hyperbolic system. The **tail** of this inducing scheme predicts the rate of mixing.
- Upper bounds for mixing are relatively easy. Lower bounds (i.e., upper bounds are sharp) are much harder, and require precise estimates (often regular variation) of the tails

  $$\mu(y \in Y : \tau(y) \geq n) \sim n^{-\beta} \ell(n),$$

  where $\ell(n)$ is a **slowly varying function** (e.g. constant, logarithmic).

Intermittent Maps

The Pomeau-Manneville map $f : [0,1] \to [0,1]$ is defined as

$$f(x) = \begin{cases} x(1 + (2x)^\alpha) & x \in [0, \frac{1}{2}]; \\ 2x - 1 & x \in (\frac{1}{2}, 1]. \end{cases} \qquad \alpha > 0.$$

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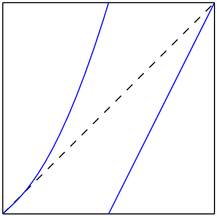
There is a finite resp. infinite absolutely continuous measure (acim) if $\alpha < 1$ resp. $\alpha \geq 1$. Due to the neutral fixed point at 0, the density $\frac{d\mu}{d\,Leb}$ is always unbounded.



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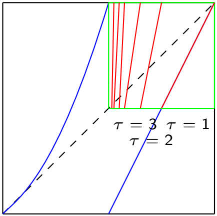

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which is the key estimates for e.g. mixing rates:

$$\frac{\rho_n(v,w)}{\int v \int w} = \begin{cases} 1 + \frac{1}{\mathbb{E}(\tau)} \sum_{j>n} \mu(\tau > j) + O(d_n) & \beta > 1 \\ (d_1 n^{\beta-1} + \dots + d_q n^{q(\beta-1)}) + O(d_n) & \beta \in (\frac{1}{2}, 1), \end{cases}$$

for known $d_1, \dots, d_q$, $q = \lfloor \frac{2\beta-1}{2-2\beta} \rfloor$, and error terms $d_n = d_n(\beta, \|v\|, \|w\|)$, and for sufficiently regular observables v, w supported on $Y = [\frac{1}{2}, 1]$.

Intermittent Maps

An **almost Anosov map** is an Anosov map (such as **Arnol'd's catmap**)

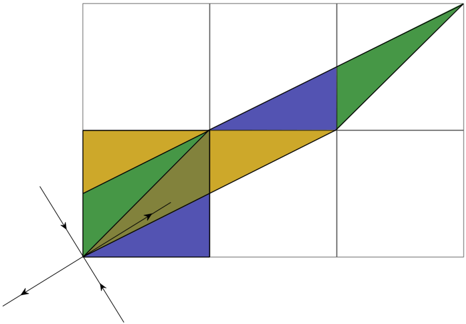
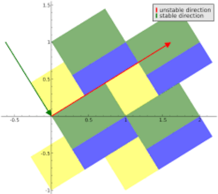


Figure: Catmap with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

with a neutral fixed point.

Intermittent Maps

Assume this neutral fixed point $p = (0, 0)$ is in the interior of an element P_0 of the Markov partition:



Using a local surgery, we can give it local formula:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(1 + a_0 x^\kappa + a_2 y^\kappa + O(|(x,y)|^{\kappa+1})) \\ y(1 - b_0 x^\kappa - b_2 y^\kappa + O(|(x,y)|^{\kappa+1})) \end{pmatrix}$$

for $a_0, a_2, b_0, b_2 \geq 0$, $\Delta := a_2 b_0 - a_0 b_2 \neq 0$, $\kappa \geq 1$.

Intermittent Maps

The task is to estimate the measure of the strips $\{\tau = n\}$, $n \geq 2$.

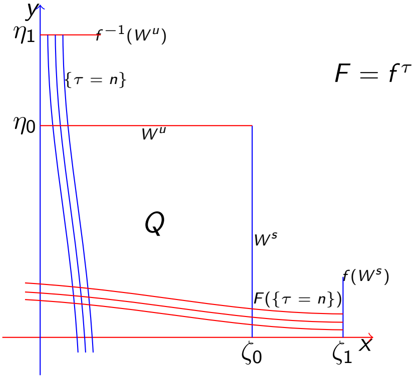


Figure: The first quadrant Q of the rectangle P_0, with stable and unstabe foliations drawn vertically and horizontally, respectively.

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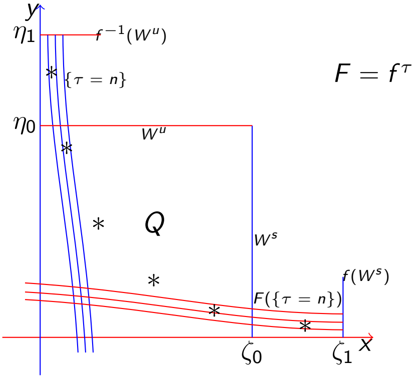

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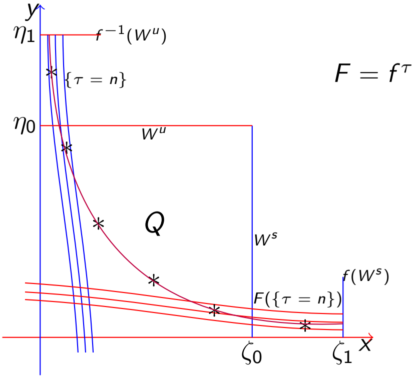

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Strategy

- Find a vector field X

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = X \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(a_0 x^\kappa + a_2 y^\kappa + O(|(x,y)|^{\kappa+1})) \\ -y(b_0 x^\kappa + b_2 y^\kappa + O(|(x,y)|^{\kappa+1})) \end{pmatrix}$$

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- Find a **first integral**

$$L(x,y) = \begin{cases} x^u y^v (\frac{a_0}{v}\ x^\kappa + \frac{b_2}{u}\ y^\kappa) & \text{if } \Delta > 0; \\ x^{-u} y^{-v} (\frac{a_0}{v}\ x^\kappa + \frac{b_2}{u}\ y^\kappa)^{-1} & \text{if } \Delta < 0. \end{cases}$$

  where $u = \frac{\kappa b_2}{\Delta}(a_0 + b_0)$, $v = \frac{\kappa a_0}{\Delta}(a_2 + b_2)$, for the truncated vector field:

$$X_{trunc} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(a_0 x^\kappa + a_2 y^\kappa) \\ -y(b_0 x^\kappa + b_2 y^\kappa) \end{pmatrix}$$

Strategy

- Use a new coordinate $M = y/x$ to rewrite the system to

$$\begin{cases} \dot{L} = 0 \\ \dot{M} = -M(c_0 + c_2 M^\kappa)x^\kappa \end{cases} \qquad c_i = a_i + b_i. \tag{1}$$

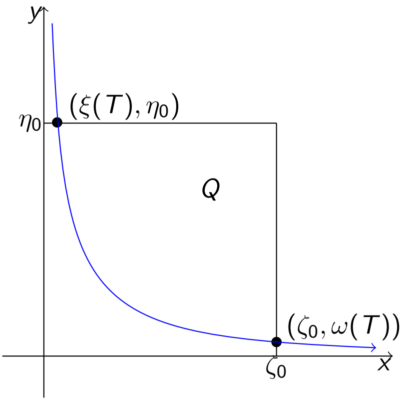
- Solve x from

$$L(x,y) = x^u y^v \left(\frac{a_0}{v}x^\kappa + \frac{b_2}{u}y^\kappa\right) = \xi^u \eta^v \left(\frac{a_0}{v}\xi^\kappa + \frac{b_2}{u}\eta^\kappa\right) = L(\xi,\eta)$$

to find an approximate solution to (1)

Strategy

Let $(\xi(T), \eta_0)$ and $(\zeta_0, \omega(T))$ be the entry and exit point of a flow-line that takes exactly time T to pass through Q.



Strategy

Theorem

There are functions $\xi_0(\eta), \omega_0(\eta), \xi_1(\eta), \omega_1(\eta) > 0$ *independent of* T *such that*

$$\xi(\eta, T) = \xi_0(\eta) T^{-\beta_2} \left(1 - \xi_1(\eta) T^{-1} + O(T^{-2}, T^{-\kappa\beta_2})\right)$$

and

$$\omega(\eta, T) = \omega_0(\eta) T^{-\beta_0} \left(1 - \omega_1(\eta) T^{-1} + O(T^{-2}, T^{-\kappa\beta_0})\right).$$

for $\beta_0 := \frac{a_0+b_0}{\kappa a_0}$ *and* $\beta_2 := \frac{a_2+b_2}{\kappa b_2}$.

Strategy

- Estimate the effects of the extra error terms $O(|(x, y)|^{\kappa+1})$ in the vector field X on L and the eventual solution.

Theorem

For the non-truncated vector field, we obtain

$$\xi(\eta, T) = \xi_0(\eta) T^{-\beta_2} \left(1 + O(\max\{n^{-\beta_*}, T^{-1} \log T)\}\right),$$

where $\beta_* = \frac{1}{\kappa} \min\left\{1, \frac{a_2}{b_2}, \frac{b_0}{a_0}\right\}$ *and an analogous formula for* $\omega(\eta, T)$.

The tails of the original system (w.r.t. the SRB-measure) follow.

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