Properties of Fibonacci-like Inverse Limit Spaces

Henk Bruin

University of Surrey

Throughout, unimodal maps $T : I \to I$ are restricted to the **core** $I = [c_2, c_1]$, $c_n = T^n(c)$ where c is the **critical point**.

Definition: The **inverse limit space** is

$$\begin{aligned} X &:= (I, T) \\ &= \{(\dots x_{-3}, x_{-2}, x_{-1}, x_0) : \\ &\qquad x_i = T(x_{i+1}) \in I \text{ for all } i < 0\}. \end{aligned}$$

equipped with product topology. Let

$$\pi_n : X \to I, \quad \pi_n(x) = x_n$$

be the n**-th projection**.

The **induced homeomorphism** is

$$\hat{T}(\dots x_{-3}, x_{-2}, x_{-1}, x_0) \mapsto$$
$$(\dots x_{-3}, x_{-2}, x_{-1}, x_0, T(x_0))$$

Basic question (Ingram):

If T and $\tilde{T}$ are non-conjugate, are X and $\tilde{X}$ non-homeomorphic?

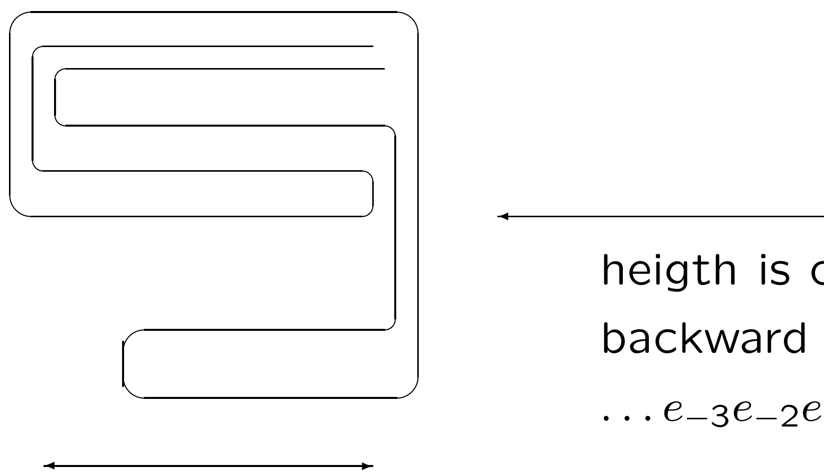
The classification of X when orb(c) is finite is complete.
(Barge & Diamond '95, Swanson & Volkmer '98, Bruin '00, Kailhofer '03, Stimač '04, Block et al. '05).

What if orb(c) is infinite?

At least uncountably many non-homeomorphic inverse limits.
(Barge & Diamond '98, Brucks & Bruin '99, Raines '04).

Abundance of subcontinua (sometimes very exotic, with "self-similarity").
(Barge, Brucks & Diamond '96, Brucks & Bruin '99)

The Inverse Limit



heigth is coded by backward itinerary $\dots e_{-3}e_{-2}e_{-1}e_0 \in \{0,1\}^{\mathbf{Z}_-}$

width is $[c_L, c_R]$

where

- $\begin{matrix} R \\ L \end{matrix} = \sup\{n > 0 : e_{-n+1} \dots e_0 = \nu_1 \dots \nu_n,$ $\#$ of $1s$ in $\nu_1 \dots \nu_n$ is $\begin{matrix} \text{even} \\ \text{odd} \end{matrix}\}$

where

$$\nu = \nu_1\nu_2\nu_3\nu_4 \dots \in \{0,1\}^{\mathbf{N}}$$

is the **kneading sequence.**

Cutting Times

The **kneading sequence** is

$$\nu = \nu_1 \nu_2 \nu_3 \nu_4 \cdots \in \{0, 1\}^{\mathbf{N}}$$

Define the **cutting times** by

$$S_0 = 1, \quad S_k = \min\{j > S_{k-1} : \nu_j \neq \nu_{j-S_{k-1}}\}.$$

There is a map (called **kneading map**)

$$Q : \mathbf{N} \to \mathbf{N} \cup \{0\}$$

such that

$$S_k - S_{k-1} = S_{Q(k)}.$$

Examples:

The **Feigenbaum-Coullet-Tresser** map:

$$Q(k) = k - 1, \ S_k = 2^k.$$

Its inverse limit is known (Barge & Ingram).

The **Fibonacci** map:

$$Q(k) = \max\{0, k - 2\}.$$

The S_k are the Fibonacci numbers.

Fibonacci-like maps:

$$Q(k) = \max\{0, k - d\},$$

This implies:

- $\omega(c)$ is minimal Cantor set.
- $|c_L - c_R| \to 0$ as L or $R \to \infty$.

Theorem 1 *The inverse limit of a Fibonacci-like map has*

- *a Cantor set of endpoints.*

- *countably many disjoint non-arc subcontinua, all of which are*

  $$\sin \frac{1}{x}\text{-}\textit{curves}$$

  arranged in $d - 1$ $\hat{T}$*-orbits.*

- **conjecture:** *and no asymptotic composants.*

Conjecture: If Q is eventually injective ($Q(k) \neq Q(l)$ for all $k \neq l$ suff. large), then there are no asymptotic composants.

Construction of the Tree

1. Start with $\bullet_2$ ——— $\bullet_1$

2. Define

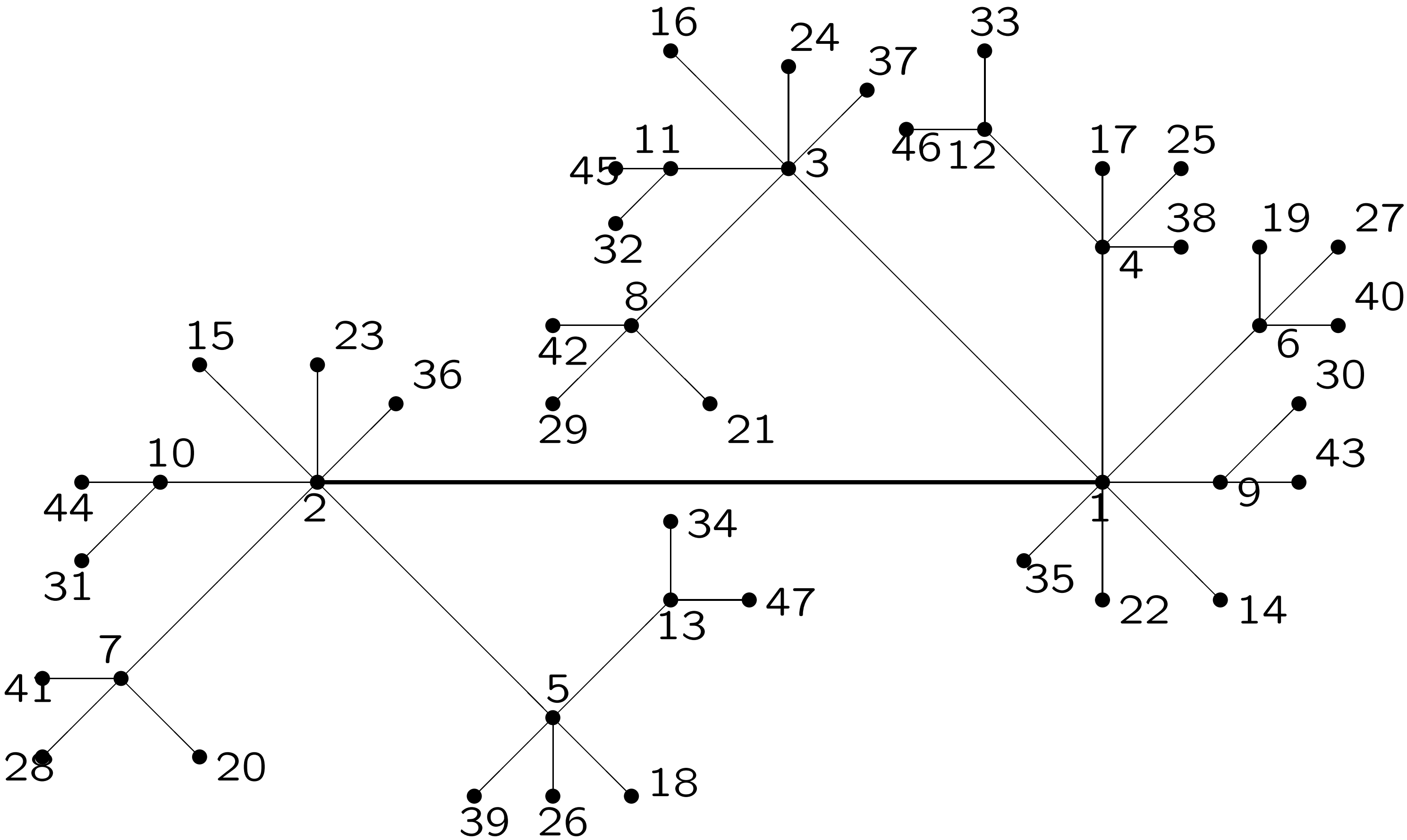
   $$\beta(n) = n - \max\{S_k < n\}$$

   and attach

   $$\bullet_n \text{ ——— } \bullet_{\beta(n)}$$

   for $n = 3, 4, 5, \ldots$.

3. $\bullet_n$ ——— $\bullet_m$ is shorter as $|b - a|$ is larger. (Analogous to p-adic topology.)



Tree of the Fibonacci map.

How to Walk the Tree?

- ν is public key;
  $e = \ldots e_{-3}e_{-2}e_{-1}e_0$ is personal key.

- $$\begin{matrix} R \\ L \end{matrix} = \sup\{n > 0 : e_{-n+1} \ldots e_0 = \nu_1 \ldots \nu_n,$$
  $$\# \text{ of } 1s \text{ in } \nu_1 \ldots \nu_n \text{ is } \begin{matrix} \text{even} \\ \text{odd} \end{matrix}\}$$

- 1. Compute R, move to node R and swap entry e_{-R}.

  2. Compute L, move to node L and swap entry e_{-L}.

  3. Goto 1.

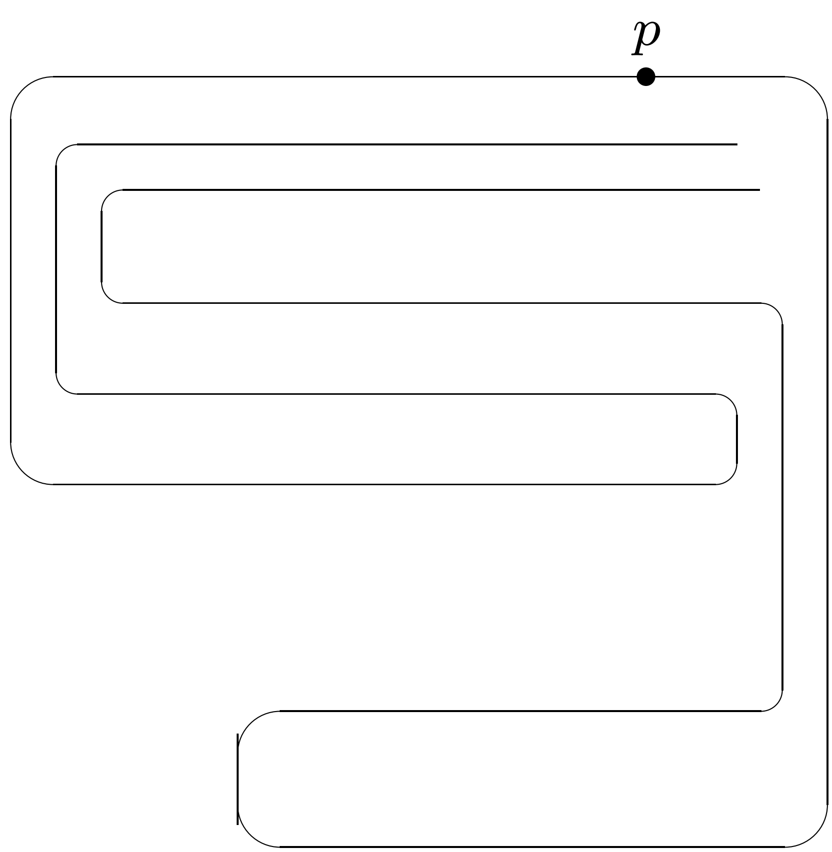
- R or $L = \infty$ corresponds to endpoints.

Properties of a Walk.

Let $\{F_j\}_{j \in \mathbf{Z}}$ be the values $\dots RLRL \cdot RLRLRL \dots$ of a walk.

- The walk is invertible. No merging composants.
- $|F_j - F_{j-1}| = S_{k_j}$ is a cutting time.
- If $F_j < F_{j+1} < F_{j-1}$ then $k_j = Q(k_{j+1})$.
- If $F_{j-1} < F_j < F_{j+1}$ then $Q(k_j + 1) \geq Q(k_{j+1} + 1)$.
- If $F_i = F_j$, then there is $i < a < j$ such that $F_a > F_i$.

A typical example of this is when T has a periodic critical point of period 3. The below picture illustrates the resulting inverse limit space; p is the fixed point of $\widehat{f}$.



This representation has a single infinite Wada channel.