Equilibrium states for multimodal maps

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Multimodal maps.

In this talk the interval map $f : I \to I$ will

- be C^2 multimodal (i.e., critical set

$$\mathrm{Cr} = \{c : f'(c) = 0\}$$

  is finite);

- have non-flat critical points c:

$$f(x) = f(c) + O(|x-c|^{\ell_c})$$

  for $x \approx c$, and critical order $\ell_c \in (1, \infty)$. Take $\ell_{\max} = \max\{\ell_c\}$.

- be transitive.

Equilibrium states.

First fix the potential

$$\varphi_t = -t \log |Df|.$$

The **pressure** is sup of free energies:

$$\underbrace{P(\varphi_t)}_{\text{pressure}} = \sup_{\nu \in \mathcal{M}_{erg}} \underbrace{h_\nu(f) + \int \varphi_t d\nu}_{\text{free energy}}$$

A measure μ with free energy $= P(\varphi_t)$ is an **equilibrium state**. Also take

$$\mathcal{M}_+ = \{\nu \in \mathcal{M}_{erg} : \underbrace{\lambda(\nu)}_{\text{Lyap. exp.}} > 0, \nu(\text{orb}(\text{Cr})) = 0\}$$

Questions to ask:

- Existence and uniqueness of equilibrium states?

- How smooth is $t \mapsto P(\varphi_t)$? Are there **phase transitions**?

Basic idea: Take the (unique) equilibrium state of inducing scheme. Project it to original system, and show that it is still an equilibrium state.

Basic problems:

- How to construct an inducing scheme?

- How to analyse potential on inducing scheme?

  Answer: **Use induced shifted potential** and thermodynamics on **infinite alphabet shift** (Sarig).

- Can different inducing schemes may give different answers?

- Is the equilibrium state global? Could there be equilibrium states that are not compatible to any inducing scheme?

Non-Collet-Eckmann case.

Theorem 1 *Suppose that for some* $t_0 \in (0,1)$, $C > 0$ *and* $\beta > \ell_{max}(1 + \frac{1}{t_0}) - 1$,

$$|Df^n(f(c))| \geq Cn^{\beta} \quad \forall\ c \in Cr,\ n \geq 1.$$

Then there exists $t_1 \in (t_0, 1)$ *such that:*

- *for every* $t \in (t_1, 1]$, (I, f, φ_t) *has an equilibrium state* $\mu_{\varphi_t} \in \mathcal{M}_+$*;*

- *for* $t_1 < t < 1$, μ_{φ_t} *is the unique equilibrium state in* $\mathcal{M}_{erg}$ *and compatible to an inducing scheme with exponential tails;*

- *for* $t = 1$ *there may be other equilibrium states in* $\mathcal{M}_{erg} \backslash \mathcal{M}_+$*. However,* $\mu_{\varphi_1} \in \mathcal{M}_+$ *is compatible to an inducing scheme with polynomial tails;*

- *(Unimodal case:)* $t \mapsto P(\varphi_t)$ *is analytic on* $(t_1, 1)$*.*

Collet-Eckmann case.

Theorem 2 *Suppose that for some* $C, \alpha > 0$,

$$|Df^n(f(c))| \geq Ce^{\alpha n} \quad \forall\ c \in \mathit{Cr},\ n \geq 1.$$

Then there exist $t_1 < 1 < t_2$ *such that for* $t \in (t_1, t_2)$

- f *has a unique equilibrium state* μ_{φ_t};
- $\mu_{\varphi_t} \in \mathcal{M}_+$;
- μ_{φ_t} *is compatible to an inducing scheme with exponential tails;*
- $t \mapsto P(\varphi_t)$ *is analytic on* (t_1, t_2).

Potentials with summable variations

Let $\mathcal{P}$ be the partition induced by Cr and $\mathcal{P}_n = \vee_{k=0}^{n-1} T^{-k}(\mathcal{P}_1)$.

The n-**th variation** of $\varphi : X \to \mathbb{R}$ is

$$V_n(\varphi) = \sup_{C \in \mathcal{P}_n} \sup_{x,y \in C} |\varphi(x) - \varphi(y)|.$$

For $C \in \mathcal{P}_1$ set

$$Z_n(\varphi, C) = \sum_{T^n(x) = x \in C} e^{\varphi_n(x)},$$

where $\varphi_n(x) = \sum_{k=0}^{n-1} \varphi \circ T^k(x)$.

The **Gurevic pressure** is

$$P_G(\varphi) = \lim(\sup)_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, C).$$

The potential φ is **recurrent** if

$$\sum_n \lambda^{-n} Z_n(\varphi, C) = \infty,$$

for $\lambda = \exp P_G(\varphi)$.

Theorem Let $\varphi : I \to \mathbb{R}$ be a recurrent potential with Gurevic pressure $P_G(\varphi) < \infty$ and

$$\sum_n nV_n(\varphi) < \infty.$$

Then there is at most one equilibrium state μ_φ within $\mathcal{M}_+$. If in addition

$$\sup \varphi - \inf \varphi < h_{top}(f),$$

then φ is indeed recurrent, and there exists a unique equilibrium state μ_φ in $\mathcal{M}_+$, and $t \mapsto P(t\varphi)$ is analytic on $[-1, 1]$.

Remarks:

- Similar results by Hofbauer & Keller and Paccaut using transfer operators.

- Condition $\sup \varphi - \inf \varphi < h_{top}(f)$ or at least $\sup \varphi < P(\varphi)$ essential for uniqueness. (Counter-example for MP map.)

- $t \mapsto P(-t \log |Df|)$ is analytic on $(-\varepsilon, \varepsilon)$.

Inducing schemes.

An *inducing scheme* (X, F, τ) consists of:

- $X \supset \cup_i X_i$ such that F maps each interval X_i diffeomorphically onto X.
- $F|_{X_i} = f^{\tau_i}$ for some $\tau_i \in \mathbb{N} := \{1, 2, 3 \dots\}$.

$\tau : \cup_i X_i \to \mathbb{N}$ is the *inducing time*. It may or may not be the first return time to X.

Compatible measures.

A measure μ *compatible* is to (X, F, τ) if

- $\mu(X) > 0$ and $\mu(X \setminus \cap_n F^{-n}(\cup_i X_i)) = 0$, and

- there exists a measure μ_F which projects to μ via

$$\mu(A) = \frac{1}{\Lambda} \sum_i \sum_{k=0}^{\tau_i - 1} \mu_F(X_i \cap f^{-k}(A)),$$

where $\Lambda := \int_X \tau \; d\mu_F < \infty$.

Inducing schemes exist.

Theorem 3 *$\mu \in \mathcal{M}_+$ if and only if there exists an inducing scheme to which μ is compatible.*

The scheme can be chosen to

- *be a first return map on the Hofbauer tower, and*
- *have uniformly bounded distortion on its branches.*

However, there is no inducing scheme to which all $\mu \in \mathcal{M}_+$ are compatible.

Symbolic maps with BIP property:

Theorem (Due to Omri Sarig)
If (X, T) is topologically mixing and

$$\sum_{n \geq 1} V_n(\Phi) < \infty,$$

then Φ has an invariant Gibbs measure if and only if A has the BIP property and $P_G(\Phi) < \infty$. Moreover the Gibbs measure μ has the following properties

(a) If $h_\mu(T) < \infty$ or $\int \Phi d\mu < \infty$ then μ is the unique equilibrium state (in particular, $P(\Phi) = h_\mu(T) + \int_X \Phi \, d\mu$);

(b) The Variational Principal holds, i.e.,

$$P_G(\Phi) = P(\Phi)$$

The induced map is a **first** return map on the Hofbauer tower.

All its branches have the same big image, so the **B**ig **I**mage and **P**reimage property holds.

Thus the following lemma applies:

Lemma (Sarig)
Let (X, F) be topologically mixing with potential Φ. Let $X' := X_i$ be an arbitrary 1-cylinder, and let (F', X', Φ') be the first return system to X' where

$$\Phi'(x) = \sum_{k=0}^{\tau(x)-1} \Phi(x)$$

is the induced potential.

If both Φ and Φ' have summable variations, then Φ is recurrent and $P_G(\Phi) = 0$ if and only if Φ' is recurrent and $P_G(\Phi') = 0$.

The global existence of equilibrium states

- a first return maps $(\hat{X}, \hat{F})$ in the Hofbauer tower has a unique equilibrium state $\widehat{\mu}_{\hat{X}}$.

- $\widehat{\mu}_{\hat{X}}$ has positive Lyapunov exponent and is Gibbs; hence it gives positive mass to open sets.

- the Hofbauer tower is transitive; so $(\hat{X}, \hat{F}, \widehat{\mu})$ 'communicates' with any other inducing scheme that Theorem 3 applies to.

- A fortiori: every inducing scheme is compatible to the equilibrium state:

Lemma 4 *If μ_φ is an equilibrium state, compatible to an inducing scheme (X, F) then it is also is compatible to any other 'natural' inducing scheme (X', F').*