On the shape of isentropes for interval maps

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joint with

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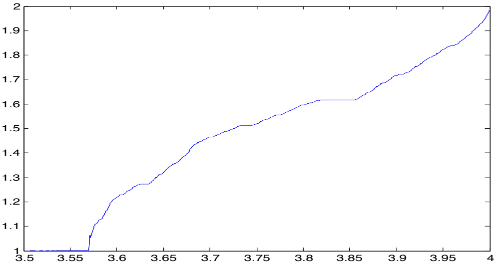
Madrid, July 2014

Let $f_a : [0,1] \to [0,1]$, $x \mapsto ax(1-x)$ be the quadratic family.

$\exp(h_{top}(f_a))$



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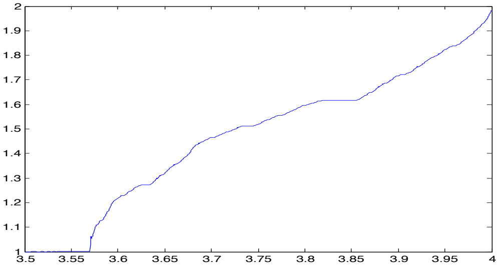

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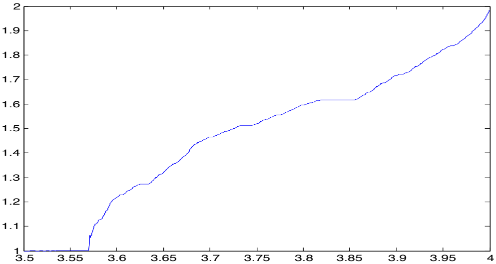

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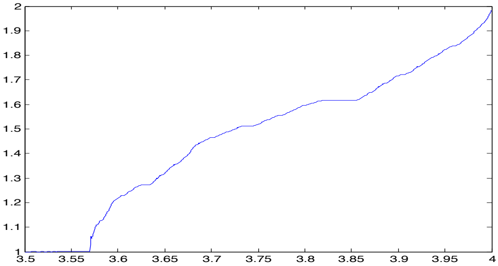

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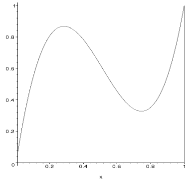
The entropy map $a \mapsto h_{top}(f_a)$ is:

- Continuous - but what is the modulus of continuity?
- Monotone - but not strictly.

Multimodal Maps

What about entropy for multimodal maps, i.e., maps with several critical points? Especially for the families of cubic, quartic, quintic, ... polynomials.



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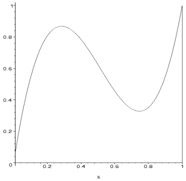

We use the following setting:

P^d is the set of degree $d+1$ polynomials $f : [0,1] \to [0,1]$ s. t.

- f has d distinct critical points, all lying in $[0,1]$.
- $f(0) = 0$ and $f(1) \in \{0,1\}$.

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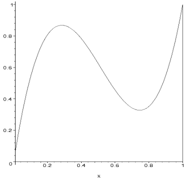

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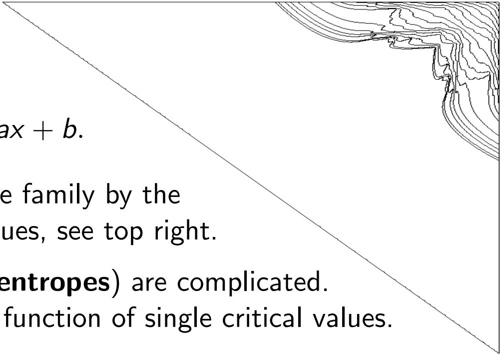
The dimension of parameter A space is d.
Monotonicity means: all isentropes

$$L_h := \{\mathbf{a} \in A : h_{top}(f_{\mathbf{a}}) = h\}$$

are connected.

Entropy in the cubic family

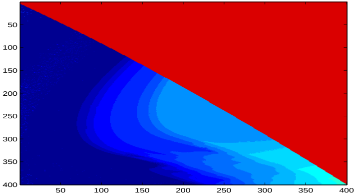


The general cubic family

$$x \mapsto x^3 - ax + b.$$

One can also parametrize the family by the height of the two critical values, see top right.

Level sets of the entropy (**isentropes**) are complicated. Entropy is not monotone as function of single critical values.



The cubic family

$$x \mapsto x^3 - ax + b.$$

Isentropes in blue colour:

Monotonicity of for degree d Polynomials

The unimodal case is by now standard:

Theorem (Milnor & Thurston 1970s, Douady & Hubbard 1980s)

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The general result:

Theorem (Bruin and van Strien, 2009)

Isentropes in P^d are connected.

The Shape of Isentropes

Monotonicity doesn't mean that isentropes are simple sets. We know that:

- For many values of the entropy h, L_h is not locally connected.
- Entropy is **not** a monotone function of each single critical values.

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Question (Milnor): Are the isentropes contractible?

Question (Thurston): Is there a dense set of $h \in [0, \log d]$ such that hyperbolic maps are dense in L_h?

The Shape of Isentropes

The following is based on a result by Friedman & Tresser, showing that the "boundary of chaos" for circle endomorphisms is not locally connected.

Theorem (Bruin & van Strien, 2013)

For any $d \geq 4$*, there is a dense set* $H \subset [0, \log(d-1)]$ *such that for each* $h \in H$*, the isentrope* L_h *of* P^d *is not locally connected.*

Sketch of Proofs

- f_a is a parameterised family of maps in P^d, $d \geq 4$, unfolding a saddle node bifurcation.

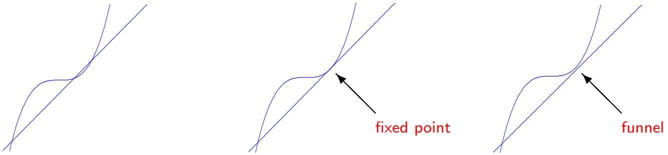

Figure: Unfolding a saddle node bifurcation. When the fixed point disappears, a funnel is left. Points take a long time to iterate through the funnel.

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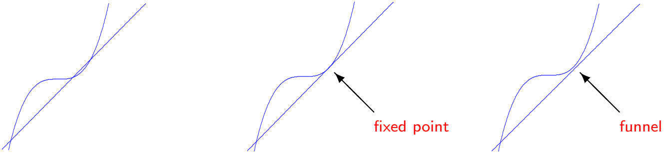

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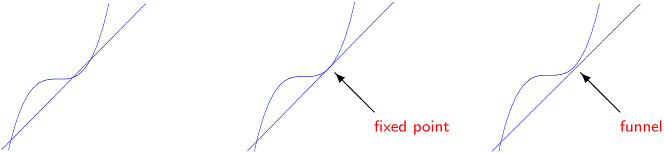


Figure: Unfolding a saddle node bifurcation. When the fixed point disappears, a funnel is left. Points take a long time to iterate through the funnel.

- $d - 2$ critical points are attracted to periodic points.
- 2 critical points c_η and $c_{\eta+1}$ belong to in interval J that under iteration of f passes along the funnel.

- There is a sequence $a_k \to a_\infty$ such that $f^{N+k}(J) \subset J$. For all these a_k,

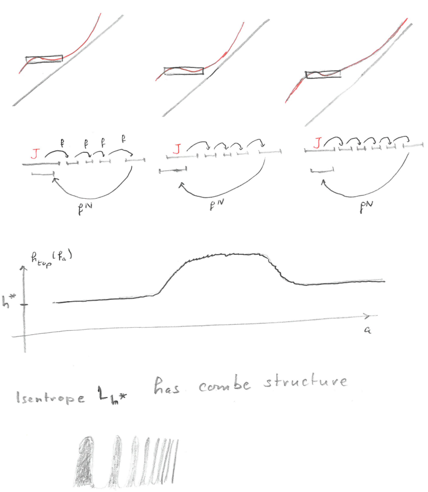
$$h_{top}(f_{a_k}) = h^* \text{ is constant.}$$

- Interspersed is a sequence $b_k \to a_\infty$ such that $f^{N+k}(J) \not\subset J$, and

$$h_{top}(f_{b_k}) > h^*.$$

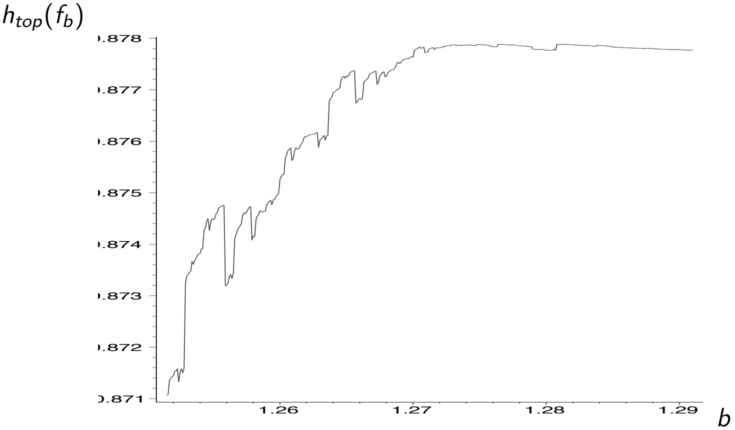
The result is a comb structure of isentrope L_{h^*}: L_{h^*} is not locally connected.

Sketch of Proofs



Non-monotonicity of entropy in single critical value.

We can prove in the case $d \geq 3$ that the entropy is not monotone on slices in parameter space. Below, the second critical value in the cubic map $x \mapsto x^3 - ax + b$ is fixed, the first, i.e., b, varies.



Non-monotonicity of entropy in single critical value.

Theorem (Non-monotonicity w.r.t. natural parameters)

Let $f_v \in P^d$ denote the polynomial map with critical values $v = (v_1, \ldots, v_d)$. For $d \geq 2$, there are fixed values of $v_2, \ldots, v_b$ such that the map

$$v_1 \mapsto h_{top}(f_v)$$

is not monotone.

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