Self-repairing discontinuities for interval maps

Henk Bruin (University of Vienna)

joint with
Carlo Carminati (University of Pisa)

explaining observations in a paper by

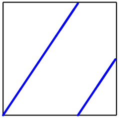
V. Botella-Soler, J. A. Oteo, J. Ros, and P. Glendinning

Richmond, March 2014

β-transformations

The β-transformation is defined as

$$x \mapsto \beta x \pmod 1$$

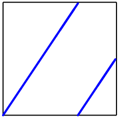


For $|\beta| > 1$, T_β has an acip μ.

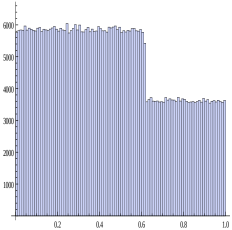
β-transformations

The β-transformation is defined as

$$x \mapsto \beta x \pmod 1$$



For $|\beta| > 1$, T_β has an acip μ.



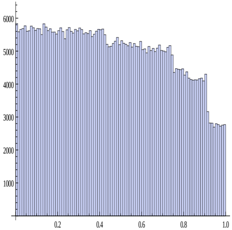
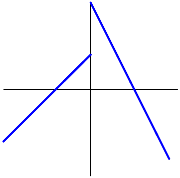


Figure: Density $\frac{d\mu}{dx}$ for $\beta = \frac{1}{2}(\sqrt{5}+1)$ and $\beta = \sqrt[3]{7}$.

The density is **only** locally constant, if there is a Markov partition.

The map T_β

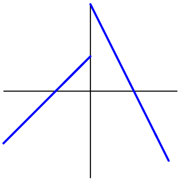
$$T_\beta(x) = \begin{cases} T_\beta^-(x) = x + 2 & \text{if } x \leq 0, \\ T_\beta^+(x) = \beta - 2x & \text{if } x \geq 0. \end{cases}$$



T_β preserves the $[\beta - \max\{2, \beta\}, max\{2, \beta\}]$ and some iterate is uniformly expanding. Therefore T_β admits an acip.

The map T_β

$$T_\beta(x) = \begin{cases} T_\beta^-(x) = x + 2 & \text{if } x \le 0, \\ T_\beta^+(x) = \beta - 2x & \text{if } x \ge 0. \end{cases}$$



T_β preserves the $[\beta - \max\{2, \beta\}, max\{2, \beta\}]$ and some iterate is uniformly expanding. Therefore T_β admits an acip.

Figure: Invariant density for the T_β: left $\beta = \frac{1}{2}(\sqrt{5} + 1)$ right: $\beta = \sqrt[3]{7}$.

Markov partitions and Entropy

The interval partition $\{P_i\}$ is a Markov partition for T if

$$T(P_i) \cap P_j \neq \emptyset \text{ implies } T(P_i) \supset P_j.$$

Markov partitions and Entropy

The interval partition $\{P_i\}$ is a Markov partition for T if

$$T(P_i) \cap P_j \neq \emptyset \text{ implies } T(P_i) \supset P_j.$$

The transition matrix $\Pi = \Pi_{i,j}$ is defined as:

$$\Pi_{i,j} = \begin{cases} 1 & \text{if } T(P_i) \supset P_j, \\ 0 & \text{if } P_j \cap T(P_i) = \emptyset, \\ \text{No other possibility, because } \{P_i\} \text{ is Markov} \end{cases}$$

Markov partitions and Entropy

The interval partition $\{P_i\}$ is a Markov partition for T if

$$T(P_i) \cap P_j \neq \emptyset \text{ implies } T(P_i) \supset P_j.$$

The transition matrix $\Pi = \Pi_{i,j}$ is defined as:

$$\Pi_{i,j} = \begin{cases} 1 & \text{if } T(P_i) \supset P_j, \\ 0 & \text{if } P_j \cap T(P_i) = \emptyset, \\ \text{No other possibility, because } \{P_i\} \text{ is Markov} \end{cases}$$

The topological entropy is

$$h_{top}(T) = \log \sigma$$

for σ the leading eigenvalue of Π.

Markov partitions and Entropy

Scale Π by the slopes $t_i = |DT_{|P_i}|$ to obtain a matrix

$$A_{i,j} = \frac{1}{t_i} \Pi_{i,j}.$$

Then $\ell_i = |P_i|$ and $\rho_i = \frac{d\mu}{dx}_{|P_i}$ satisfy $\sum_i \rho_i \ell_i = 1$ and

$$\begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix}^T A = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix}^T \quad \text{and} \quad A \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix}$$

Markov partitions and Entropy

Scale Π by the slopes $t_i = |DT_{|P_i}|$ to obtain a matrix

$$A_{i,j} = \frac{1}{t_i}\Pi_{i,j}.$$

Then $\ell_i = |P_i|$ and $\rho_i = \frac{d\mu}{dx}_{|P_i}$ satisfy $\sum_i \rho_i \ell_i = 1$ and

$$\begin{pmatrix}\rho_1\\ \vdots \\ \rho_N\end{pmatrix}^T A = \begin{pmatrix}\rho_1\\ \vdots \\ \rho_N\end{pmatrix}^T \quad \text{and} \quad A\begin{pmatrix}\ell_1\\ \vdots \\ \ell_N\end{pmatrix} = \begin{pmatrix}\ell_1\\ \vdots \\ \ell_N\end{pmatrix}$$

Rokhlin's formula gives the metric entropy:

$$h_\mu(T) = \sum_{i=1}^{N} \max\{\log(t_i), 0\}\mu(P_i)$$

Not Markov but Matching

For the family T_β, there is no Markov partition in general, but something called matching takes can occur:

Definition: There is matching if there are iterates $\kappa_\pm > 0$ such that

$$T^{\kappa_-}(0^-) = T^{\kappa_+}(0^+) \text{ and derivatives } DT^{\kappa_-}(0^-) = DT^{\kappa_+}(0^+)$$

Not Markov but Matching

For the family T_β, there is no Markov partition in general, but something called matching takes can occur:

Definition: There is matching if there are iterates $\kappa_\pm > 0$ such that

$$T^{\kappa_-}(0^-) = T^{\kappa_+}(0^+) \text{ and derivatives } DT^{\kappa_-}(0^-) = DT^{\kappa_+}(0^+)$$

The pre-matching partition plays the role of Markov partition:

$$\{\{T^j(0^-)\}_{j=0}^{\kappa_- -1}\} \cup \{\{T^j(0^+)\}_{j=0}^{\kappa_+ -1}\};$$

Not Markov but Matching

For the family T_β, there is no Markov partition in general, but something called matching takes can occur:

Definition: There is matching if there are iterates $\kappa_\pm > 0$ such that

$$T^{\kappa_-}(0^-) = T^{\kappa_+}(0^+) \text{ and derivatives } DT^{\kappa_-}(0^-) = DT^{\kappa_+}(0^+)$$

The pre-matching partition plays the role of Markov partition:

$$\{T^j(0^-)\}_{j=0}^{\kappa_- -1}\} \cup \{T^j(0^+)\}_{j=0}^{\kappa_+ -1}\};$$

Theorem: If T has matching, then $\rho = \frac{d\mu}{dx}$ is constant on each element of the pre-matching partition.

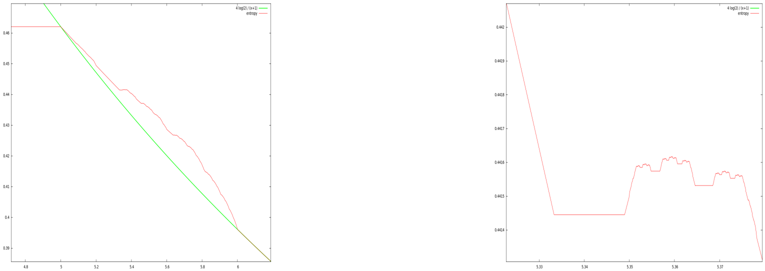


Figure: Entropy $h_\mu(T_\beta)$ for $\beta \in [4.6, 6]$ (l) and $\beta \in [5.29, 5.33]$ (r).

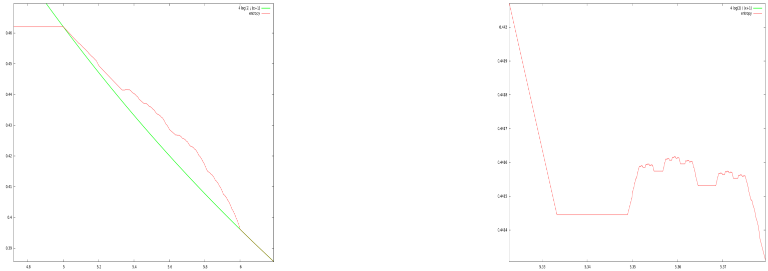


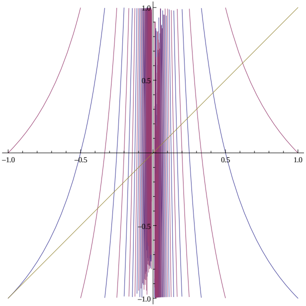
Figure: Entropy $h_\mu(T_\beta)$ for $\beta \in [4.6, 6]$ (l) and $\beta \in [5.29, 5.33]$ (r).

Definition: The matching is neutral if $\kappa_- = \kappa_+$.

Theorem: On every parameter interval where matching occurs, topological and metric entropy are monotone, and constant if the matching is neutral.

The α-continued fraction map T_α.

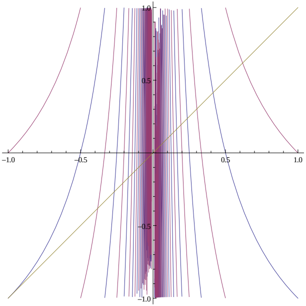
A generalization of the Gauß map stems from Nakada (and Natsui).



The α-continued fraction map T_α.

A generalization of the Gauß map stems from Nakada (and Natsui).

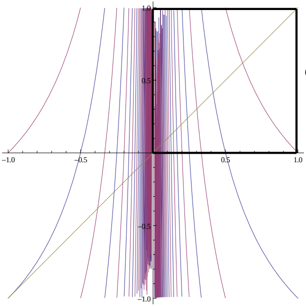
branches: $|\frac{1}{x}| + n$, $n \in \mathbb{Z}$, on $[-1, 1]$



The α-continued fraction map T_α.

A generalization of the Gauß map stems from Nakada (and Natsui).

branches: $|\frac{1}{x}| + n$, $n \in \mathbb{Z}$, on $[-1, 1]$

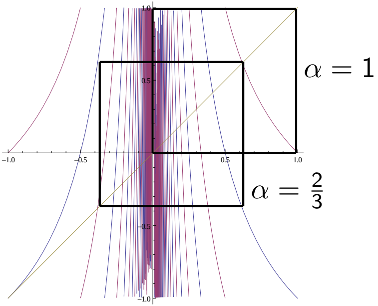


$\alpha = 1$

The α-continued fraction map T_α.

A generalization of the Gauß map stems from Nakada (and Natsui).

branches: $|\frac{1}{x}| + n$, $n \in \mathbb{Z}$, on $[-1, 1]$



The α-continued fraction map T_α.

A generalization of the Gauß map stems from Nakada (and Natsui).

branches: $|\frac{1}{x}| + n$, $n \in \mathbb{Z}$, on $[-1, 1]$

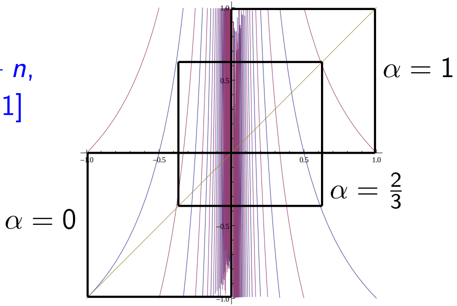


Figure: $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha], x \mapsto |\frac{1}{x}| - \lfloor \frac{1}{x} + 1 - \alpha \rfloor$.

All of them have invariant densities (infinite if $\alpha = 0$).

The α-continued fraction map T_α.

A generalization of the Gauß map stems from Nakada (and Natsui).

branches: $|\frac{1}{x}| + n$, $n \in \mathbb{Z}$, on $[-1, 1]$

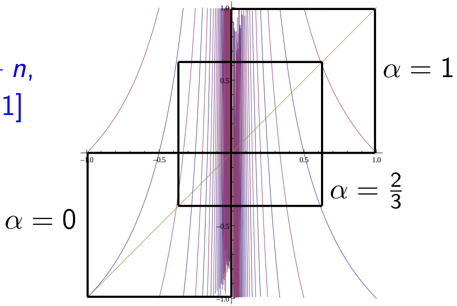


Figure: $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha], x \mapsto |\frac{1}{x}| - \lfloor \frac{1}{x} + 1 - \alpha \rfloor$.

All of them have invariant densities (infinite if $\alpha = 0$).

Matching of the orbits of α and $\alpha - 1$ occurs for a.e. $\alpha \in [0, 1]$.

α-continued fractions and the Mandelbrot set

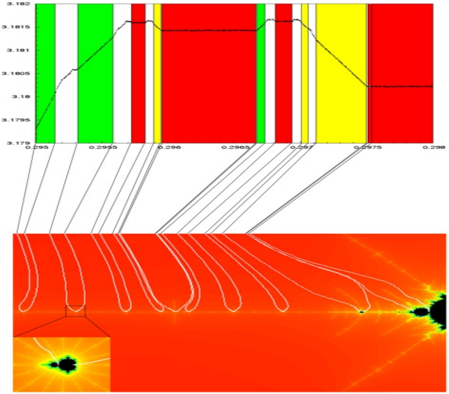


Figure: From a paper by Bonanno, Carminati, Isola and Tiozzo: The non-matching set and the real antenna of Mandelbrot set

Matching is Lebesgue typical

Theorem: T_β has matching for For Lebesgue-a.e. β

Matching is Lebesgue typical

Theorem: T_β has matching for For Lebesgue-a.e. β

Observations towards the proof:

- Let $r_n(x) = \#\{0 \leq i < n : T^n(x) > 0\}$. If $r_m(0^-) = r_n(0^+)$ then $T^m(0^-) - T^n(0^+)$ are a multiple of 2 apart.

Matching is Lebesgue typical

Theorem: T_β has matching for For Lebesgue-a.e. β

Observations towards the proof:

- Let $r_n(x) = \#\{0 \leq i < n : T^n(x) > 0\}$. If $r_m(0^-) = r_n(0^+)$ then $T^m(0^-) - T^n(0^+)$ are a multiple of 2 apart.
- Let $J_\beta = [\frac{\beta-2}{2}, 2]$. For $x \in J_\beta$, both x and $T_\beta(x) \in (0,2]$.
- Of any two successive returns to $(0,\infty)$, at least one is in $(0,2]$.

Matching is Lebesgue typical

Theorem: T_β has matching for For Lebesgue-a.e. β

Observations towards the proof:

- Let $r_n(x) = \#\{0 \leq i < n : T^n(x) > 0\}$. If $r_m(0^-) = r_n(0^+)$ then $T^m(0^-) - T^n(0^+)$ are a multiple of 2 apart.
- Let $J_\beta = [\frac{\beta-2}{2}, 2]$. For $x \in J_\beta$, both x and $T_\beta(x) \in (0, 2]$.
- Of any two successive returns to $(0, \infty)$, at least one is in $(0, 2]$.
- Therefore, if $T^m(0^-) \in J_\beta$, either $T^m(0^-)$ or $T^{m+1}(0^-)$ will match with $\mathrm{orb}(0^+)$.

Matching is Lebesgue typical

Theorem: T_β has matching for For Lebesgue-a.e. β

Observations towards the proof:

- Let $r_n(x) = \#\{0 \leq i < n : T^n(x) > 0\}$. If $r_m(0^-) = r_n(0^+)$ then $T^m(0^-) - T^n(0^+)$ are a multiple of 2 apart.
- Let $J_\beta = [\frac{\beta-2}{2}, 2]$. For $x \in J_\beta$, both x and $T_\beta(x) \in (0,2]$.
- Of any two successive returns to $(0,\infty)$, at least one is in $(0,2]$.
- Therefore, if $T^m(0^-) \in J_\beta$, either $T^m(0^-)$ or $T^{m+1}(0^-)$ will match with $\mathrm{orb}(0^+)$.
- Hence we need to estimate the measure of the set of β such that $\mathrm{orb}(0^-)$ avoids J_β, and in particular is not dense.

On the proof of "Matching is Lebesgue typical"

Proof by	The critical orbit is	for the family of
Brucks & Misurewicz	dense a.s.	tents
Schmeling	Birkhoff typical a.s.	β-transformations
Bruin	Birkhoff typical a.s.	tents
Benedics & Carleson	Birkhoff typical for positive measure	logistic maps
Faller	Birkhoff typical a.s.	shifted β-transformations
Schnellman	Birkhoff typical a.s.	expanding maps

On the proof of "Matching is Lebesgue typical"

Proof by	The critical orbit is	for the family of
Brucks & Misurewicz	dense a.s.	tents
Schmeling	Birkhoff typical a.s.	β-transformations
Bruin	Birkhoff typical a.s.	tents
Benedics & Carleson	Birkhoff typical for positive measure	logistic maps
Faller	Birkhoff typical a.s.	shifted β-transformations
Schnellman	Birkhoff typical a.s.	expanding maps
Bruin & Carminati	dense a.s.	T_β

Non-matching?

Question: Is there always matching?

Non-matching?

Question: Is there always matching?

No, for $\beta = 5$, $\beta = 4\frac{11}{12}$ and $\beta = 4\frac{15}{16}$, there is no matching.

Non-matching?

Question: Is there always matching?

No, for $\beta = 5$, $\beta = 4\frac{11}{12}$ and $\beta = 4\frac{15}{16}$, there is no matching.

Here 0^- and 0^+ eventually map to the same point, but "out-of-phase", so the derivatives don't match.

Non-matching?

Question: Is there always matching?

No, for $\beta = 5$, $\beta = 4\frac{11}{12}$ and $\beta = 4\frac{15}{16}$, there is no matching.

Here 0^- and 0^+ eventually map to the same point, but "out-of-phase", so the derivatives don't match.

Theorem: The non-matching set E has Hausdorff dimension 1. The left neighborhood of $\beta = 6$ is responsible for this:

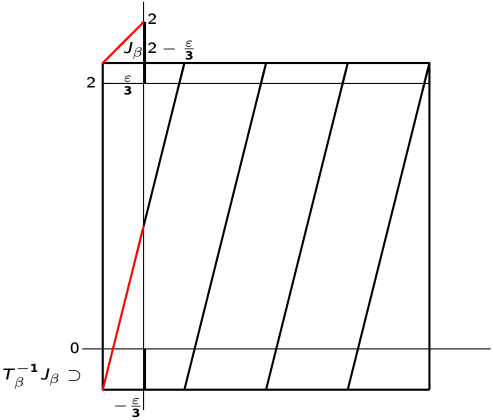
$$\dim_H(E \setminus (6-\varepsilon, 6)) < 1 \text{ for every } \varepsilon > 0.$$

Hausdorff dimension proof

Let $\beta = 6 - \varepsilon$ and $F : [-\frac{\varepsilon}{3}, 2 - \frac{\varepsilon}{3}] \to [-\frac{\varepsilon}{3}, 2]$ the first entrance map.

Hausdorff dimension proof

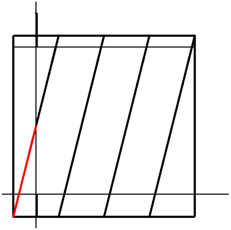
Let $\beta = 6 - \varepsilon$ and $F : [-\frac{\varepsilon}{3}, 2 - \frac{\varepsilon}{3}] \to [-\frac{\varepsilon}{3}, 2]$ the first entrance map.



Up to the interval $[-\frac{\varepsilon}{3}, 0]$ which maps directly into J_β, this is a *quadrupling map*.

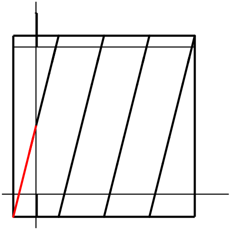
Hausdorff dimension proof

Let K_ε be the set of points that remain in $[0, 2 - \frac{\varepsilon}{3}]$ for all iterates of F.



Hausdorff dimension proof

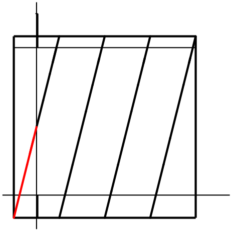
Let K_ε be the set of points that remain in $[0, 2 - \frac{\varepsilon}{3}]$ for all iterates of F.



- $\dim_H(K_\varepsilon) \to 1$ as $\varepsilon \to 0$.

Hausdorff dimension proof

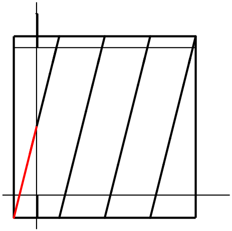
Let K_ε be the set of points that remain in $[0, 2 - \frac{\varepsilon}{3}]$ for all iterates of F.



- $\dim_H(K_\varepsilon) \to 1$ as $\varepsilon \to 0$.
- If $orb(0^\pm)$ remain in K_ε, then there is no matching.

Hausdorff dimension proof

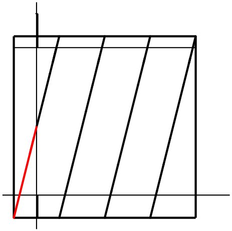
Let K_ε be the set of points that remain in $[0, 2 - \frac{\varepsilon}{3}]$ for all iterates of F.



- $\dim_H(K_\varepsilon) \to 1$ as $\varepsilon \to 0$.
- If *orb*$(0^\pm)$ remain in K_ε, then there is no matching.
- In fact, *orb*$(0^+) \subset K_\varepsilon$ iff *orb*$(0^+) \subset K_\varepsilon$.

Hausdorff dimension proof

Let K_ε be the set of points that remain in $[0, 2 - \frac{\varepsilon}{3}]$ for all iterates of F.

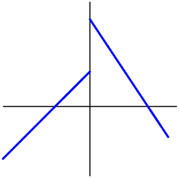


- $\dim_H(K_\varepsilon) \to 1$ as $\varepsilon \to 0$.
- If $orb(0^\pm)$ remain in K_ε, then there is no matching.
- In fact, $orb(0^+) \subset K_\varepsilon$ iff $orb(0^+) \subset K_\varepsilon$.
- $\dim_H\{\beta : \mathrm{orb}(0^-) \in K_\varepsilon\} = \dim_H(K_\varepsilon) \to 1$ as $\varepsilon \to 0$.

Other slopes

Generalize to slope s

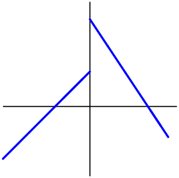
$$T_\beta(x) = \begin{cases} T_\beta^-(x) = x + s & \text{if } x \leq 0, \\ T_\beta^+(x) = \beta - sx & \text{if } x \geq 0. \end{cases}$$



Other slopes

Generalize to slope s

$$T_\beta(x) = \begin{cases} T_\beta^-(x) = x + s & \text{if } x \le 0, \\ T_\beta^+(x) = \beta - sx & \text{if } x \ge 0. \end{cases}$$



For $s = \frac{1}{2}(\sqrt{5}+1)$ and $\sqrt{2}+1$ and some other, large intervals of matching has been observed.

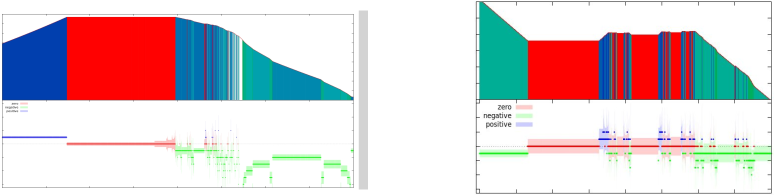
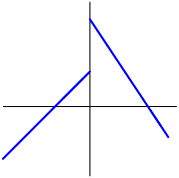


Figure: $h_\mu(T_\beta)$ for $s = \frac{\sqrt{5}+1}{2}$, $\beta \in [4.6, 6]$ (l) and $\beta \in [5.29, 5.33]$ (r).

Other slopes

Generalize to slope s

$$T_\beta(x) = \begin{cases} T_\beta^-(x) = x + s & \text{if } x \le 0, \\ T_\beta^+(x) = \beta - sx & \text{if } x \ge 0. \end{cases}$$



For $s = \frac{1}{2}(\sqrt{5}+1)$ and $\sqrt{2}+1$ and some other, large intervals of matching has been observed.

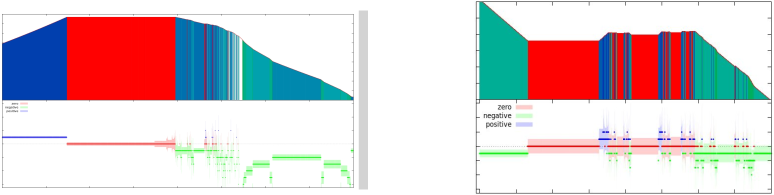


Figure: $h_\mu(T_\beta)$ for $s = \frac{\sqrt{5}+1}{2}$, $\beta \in [4.6, 6]$ (l) and $\beta \in [5.29, 5.33]$ (r).

Note that these slopes are quadratic Pisot numbers.

Other slopes

This is no coincidence.

Other slopes

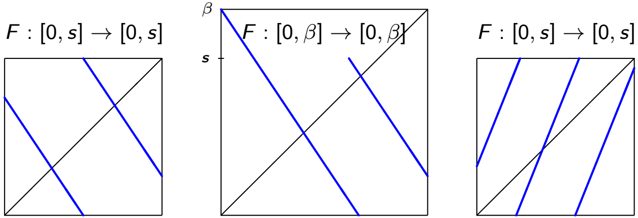
This is no coincidence. T_β preserves the ring $H = \mathbb{Z}[\beta, \beta s, s]$.

Other slopes

This is no coincidence. T_β preserves the ring $H = \mathbb{Z}[\beta, \beta s, s]$.
For matching, we need

$$\#\{0 \le i < \kappa^- : T^i(0^-) > 0\} = \#\{0 \le i < \kappa^- : T^i(0^-) > 0\},$$

so we look at the first return map F:



matching | any matching?? | typical matching?

Figure: Return map F for $\beta < s$, $s < \beta < 3 + \sqrt{5}$, and $\beta > 3 + \sqrt{5}$.

Other slopes

This is no coincidence. T_β preserves the ring $H = \mathbb{Z}[\beta, \beta s, s]$.
For matching, we need

$$\#\{0 \le i < \kappa^- : T^i(0^-) > 0\} = \#\{0 \le i < \kappa^- : T^i(0^-) > 0\},$$

so we look at the first return map F:

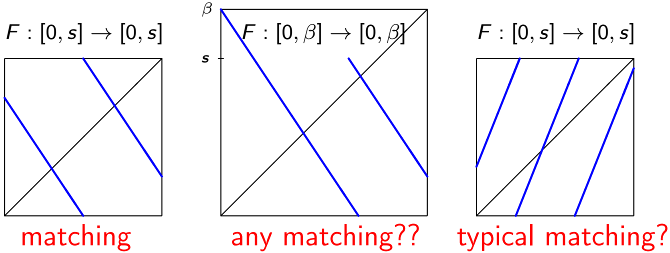


Figure: Return map F for $\beta < s$, $s < \beta < 3 + \sqrt{5}$, and $\beta > 3 + \sqrt{5}$.

F act affinely on H. Restricted to $\mathrm{orb}(0^\pm)$, we need to iterate

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \tau_n \\ 0 \end{pmatrix}$$

Other slopes

F act affinely on H. Restricted to $\mathrm{orb}(0^\pm)$, we need to iterate

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} + \begin{pmatrix} \tau_n(0^\pm) \\ 0 \end{pmatrix},$$

where $\tau_n(0^\pm)$ is the branch number containing $F^n(0^\pm)$, starting with

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } 0^- \qquad \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } 0^+$$

Other slopes

F act affinely on H. Restricted to $\mathrm{orb}(0^{\pm})$, we need to iterate

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} + \begin{pmatrix} \tau_n(0^{\pm}) \\ 0 \end{pmatrix},$$

where $\tau_n(0^{\pm})$ is the branch number containing $F^n(0^{\pm})$, starting with

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } 0^- \qquad \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } 0^+$$

Matching occurs if there is n such that:

$$\begin{pmatrix} a_n(0^-) \\ b_n(0^-) \end{pmatrix} = \begin{pmatrix} a_n(0^+) \\ b_n(0^+) \end{pmatrix}$$

Other slopes

F act affinely on H. Restricted to $\mathrm{orb}(0^{\pm})$, we need to iterate

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} + \begin{pmatrix} \tau_n(0^{\pm}) \\ 0 \end{pmatrix},$$

where $\tau_n(0^{\pm})$ is the branch number containing $F^n(0^{\pm})$, starting with

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } 0^- \qquad \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } 0^+$$

Matching occurs if there is n such that:

$$\begin{pmatrix} a_n(0^-) \\ b_n(0^-) \end{pmatrix} = \begin{pmatrix} a_n(0^+) \\ b_n(0^+) \end{pmatrix}$$

Question: Does this happen Lebesgue typically for $s = \frac{\sqrt{5}+1}{2}$?

C. Bonanno, C. Carminati, S. Isola, G. Tiozzo, *Dynamics of continued fractions and kneading sequences of unimodal maps*, Discrete Contin. Dyn. Syst. **33** (2013), no. 4, 1313–1332.

V. Botella-Soler, J. A. Oteo, J. Ros, P. Glendinning, *Families of piecewise linear maps with constant Lyapunov exponents*, J. Phys. A: Math. Theor. **46** 125101

C. Carminati, G. Tiozzo, *Tuning and plateaux for the entropy of α-continued fractions*, Nonlinearity **26** (2013), no. 4, 1049–1070.

H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. **4** (1981), 399–426

H. Nakada, R. Natsui, *The non-monotonicity of the entropy of α-continued fraction transformations*, Nonlinearity, **21** (2008), 1207–1225.