# Matching for translated $\beta$-transformations.

Henk Bruin (University of Vienna)

joint with

Carlo Carminati (University of Pisa)

Charlene Kalle (University of Leiden)

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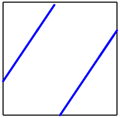
# Translated $\beta$-transformations

The translated $\beta$-transformation is defined as

$$T_{\beta,\alpha} : x \mapsto \beta x + \alpha \pmod 1$$



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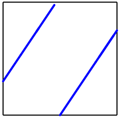

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Figure: Density $\frac{d\mu}{dx}$ for $\beta = \frac{1}{2}(\sqrt{5}+1)$ and $\beta = \sqrt[3]{7}$.

The density is **only?** locally constant, if there is a Markov partition.

# Not Markov but Matching

For the family $T_\alpha$, there is no Markov partition in general, but something called matching can occur:

Definition: There is matching if there is an iterate $\kappa > 0$ such that

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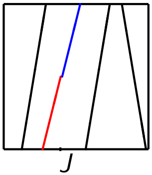
This is a general theorem: If a piecewise affine expanding interval map $T : [0,1] \to [0,1]$ has matching at all its discontinuity points, then $\frac{d\mu}{dx}$ is constant on each element of the pre-matching partition.

# Matching and piecewise constant densities

Sketch of proof: 


- Take a nice interval $J$ disjoint from the matching set. (nice means that $\mathrm{orb}(\partial J) \cap J^\circ = \emptyset$).
- Consider the first return map $R$ to $J$; it has only onto linear (or Möbius) branches.
- Hence the $R$-invariant denstity is constant (or Möbius).
- The $T$-invariant density coincides with $T$-invariant density (up to a scaling factor).

# Typical matching for $T_\alpha$: Quadratic Pisot Numbers

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The quadratic Pisot numbers are those $\beta > 1$ satisfying

$$\beta^2 - k\beta \pm d = 0 \qquad \text{with } \begin{cases} k > d+1 & \text{if } d > 0, \\ k > d-1 & \text{if } d < 0. \end{cases}$$

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Hence $\dim_H(A_\beta) = 0$ if $d = \pm 1$ (quadratic Pisot units). We conjecture that this is the only situation where $\dim_H(A_\beta) = 0$.

# Proof: Matching for quadratic Pisot slope $T_\alpha$

There are integers $a_j, b_j$ (depending on $n$) such that

$$\begin{aligned} T_\alpha^n(0) &= (\beta^{n-1} + \cdots + 1)\alpha - a_{n-2}\beta^{n-2} - \cdots - a_1\beta - a_0, \\ T_\alpha^n(1) &= (\beta^{n-1} + \cdots + 1)\alpha + \beta^n - b_{n-1}\beta^{n-1} - \cdots - b_1\beta - b_0. \end{aligned}$$

Therefore matching at (minimal) iterate $n$ requires

$$0 = T_\alpha^n(1) - T_\alpha^n(0) = \beta^n + \sum_{j=0}^{n-1} \beta^j (b_j - a_j).$$

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The integers $b_j, a_j$ depend on $\alpha$, but change only at a finite set.
Hence, if matching occurs, it occurs on an entire interval.

# Proof: Matching for quadratic Pisot-slope $T_\alpha$

Since $\beta$ is an algebraic integer of order $n$, we can write

$$T_\alpha^j(0) - T_\alpha^j(1) = \sum_{k=1}^{n} \frac{e_k(j)}{\beta^k} \qquad e_k(j) \in \mathbb{Z}.$$

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**Proof.**

If $|T_\alpha^j(0) - T_\alpha^j(1)| = 1/\beta$, then $T_\alpha^j(0)$ and $T_\alpha^j(1)$ belong to neighbouring branch-domains of $T_\alpha$, and their images are the same. □

# Proof: Matching for quadratic Pisot slope $T_\alpha$

Sketch of proof for $\beta^2 - k\beta - d = 0$, $k \in \mathbb{N}$, so $k - 1 < \beta < k$ and $T_\alpha$ has $k$ or $k + 1$ branches.

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Lemma: If $\alpha \in [k-\beta, 1)$, then $T_\alpha$ has $k+1$ branches, but there is matching after two steps.

Hence, take $\alpha \in [0, k-\beta)$ and call the domains of the branches $\Delta_0, \ldots, \Delta_k$. Compute

$$T_\alpha(1) = \beta + \underbrace{\alpha}_{=T(0)} - (k-1) = T_\alpha(0) + \underbrace{\beta - (k-1)}_{\gamma}.$$

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Lemma: If $T^\ell(0) \in \Delta_i$ and $T^\ell(1) \in \Delta_{i+(k-1)-d}$ for $1 \le \ell < n, i = i(\ell)$, then

$$T^n(1) - T^n(0) = \gamma.$$

# Proof: Matching for quadratic Pisot slope $T_\alpha$

Lemma: If $T^{n-1}(0) \in \Delta_i$ and $T^{n-1}(1) \in \Delta_{i+k-d}$ then the distance $|T^n(1) - T^n(0)| = \frac{d}{\beta}$ and there is matching in 2 steps.

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Hence, to avoid matching, $T^\ell(0)$ has to avoid the sets

$$\begin{aligned} V_i &:= \{x \in \Delta(i) : x + \gamma \in \Delta(i+k-d)\} \\ &= \Big[\frac{i+k-d-\alpha}{\beta_k} - \gamma \,,\ \frac{i+1-\alpha}{\beta_k}\Big). \end{aligned}$$

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Lemma: If

$$T^n(0) \in V = \cup_{i=0}^{d-1} V_i$$

then there is matching in two steps.

# Proof: Matching for quadratic Pisot slope $T_\alpha$

Lemma: The map $g_\alpha : [0, 1-\beta] \to [0, 1-\beta]$,

$$g_\alpha(x) := \begin{cases} k-\beta & \text{if } x \in V, \\ T_\alpha(x) & \text{otherwise.} \end{cases}$$

is a non-decreasing degree $d$ circle endomorpism, and $g^n(0) \in V$ for some $n > 1$ precisely if $k-\beta$ is periodic.

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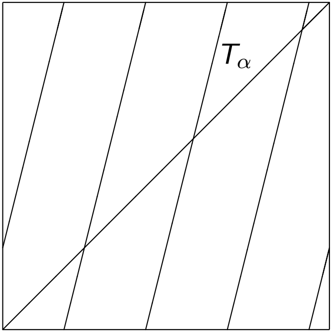


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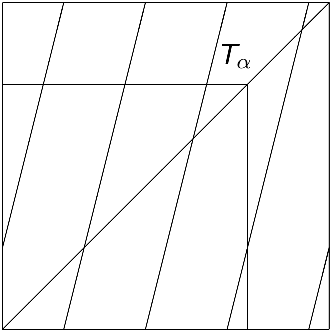


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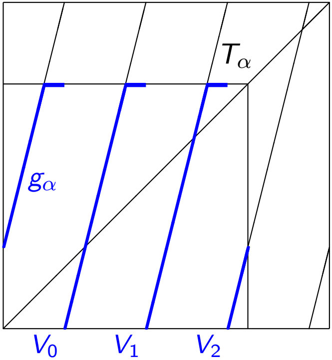


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Recall that $V$ is the union of plateaux of the map

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Lemma: Define

$$X_\alpha = \{x \in \mathbb{S}^1 \,:\, g_\alpha^n(x) \notin V \text{ for all } n \geq 0\}.$$

If there is no matching, then $\dim_H(X_\alpha) = \frac{\log d}{\log \beta}$.

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Idea of Proof.
For each $n$, we cover $X_\alpha$ by $O(d^n)$ intervals of length $\beta^{-n}$. □

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- On the other hand, the intervals $U$ in the cover of the previous lemma move with fixed speed (independent of $n$).
- Therefore, for each $n$, the set $A_\alpha$ can be covered by $O(d^n)$ intervals of length $O(\beta^{-n})$.

□

# Matching for non-Pisot Units

The examples we have of prevalent matching all relate to $\beta$ being a Pisot unit. However, matching can occur at non-Pisot units, e.g., the quartic Salem number satisfying

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Numerical simulations give the following table

| $\beta$ | minimal polynomial | $\dim_B(\mathcal{E}_\beta)$ |
|---|---|---|
| tribonacci | $\beta^3 - \beta^2 - \beta - 1 = 0$ | 0.66... |
| tetrabonacci | $\beta^4 - \beta^3 - \beta^2 - \beta - 1 = 0$ | 0.76... |
| plastic | $\beta^3 - \beta - 1 = 0$ | 0.93... |

# Matching for non-Quadratic Pisot Units

There is another frequently used class of Pisot units, namely leading solutions $\beta_k$ of

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for $k \geq 3$.

**Theorem (Non-Quadratic Pisot Units)**

*For $\beta_3$, there is prevalent matching. For the non-matching set $\dim_H(A_\beta) \in (0, 1)$.*

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## Theorem (Non-Quadratic Pisot Units)

*For $\beta_3$, there is prevalent matching. For the non-matching set* $\dim_H(A_\beta) \in (0, 1)$.

We expect the same result for $\beta_k$, $k \geq 4$, but at the moment, we have no proof.

# Matching for non-Quadratic Pisot Units

**Lemma:** For every $k \geq 2$ and $j \geq 0$ we have

$$|T_\alpha^j(0) - T_\alpha^j(1)| \in \left\{ \frac{e_1}{\beta} + \frac{e_2}{\beta^2} + \cdots + \frac{e_k}{\beta^k} \ : \ e_1, \ldots, e_k \in \{0, 1\} \right\}.$$

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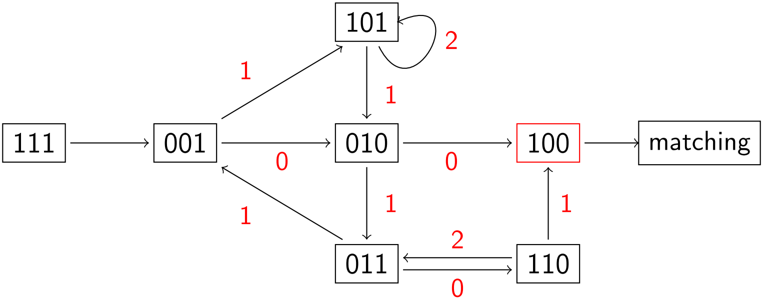


Figure: The transition graph for the tribonacci number $\beta_3$. The red numbers indicate the difference in branch between $T_\alpha^j(0)$ and $T_\alpha^j(1)$.

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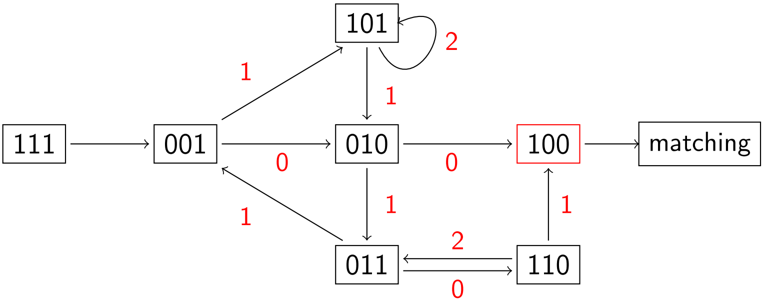


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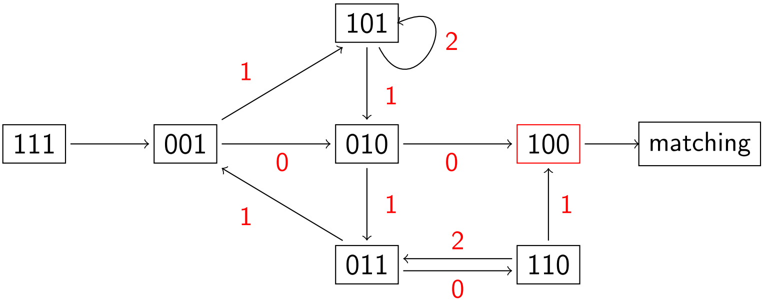


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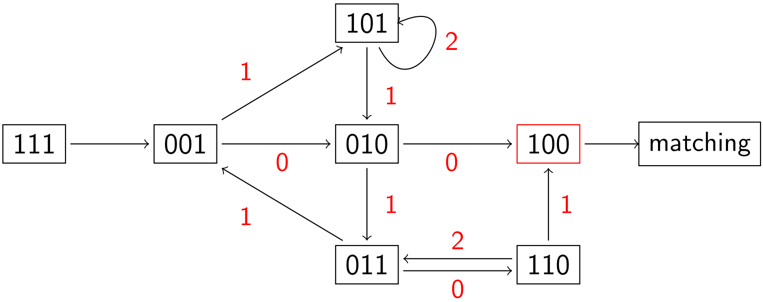


- The diagram expresses only a the “fiber part” of a skew-product. So it is more complicated than a SFT.
- There are linked non-trivial loops that give a Cantor set of positive Hausdorff dimension inside the bifurcation set.
- Abundancy of paths to matching gives upper bound $< 1$.

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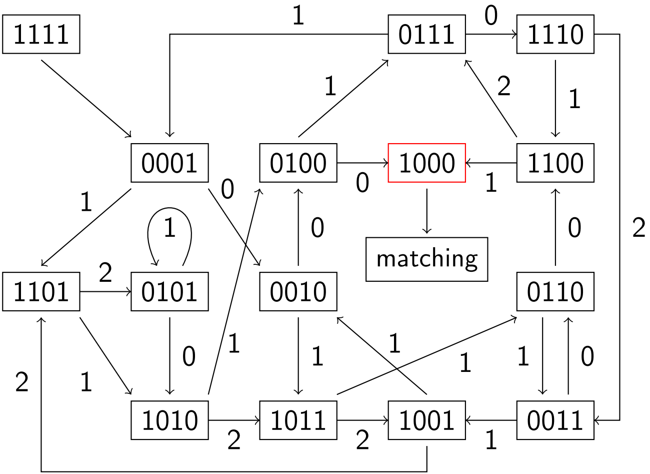


Figure: The transition graph for the Pisot number $\beta_4$ is similar but too complicated to handle.

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