Matching of discontinuous interval maps.

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joint with

Carlo Carminati (University of Pisa)
Stefano Marmi (Scuola Normale di Pisa)
Alessandro Profeti (University of Pisa)

and results in papers by

Dajani (Utrecht) & Kalle (Leiden)
Bonanno, Carminati (Pisa), Isola (Bologna) & Tiozzo (Yale)

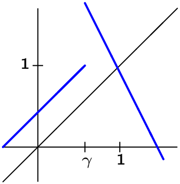
Krakow, June 2019

Figure: Kiev 1998

The map of Botero-Soler et al. generalized

For fixed slope $1 < s \in \mathbb{N}$, take:

$$Q_\gamma(x) = \begin{cases} x+1, & x \le \gamma \\ 1 + s(1-x), & x > \gamma \end{cases}$$



Matching

Let $T = T_\gamma : [0,1] \to [0,1]$ be a family of piecewise C^1 and expanding interval maps with discontinuity at γ.

Definition: There is matching if there are iterates $\kappa_\pm > 0$ such that

$$T^{\kappa_-}(\gamma^-) = T^{\kappa_+}(\gamma^+) \text{ and derivatives } DT^{\kappa_-}(\gamma^-) = DT^{\kappa_+}(\gamma^+)$$

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Despite T_γ not being a Markov map, matching has the following similar effect:

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In fact, if T is piecewise affine, then ρ is constant on each element of the pre-matching partition.

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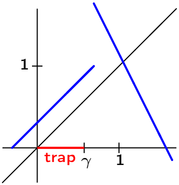

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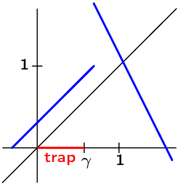

Matching properties of Q_γ:

- $x = 1$ is fixed for all s and $\gamma < 1$;
- For integer $s \ge 2$, every point ps^{-m}, $p, m \in \mathbb{N}$, eventually maps to 1;

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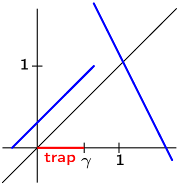

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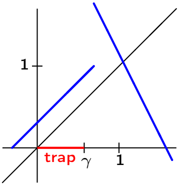

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- matching occurs when $Q_\gamma^n(\gamma^+) \in [0, \gamma) =$ trap for some $n \geq 1$.

Signed binary digit map

$$S_\alpha(x) = \begin{cases} 2x+\alpha & x \in [-1,-\frac{1}{2}); \\ 2x & x \in (-\frac{1}{2},\frac{1}{2}); \\ 2x-\alpha & x \in (\frac{1}{2},1]. \end{cases} \qquad \alpha \in [1,2].$$

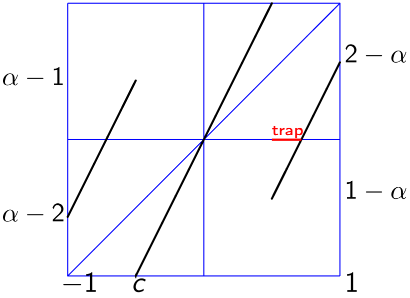

Figure: The signed binary expansion map (Dajani & Kalle).

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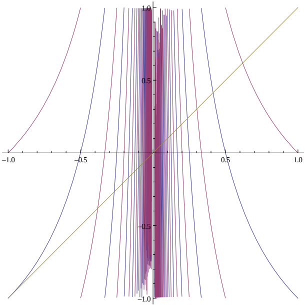
Matching of $\frac{1}{2}^+$ and $\frac{1}{2}^-$ (and by symmetry of $(-\frac{1}{2})^-$ and $(-\frac{1}{2})^+$) occurs

- for $\alpha \in [\frac{3}{2}, 2]$,
- whenever $S_\alpha^\kappa(1) \in [\frac{1}{2}, \frac{\alpha}{2}] = \text{trap}$, and
- the matching exponent is the minimal such κ.

The α-continued fraction map T_α.

A generalization of the Gauß map stems from Nakada (and Natsui).



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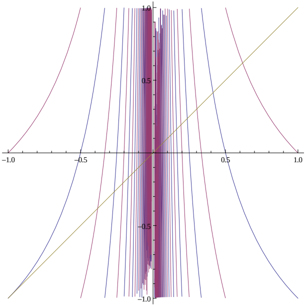

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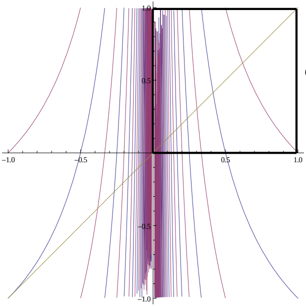

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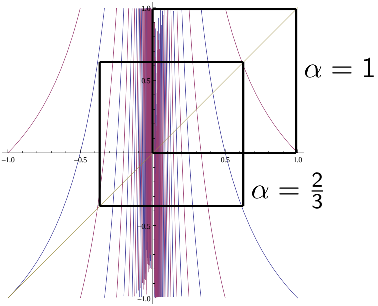

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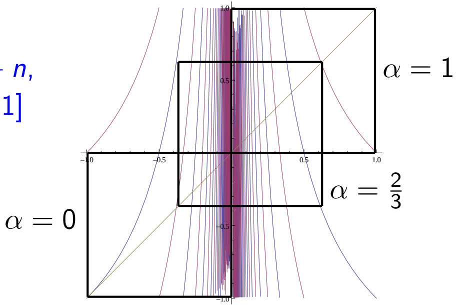

Figure: $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha], x \mapsto |\frac{1}{x}| - \lfloor \frac{1}{x} + 1 - \alpha \rfloor$.

All of them have invariant densities (infinite if $\alpha = 0$).

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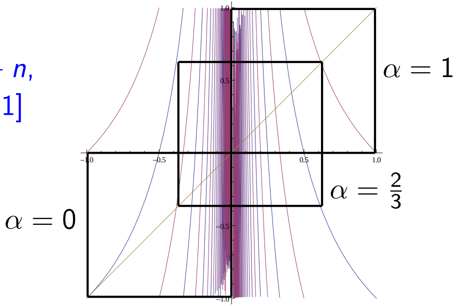

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Matching of the orbits of α and $\alpha - 1$ occurs for a.e. $\alpha \in [0, 1]$.

α-continued fractions and the Mandelbrot set

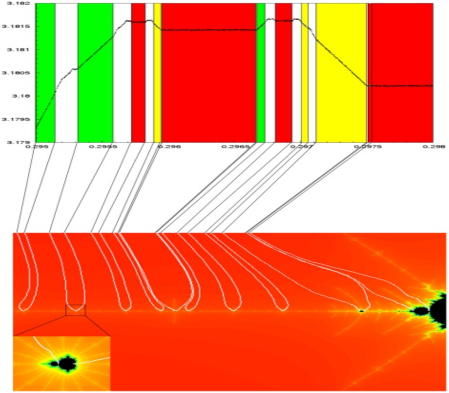

Figure: From a paper by Bonanno, Carminati, Isola and Tiozzo: The non-matching set and the real antenna of Mandelbrot set

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For the quadratic family $z \mapsto z^2 + c$, replace matching by:

the ray of parameter angle θ lands at the real antenna,

of Mandelbrot set and you get the same result w.r.t. θ.

Monotonicity of entropy for Q_γ

Theorem: Let $\Delta = \kappa_+ - \kappa_-$. The topological and metric entropy

$$h_\mu(Q_\gamma) \text{ and } h_{top}(Q_\gamma) \text{ are } \begin{cases} \text{decreasing} & \text{if } \Delta < 0; \\ \text{constant} & \text{if } \Delta = 0; \\ \text{increasing} & \text{if } \Delta > 0, \end{cases}$$

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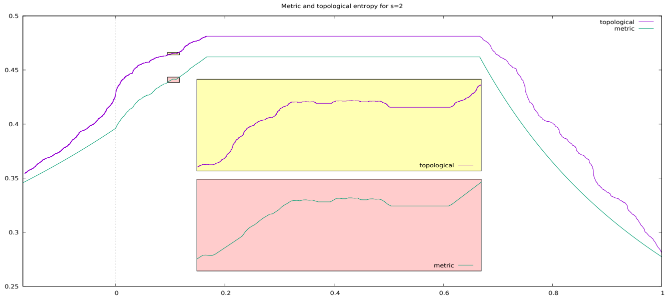

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Question: Is there a meta-theorem (more precise than "renormalization") explaining these joint features of such interval maps?

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- For binary string u, let $\check{u}$ be the bitwise negation of u.
- For $\xi \in \mathbb{Q}_{\mathrm{dyd}} \setminus \{1\}$ and let w be the shortest even binary expansion of ξ and v be the shortest odd binary expansion of $1 - \xi$.

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- For $\xi \in \mathbb{Q}_{\mathrm{dyd}} \setminus \{1\}$ and let w be the shortest even binary expansion of ξ and v be the shortest odd binary expansion of $1 - \xi$.
- Define the interval $I_\xi := (\xi_L, \xi_R)$ containing ξ where,
- $\xi_L := .\overline{\check{v} v}$, $\quad \xi_R := .\overline{w}$.
- Also define the "degenerate" interval $I_1 := (2/3, +\infty)$.

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If $\xi = 1/2$ then $w = 10$, $v = 1$ and $\xi_L = .\overline{01}$, $\xi_R = .\overline{10}$.

(01) $w = u01 \Rightarrow \xi_L = .\overline{u001\check{u}110}$;

(11) $w = u11 \Rightarrow \xi_L = .\overline{u101\check{u}010}$;

(010) $w = u010 \Rightarrow \xi_L = .\overline{u00\check{u}11}$;

(110) $w = u110 \Rightarrow \xi_L = .\overline{u10\check{u}01}$.

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ξ	ξ_R	ξ_L
$\frac{1}{2} = .10$	$\frac{2}{3} = .\overline{10}$	$\frac{1}{3} = .\overline{01}$
$\frac{1}{4} = .01$	$\frac{1}{3} = .\overline{01}$	$\frac{2}{9} = .\overline{001110}$
$\frac{7}{32} = .001110$	$\frac{2}{9} = .\overline{001110}$	$\frac{7}{33} = .\overline{0011011001}$
$\frac{3}{16} = .0011$	$\frac{1}{5} = .\overline{0011}$	$\frac{2}{11} = .\overline{0010111010}$
$\frac{9}{64} = .001001$	$\frac{1}{7} = .\overline{001}$	$\frac{4334}{16383} = .\overline{00100011101110}$
$\frac{1}{8} = .0010$	$\frac{2}{15} = .\overline{0010}$	$\frac{1}{9} = .\overline{000111}$

Matching intervals for Q_γ

Theorem:

- All matching intervals have the form I_ξ, where $\xi \in \mathbb{Q}_{\mathrm{dyd}}$ are precisely the pseudo-centers of the components of $[0, \frac{2}{3}] \setminus \mathcal{E}$.

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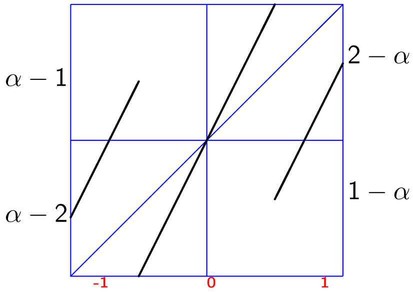
Example: All matching intervals in $(\frac{1}{6}, \frac{2}{3})$ have matching index $\Delta = 0$. Hence, entropy is constant on $[\frac{1}{6}, \frac{2}{3}]$.

Matching intervals for S_α



Let $d_1 d_2 d_3 \cdots \in \{-1, 0, 1\}^{\mathbb{N}}$ be the itinerary of 1:

$$d_i = \begin{cases} -1 & S_\alpha^{i-1}(1) \in [-1, -\frac{1}{2}); \\ 0 & S_\alpha^{i-1}(1) \in [-\frac{1}{2}, \frac{1}{2}]; \\ 1 & S_\alpha^{i-1}(1) \in (-\frac{1}{2}, 1]. \end{cases}$$

Note that the matching index $\kappa = \min\{n : d_n = -1\} - 1$.

Matching intervals for S_α

For $v = ua \in \{0,1\}^\kappa$, define

$$\psi_S(ua) = ua\check{u}1 \qquad a \in \{0,1\}.$$

Definition: A word $v = v_1 \dots v_\kappa \in \{0,1\}^\kappa$ is called primitive if

- $v_1 = v_2 = v_\kappa = 1$;
- $v_j \dots v_\kappa \preceq x_1 \dots x_{\kappa-j+1}$ (shift-maximal in lexicogr. order);
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Theorem: Let α have matching exponent κ and itinerary $d_1 d_2 d_3 \dots$ Then the matching interval of α is

$$J = \left(\frac{1 + 2^{-\kappa}}{\sum_{j=1}^{\kappa} d_j 2^{-j} + 2^{-\kappa}}, \frac{1 + 2^{-\kappa}}{\sum_{j=1}^{\kappa} d_j 2^{-j} - 2^{-\kappa}} \right)$$

if and only if $d_1 \dots d_\kappa$ is a primitive word.

Matching intervals for Q_γ (period doubling)

Take a pseudo-center

$$\xi = \begin{cases} 0.w & \text{(even expansion)} \\ 1 - 0.v & \text{(odd expansion).} \end{cases}$$

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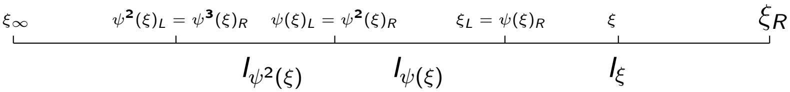
Let

$$\psi_S(\xi) := .\check{v} \qquad \text{for } \xi = 1 - 0.v$$

Note that ξ_L is also the right end-point of $I_{\psi(\xi)}$. We call this "period doubling". It repeats countably often, converging to $\xi_\infty := \lim_n \psi^n(\xi)$.



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Lemma: The pseudo-center of the next period doubling can be obtained from the previous using the substitution:

$$\chi : \begin{array}{ll} w \mapsto \check{v}v & \check{w} \mapsto v\check{v} \\ v \mapsto vw & \check{v} \mapsto \check{v}\check{w} \end{array} .$$

Thus the limit ξ_∞ has s-adic expansion

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$$\xi_\infty = .\check{v}\check{w}v\check{v}vw\check{v}\check{w}vw\check{v}v\check{v}\check{w}\ldots$$

Remark: This substitution factorizes over the Thue-Morse substitution

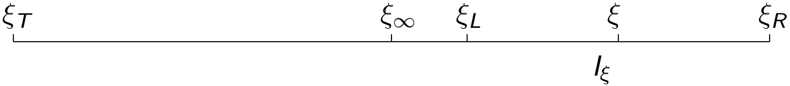
$$\chi_{\text{Thue-Morse}} : \begin{cases} 0 \mapsto 01; \\ 1 \mapsto 10 \end{cases}$$

(via the change of symbols $\pi(v) = \pi(\check{w}) = 0$, $\pi(\check{v}) = \pi(w) = 1$), which factorizes over the period-doubling (Feigenbaum) substitution

$$\chi_{\text{Feigenbaum}} : \begin{cases} 0 \mapsto 11; \\ 1 \mapsto 10. \end{cases}$$

Matching intervals for Q_γ (tuning windows)

Pseudo-center $\xi = .w$ (even expansion) and $1 - \xi = .v$ (odd exp).
The matching interval is $I_\xi = [\xi_L, \xi_R]$ for $\xi_L = .\overline{\check{v}v}$ and $\xi_R = .\overline{w}$.
The tuning interval is $T_\xi = [\xi_T, \xi_R]$ for $\xi_T = .\check{v}\overline{\check{w}}$.



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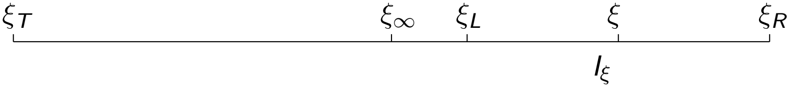
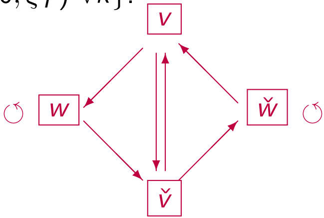


Theorem: Let $K(\xi_T) = \{x : Q_\gamma(\gamma^+) \notin [0, \xi_T)\ \forall k\}$.
Then $x \in K(\xi_T) \cap T_\xi$ if and only if

$$x = .\sigma_1\sigma_2\sigma_3\sigma_4\ldots$$

for $\sigma_1 \in \{w, \check{v}\}$, $\sigma_j \in \{w, v, \check{w}, \check{v}\}$
describing a path in the diagram.



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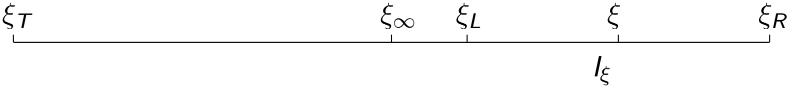

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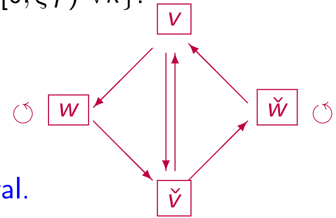
describing a path in the diagram.
If $\Delta(\xi) = 0$, all matching in T_ξ is neutral.



Matching intervals for S_α (period doubling)

Recall for $v = ua \in \{0,1\}^\kappa$, define

$$\psi(ua) = \psi_S(ua) = ua\check{u}1 \qquad a \in \{0,1\}.$$

We have again period doubling:

$$v_L = \psi(v)_R \qquad \text{provided } v \text{ is primitive.}$$

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$$v_L = \psi(v)_R \qquad \text{provided } v \text{ is primitive.}$$

Remark: Starting with $v = 11$, we get

$$\lim_{n\to\infty} \psi^n(v) = v_\infty = 110\ 1001\ 1001\ 0110\ 1001\ 0110\dots$$

which is the left-shift of the fixed point of the Thue-Morse substitution.

Matching intervals for S_α (tuning intervals)

The analogon of tuning interval for the family S_α doesn't seem to have been investigated.

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To finish: the flowers on my balcony

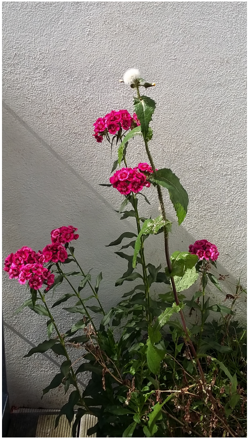

Figure: Maiden pink – Dianthus Deltoides – Goździk kropkowany

Happy Birthday Michał !