About pseudo-Markov interval maps, entropy and pseudocenters.

Henk Bruin (University of Vienna)

joint with
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explaining observations in a paper by

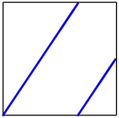
V. Botella-Soler, J. A. Oteo, J. Ros, and P. Glendinning

Maribor, January 2015

β-transformations

The β-transformation is defined as

$$x \mapsto \beta x \pmod 1$$



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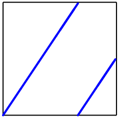


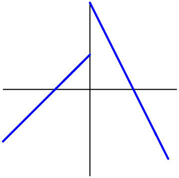
For $|\beta| > 1$, T_β has an acip μ.

Figure : Density $\frac{d\mu}{dx}$ for $\beta = \frac{1}{2}(\sqrt{5}+1)$ and $\beta = \sqrt[3]{7}$.

The density is **only** locally constant, if there is a Markov partition.

The map T_β

$$T_\beta(x) = \begin{cases} T_\beta^-(x) = x + 2 & \text{if } x \leq 0, \\ T_\beta^+(x) = \beta - 2x & \text{if } x \geq 0. \end{cases}$$



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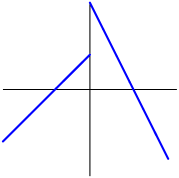

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Figure : Invariant density for the T_β: left $\beta = \frac{1}{2}(\sqrt{5}+1)$ right: $\beta = \sqrt[3]{7}$.

Markov partitions and entropy

The interval partition $\{P_i\}$ is a Markov partition for T if

$$T(P_i) \cap P_j \neq \emptyset \text{ implies } T(P_i) \supset P_j.$$

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$$\Pi_{i,j} = \begin{cases} 1 & \text{if } T(P_i) \supset P_j, \\ 0 & \text{if } P_j \cap T(P_i) = \emptyset, \\ \text{No other possibility, because } \{P_i\} \text{ is Markov} \end{cases}$$

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The topological entropy is

$$h_{top}(T) = \log \sigma$$

for σ the leading eigenvalue of Π.

Markov partitions and entropy

Scale Π by the slopes $t_i = |DT_{|P_i}|$ to obtain a matrix

$$A_{i,j} = \frac{1}{t_i}\Pi_{i,j}.$$

Then $\ell_i = |P_i|$ and $\rho_i = \frac{d\mu}{dx}_{|P_i}$ satisfy $\sum_i \rho_i \ell_i = 1$ and

$$\begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix}^T A = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix}^T \quad \text{and} \quad A \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix}$$

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Rokhlin's formula gives the metric entropy:

$$h_\mu(T) = \sum_{i=1}^{N} \max\{\log(t_i), 0\}\mu(P_i)$$

Not Markov but Matching

For the family T_β, there is no Markov partition in general, but something called matching takes can occur:

Definition: There is matching if there are iterates $\kappa_\pm > 0$ such that

$$T^{\kappa_-}(0^-) = T^{\kappa_+}(0^+) \text{ and derivatives } DT^{\kappa_-}(0^-) = DT^{\kappa_+}(0^+)$$

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Theorem: If T has matching, then $\rho = \frac{d\mu}{dx}$ is constant on each element of the pre-matching partition.

Not Markov but Matching

Remark: The density can again be found by linear algebra, using

$$\Pi_{i,j} = |T(P_i) \cap P_j| / |P_j|.$$

Then $\Pi_{i,j} \in \{0, 1\}$ except for those column numbers j such that $z := T^{\kappa_-}(c_-) = T^{\kappa_+}(c_+) \in P_j$.

Example: For $0 < \beta \leq 2$, we have matrices

$$\Pi = \begin{pmatrix} 0 & 1 & \theta \\ 0 & 0 & 1-\theta \\ 1 & 1 & \theta \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & \theta \\ 0 & 0 & 1-\theta \\ \frac{1}{2} & \frac{1}{2} & \frac{\theta}{2}, \end{pmatrix}.$$

The eigenvectors are

$$\rho = \frac{1}{10-4\theta} \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \ell = \begin{pmatrix} 2 \\ 2-2\theta \\ 2 \end{pmatrix}$$

The metric entropy is $h_\mu(T_\beta) = \frac{2\log 2}{5-2\theta}$ by the Rokhlin formula.

Monotonicity of entropy

Numerical illustration for the metric entropy:

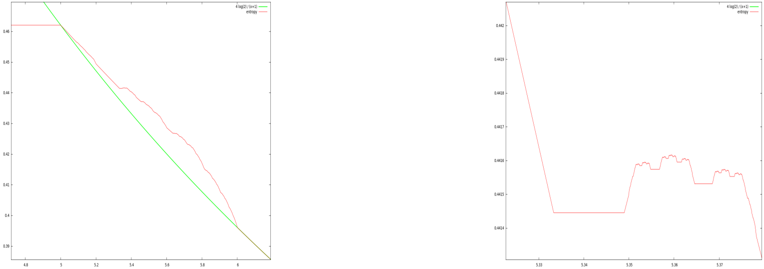


Figure : Entropy $h_\mu(T_\beta)$ for $\beta \in [4.6, 6]$ (l) and $\beta \in [5.29, 5.33]$ (r).

Monotonicity of entropy

Definition: The matching index is $\Delta = \kappa_+ - \kappa_-$.

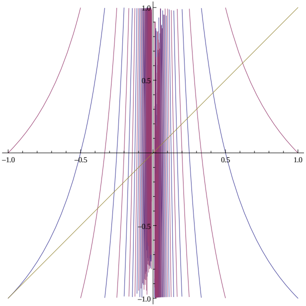
Theorem: Topological and metric entropy are

$$h_\mu(T_\beta) \text{ and } h_{top}(T_\beta) \text{ are } \begin{cases} \text{increasing} & \text{if } \Delta < 0; \\ \text{constant} & \text{if } \Delta = 0; \\ \text{decreasing} & \text{if } \Delta > 0, \end{cases}$$

as function of β within matching intervals.

The α-continued fraction map T_α.

A generalization of the Gauß map stems from Nakada (and Natsui).



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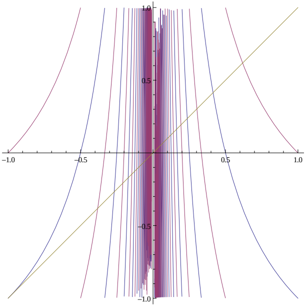

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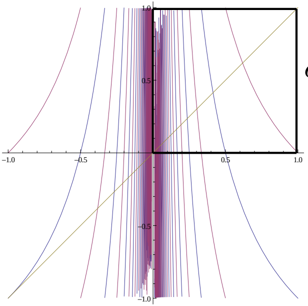

$\alpha = 1$

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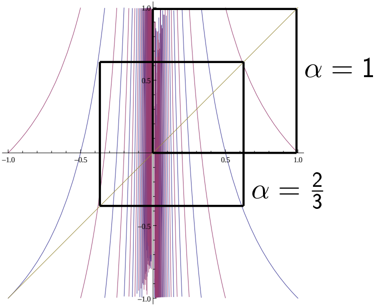

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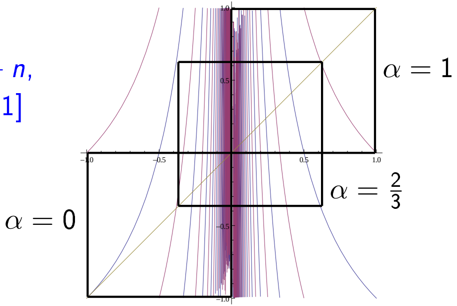

Figure : $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha], x \mapsto |\frac{1}{x}| - \lfloor \frac{1}{x} + 1 - \alpha \rfloor$.

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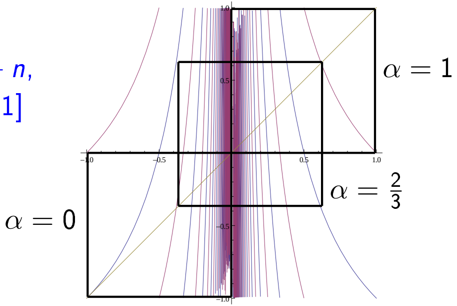

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Matching of the orbits of α and $\alpha - 1$ occurs for a.e. $\alpha \in [0, 1]$.

α-continued fractions and the Mandelbrot set

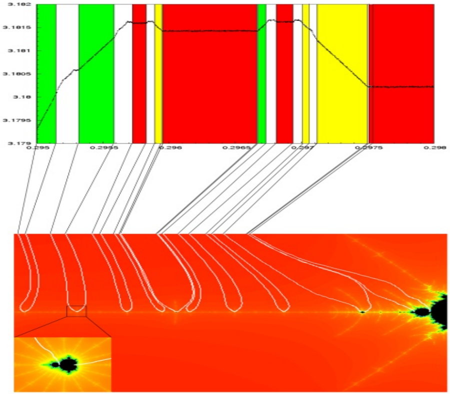
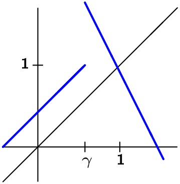


Figure : From a paper by Bonanno, Carminati, Isola and Tiozzo: The non-matching set and the real antenna of Mandelbrot set

Change of coordinates

For fixed slope $s > 1$, take:

$$Q_\gamma(x) = \begin{cases} x+1, & x \le \gamma \\ 1 + s(1-x), & x > \gamma \end{cases}$$



For $s = 2$, Q_γ is conjugate to T_β above via

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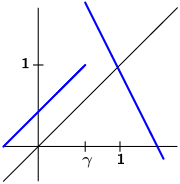

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Advantages of Q_γ:

- $x = 1$ is fixed for all $s \in \mathbb{R}$ and $\gamma < 1$;
- For integer $s \ge 2$, every point ps^{-m}, $p, m \in \mathbb{N}$, eventually maps to 1;
- therefore matching occurs whenever $\gamma = ps^{-m}$;
- matching occurs on an open dense set!

Matching is Lebesgue typical

Theorem: Q_γ has matching for Lebesgue-a.e. γ, but the set $\mathcal{E}$ of non-matching parameters has Haussdorf dimension 1.

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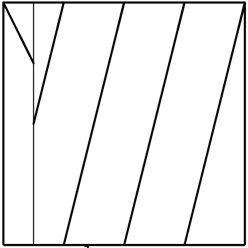
Let $g(x) := s(1-x) \bmod 1$ and $R : (0,1) \to (0,1)$ be the first return of $Q_\gamma^k(x)$ to $[0,1)$.

Lemma:

$$R(x) = \begin{cases} g(x) & \text{if } x \in (0,\gamma) \\ g^2(x) & \text{if } x \in (\gamma,1) \end{cases}$$



$\gamma = \frac{1}{8}$ $\quad s = 2$

On the proof of "Matching is Lebesgue typical"

Lemma: For fixed $\gamma \in [0,1]$, the following conditions are equivalent:

(i) $g^k(\gamma) < \gamma$ for some $k \in \mathbb{N}$;

(ii) matching holds for γ.

In other words, the bifurcation set is

$$\mathcal{E} = \{\gamma \in [0,1] \;:\; g^k(\gamma) \geq \gamma \;\forall k \in \mathbb{N}\}.$$

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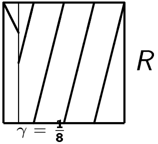

Proof of the Theorem:

- Lebesgue measure is preserved by g, so the Ergodic Theorem implies that $\inf\{g^k(\gamma) : k \geq 1\} = 0$ for a.e. γ. The previous lemma gives that each such $\gamma \notin \mathcal{E}$.

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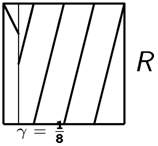

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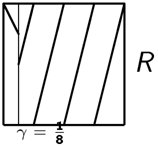

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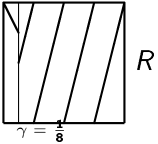

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- The Hausdorf dimension $\dim_H(K(t)) \to 1$ and $\dim_H(K(t) \cap [0,1]) \to 1$ as $t \to 0$.
- Combine this with $\mathcal{E} \cap [0,t] \supset K(t)$.

Monotonicity

Recall the Monotonicity Theorem stated for Q_γ:

Topological and metric entropy are

$$h_\mu(Q_\gamma) \text{ and } h_{top}(Q_\gamma) \text{ are } \begin{cases} \text{decreasing} & \text{if } \Delta < 0; \\ \text{constant} & \text{if } \Delta = 0; \\ \text{increasing} & \text{if } \Delta > 0, \end{cases}$$

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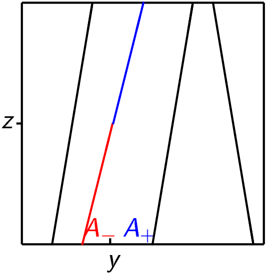
$$\text{orb}(\partial J) \cap J^\circ = \emptyset.$$

Monotonicity

Lemma: All branches of R are monotone onto, also the branches that contain a preimage $y \in Q_\gamma^{-N}(\gamma)$.

$R = Q_\gamma^\tau$ for first return time $\tau : [0,1) \to \mathbb{N}$

$$\tau(A_-) = N + \kappa_-$$
$$\tau(A_+) = N + \kappa_+$$



Corollary: R preserves Lebesgue measure m.

Monotonicity

Proof-sketch of the monotonicity theorem:

- $\int_J \tau \; dm h_m(R)$ increases by an amount proportional to $\eta := \Delta \times$ increased proportion of $|A_+|/|A_-|$.
- Abramov's Formula: $h_\mu(Q_\gamma) = \frac{1}{\int_J \tau \; dm} h_m(R)$.
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- As γ moves within a matching interval, periodic points in J don't change,
- but their period increases by Δ as A_+ absorbes them (when they previously belonged to A_-).
- Hence the topological entropy decreases accordingly.

Pseudo-centers

Motivation: Find exact formulas for matching intervals J and their matching indices Δ for slope $s = 2$.

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- Define the interval $I_\xi := (\xi_L, \xi_R)$ containing ξ where,
- $\xi_L := .\overline{\check{v}v}, \quad \xi_R := .\overline{w}.$
- Also define the "degenerate" interval $I_1 := (2/3, +\infty)$.

Pseudo-centers

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If $\xi = 1/2$ then $w = 10$, $v = 1$ and $\xi_L = .\overline{01}$, $\xi_R = .\overline{10}$.

(01) $w = u01 \Rightarrow \xi_L = .\overline{u001\check{u}110}$;

(11) $w = u11 \Rightarrow \xi_L = .\overline{u101\check{u}010}$;

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ξ	ξ_R	ξ_L
$\frac{1}{2} = .10$	$\frac{2}{3} = .\overline{10}$	$\frac{1}{3} = .\overline{01}$
$\frac{1}{4} = .01$	$\frac{1}{3} = .\overline{01}$	$\frac{2}{9} = .\overline{001110}$
$\frac{7}{32} = .001110$	$\frac{2}{9} = .\overline{001110}$	$\frac{7}{33} = .\overline{0011011001}$
$\frac{3}{16} = .0011$	$\frac{1}{5} = .\overline{0011}$	$\frac{2}{11} = .\overline{0010111010}$
$\frac{9}{64} = .001001$	$\frac{1}{7} = .\overline{001}$	$\frac{4334}{16383} = .\overline{00100011101110}$
$\frac{1}{8} = .0010$	$\frac{2}{15} = .\overline{0010}$	$\frac{1}{9} = .\overline{000111}$

Pseudo-centers

Theorem:

- All matching intervals have the form I_ξ, where $\xi \in \mathbb{Q}_{\mathrm{dyd}}$ are precisely the pseudo-centers of the components of $[0, \frac{2}{3}] \setminus \mathcal{E}$.
- The matching index is

$$\Delta(\xi) = \frac{3}{2}(|w|_0 - |w|_1),$$

where $|w|_a$ is the number of symbols a in w (the shortest even binary expansion of ξ).

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Proposition: If $g^k(\gamma) \geq \frac{1}{6}$, then $|w_0| = |w|_1$. In particular, all matching intervals in $(\frac{1}{6}, \frac{2}{3})$ have matching index $\Delta = 0$.

Pseudo-centers

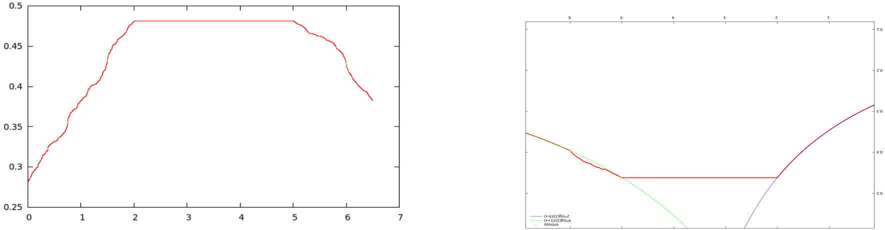

Figure : Entropies $h_{top}(T_\beta)$ and $h_\mu(T_\beta)$ for $\beta \in [0, 6.5]$.

Remark: This proposition explains constant entropy on all matching intervals in $(\frac{1}{6}, \frac{2}{3})$. Using an extra (no devil's staircase) argument:

$$h_\mu(Q_\gamma) = \log(\frac{1+\sqrt{5}}{2}) \quad \text{and } h_{top}(Q_\gamma) = \frac{2}{3}\log 2,$$

for all $\gamma \in [\frac{1}{6}, \frac{2}{3}]$.

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