Matching for translated β-transformations.

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joint with

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Survey on Matching in 1D dynamics

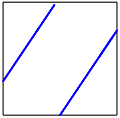
including work of
Dajani, Kalle and Bruin, Carminati, Marmi, Profeti

Conference in Hillerød, Denmark, 1993.

Translated β-transformations

The translated β-transformation is defined as

$$T_{\beta,\alpha} : x \mapsto \beta x + \alpha \pmod 1$$



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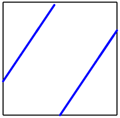

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Figure: Density $\frac{d\mu}{dx}$ for $\beta = \frac{1}{2}(\sqrt{5}+1)$ and $\beta = \sqrt[3]{7}$.

The density is **only?** locally constant, if there is a Markov partition.

Not Markov but Matching

For the family T_α, there is no Markov partition in general, but something called matching takes can occur:

Definition: There is matching if there is an iterate $\kappa > 0$ such that

$$\lim_{x \uparrow 0} T_\alpha^\kappa(x) = \lim_{x \downarrow 0} T_\alpha^\kappa(x)$$

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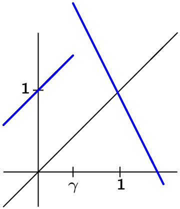
Theorem: If T_α has matching, then $\rho = \frac{d\mu}{dx}$ is constant on each element of the pre-matching partition.

The family Q_γ

The following family is conjuagte to the one studied by Botella-Soler, Oteo, Ros & Glendinning.

For fixed slope $s > 1$, take:

$$Q_\gamma(x) = \begin{cases} x + 1, & x \leq \gamma \\ 1 + s(1 - x), & x > \gamma \end{cases}$$



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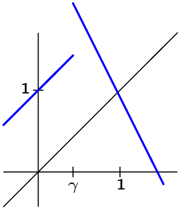


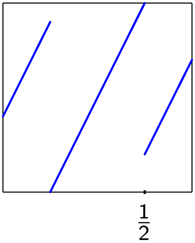
Matching occurs when there are $\kappa_\pm \in \mathbb{N}$ such that

$$\lim_{x\uparrow\gamma} Q_\gamma^{\kappa_-}(x) = \lim_{x\downarrow\gamma} Q_\gamma^{\kappa_+}(x) \quad \text{and} \quad \lim_{x\uparrow\gamma} DQ_\gamma^{\kappa_-}(x) = \lim_{x\downarrow\gamma} DQ_\gamma^{\kappa_+}(x)$$

The family S_α (Dajani & Kalle)

Dajani and Kalle study the following map:

$$S_\alpha(x) = \begin{cases} 2x + \alpha & x \in [-1, \frac{1}{2}) \\ 2x & x \in [-\frac{1}{2}, \frac{1}{2}] \\ 2x - \alpha & x \in (\frac{1}{2}, 1] \end{cases}$$



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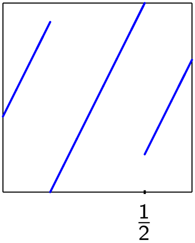

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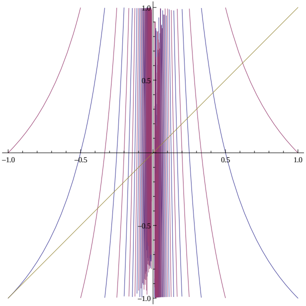
The α-continued fraction map A_α.

A generalization of the Gauß map stems from Nakada (and Natsui).



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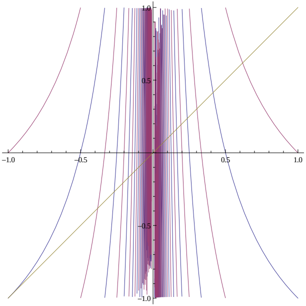

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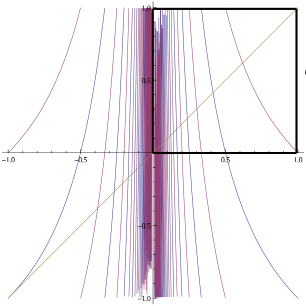

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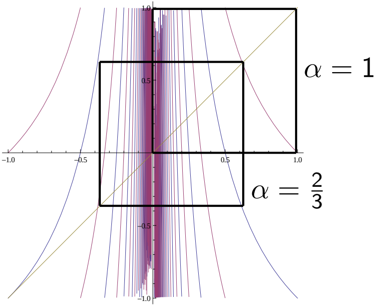

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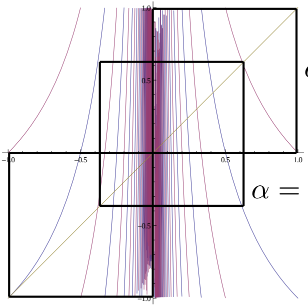

Figure: $A_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha], x \mapsto |\frac{1}{x}| - \lfloor \frac{1}{x} + 1 - \alpha \rfloor$.

All of them have invariant densities (infinite if $\alpha = 0$).

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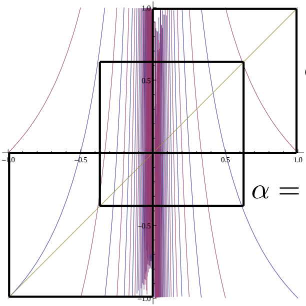

Figure: $A_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha], x \mapsto |\frac{1}{x}| - \lfloor \frac{1}{x} + 1 - \alpha \rfloor$.

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Matching occurs if $A_\alpha^\kappa(\alpha) = A_\alpha^\kappa(\alpha - 1)$ for some $\kappa \geq 1$.

Matching and piecewise constant densities

The pre-matching partition plays the role of Markov partition:

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This is a general theorem: If a piecewise affine expanding interval map $T : [0,1] \to [0,1]$ has matching at all its discontinuity points, then $\frac{d\mu}{dx}$ is constant on each element of the pre-matching partition.

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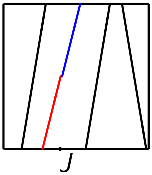
The family of α-continued fraction maps has Möbius branches. The density is piecewise Möbius, accordingly.

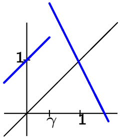
Matching and piecewise constant densities

Sketch of proof: 

- Take a nice interval J disjoint from the matching set. (nice means that $\mathrm{orb}(\partial J) \cap J^\circ = \emptyset$).
- Consider the first return map R to J; it has only onto linear (or Möbius) branches.
- Hence the R-invariant denstity is constant (or Möbius).
- The T-invariant density coincides with T-invariant density (up to a scaling factor).

Matching is typical in parameter space

Some properties of Q_γ:



- $x = 1$ is fixed for all $s \in \mathbb{R}$ and $\gamma < 1$;
- For integer $s \geq 2$, every point ps^{-m}, $p, m \in \mathbb{N}$, eventually maps to 1;
- Therefore matching occurs whenever $\gamma = ps^{-m}$;
- Matching occurs on an open dense set!

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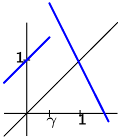

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Something far more general is true:

Theorem: The above families Q_γ, S_α and A_α have matching Lebesgue-a.e., but the set $\mathcal{E}$ of non-matching parameters has Haussdorf dimension 1.

α-continued fractions and the Mandelbrot set

The matching for the α-continued fraction is parallel to renormalization in the logistic family.

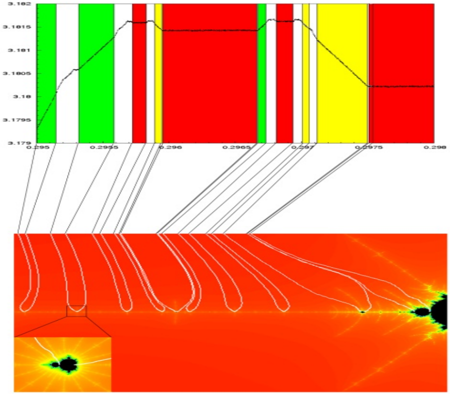


Figure: From a paper by Bonanno, Carminati, Isola and Tiozzo: The non-matching set and the real antenna of Mandelbrot set

Proof: Matching is Lebesgue typical

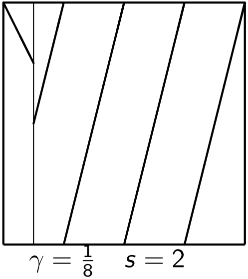
Let

$$g(x) := s(1 - x) \bmod 1$$

and $R : (0,1) \to (0,1)$ be the first return of $Q_\gamma^k(x)$ to $[0,1)$.

Lemma:

$$R(x) = \begin{cases} g(x) & \text{if } x \in (0,\gamma) \\ g^2(x) & \text{if } x \in (\gamma,1) \end{cases}$$



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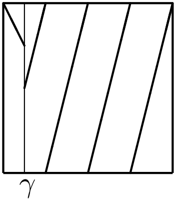
Return map R



Lemma: For fixed $\gamma \in [0,1]$, the following conditions are equivalent:

(i) $g^k(\gamma) < \gamma$ for some $k \in \mathbb{N}$;

(ii) matching holds for γ.

In other words, the bifurcation set is

$$\mathcal{E} = \{\gamma \in [0,1] \ : \ g^k(\gamma) \geq \gamma \ \forall k \in \mathbb{N}\}.$$

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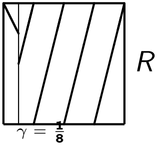

Proof of the Theorem:

- Lebesgue measure is preserved by g, so the Ergodic Theorem implies that

$$\inf\{g^k(\gamma) : k \geq 1\} = 0 \quad \text{for a.e. } \gamma.$$

The previous lemma gives that each such $\gamma \notin \mathcal{E}$.

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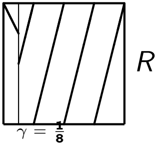

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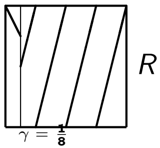

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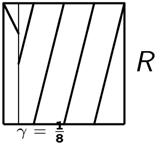

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- Combine this with $\mathcal{E} \cap [0,t] \supset K(t)$.

Typical matching for T_α: Quadratic Pisot Numbers

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The quadratic Pisot numbers are those $\beta > 1$ satisfying

$$\beta^2 - k\beta \pm d = 0 \qquad \text{with } \begin{cases} k > d+1 & \text{if } d > 0, \\ k > d-1 & \text{if } d < 0. \end{cases}$$

Theorem: If β is quadratic Pisot, then $\dim_H(A_\beta) = \frac{\log d}{\log \beta}$.

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Theorem: If β is quadratic Pisot, then $\dim_H(A_\beta) = \frac{\log d}{\log \beta}$.

Hence $\dim_H(A_\beta) = 0$ if $d = \pm 1$ (quadratic Pisot units). We conjecture that this is the only situation where $\dim_H(A_\beta) = 0$.

Proof: Matching for quadratic-Pisot-slope T_α

There are integers a_j, b_j such that

$$\begin{aligned} T_\alpha^n(0) &= (\beta^{n-1} + \cdots + 1)\alpha - a_{n-2}\beta^{n-2} - \cdots - a_1\beta - a_0, \\ T_\alpha^n(1) &= (\beta^{n-1} + \cdots + 1)\alpha + \beta^n - b_{n-1}\beta^{n-1} - \cdots - b_1\beta - b_0. \end{aligned}$$

Therefore matching at (minimal) iterate n requires

$$0 = T_\alpha^n(1) - T_\alpha^n(0) = \beta^n + \sum_{j=0}^{n-1} \beta^j (b_j - a_j).$$

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The integers b_j, a_j depend on α, but change only at a finite set. Hence, if matching occurs, it occurs on an entire interval.

Proof: Matching for quadratic-Pisot-slope T_α

Since β is an algebraic integer of order n, we can write

$$T_\alpha^j(0) - T_\alpha^j(1) = \sum_{k=1}^{n} \frac{e_k(j)}{\beta^k} \qquad e_k(j) \in \mathbb{Z}.$$

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Lemma (Sample Lemma)

If $|T_\alpha^j(0) - T_\alpha^j(1)| = 1/\beta$*, then there is matching at iterate* $j+1$*.*

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Lemma (Sample Lemma)

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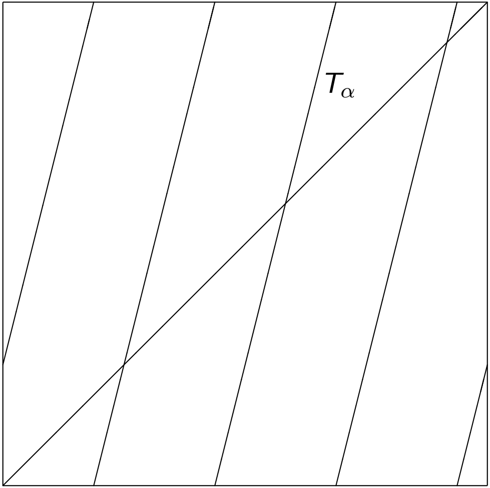
Proof.

If $|T_\alpha^j(0) - T_\alpha^j(1)| = 1/\beta$, then $T_\alpha^j(0)$ and $T_\alpha^j(1)$ belong to neighbouring branch-domains of T_α, and their images are the same. □

The maps T_α



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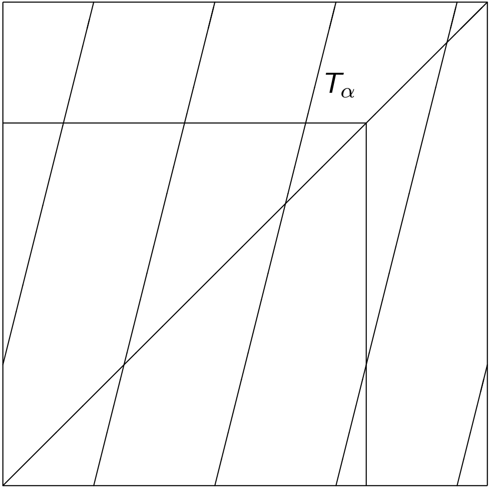

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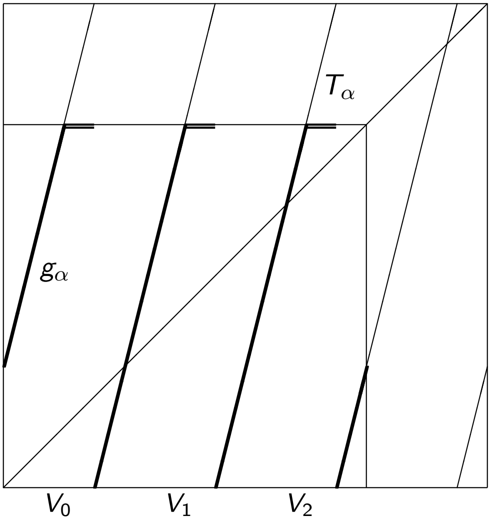

and g_α

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Lemma: If $\alpha \in [k - \beta, 1)$, then T_α has $k + 1$ branches, but there is matching after two steps.

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Lemma: If $\alpha \in [k - \beta, 1)$, then T_α has $k+1$ branches, but there is matching after two steps.

Hence, take $\alpha \in [0, k - \beta)$ and call the domains of the branches $\Delta_0, \dots, \Delta_k$. Compute

$$T_\alpha(1) = \beta + \underbrace{\alpha}_{=T(0)} - (k-1) = T_\alpha(0) + \underbrace{\beta - (k-1)}_{\gamma}.$$

Proof: Matching for quadratic-Pisot-slope T_α

We sketch the proof for $\beta^2 - k\beta - d = 0$, $k \in \mathbb{N}$, so $k-1 < \beta < k$ and T_α has k or $k+1$ branches.

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Lemma: If $T^\ell(0) \in \Delta_i$ and $T^\ell(1) \in \Delta_{i+(k-1)-d}$ for $1 \le \ell < n, i = i(\ell)$, then

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Hence, to avoid as matching, $T^\ell(0)$ has to avoid the sets

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Idea of Proof.

For each n, we cover X_α by $O(d^n)$ intervals of length β^{-n}. □

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- Therefore, for each n, the set A_α can be covered by $O(d^n)$ intervals of length $O(\beta^{-n})$.

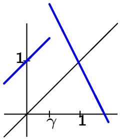
□

Monotonicity of entropy

If T has constant slope β, then the entropy

$$h_{top}(T) = h_\mu(T) = \log \beta.$$

For Q_γ (non-constant slope!)
we have the following result:



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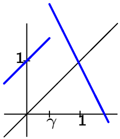


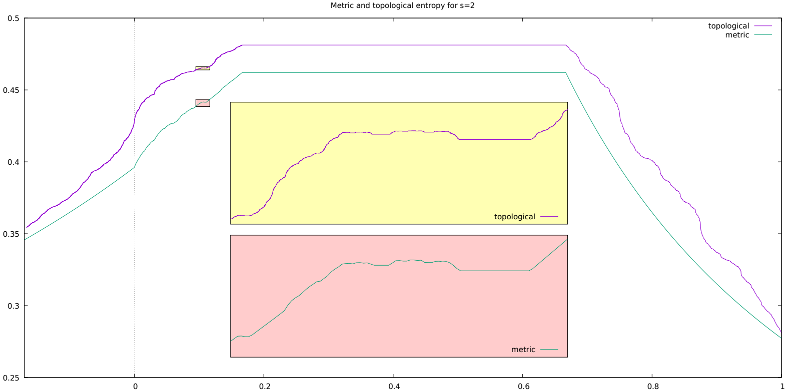
Theorem: Call $\Delta = \kappa_+ - \kappa_-$ the matching index of Q_γ.
Topological and metric entropy are

$$h_\mu(Q_\gamma) \text{ and } h_{top}(Q_\gamma) \text{ are } \begin{cases} \text{decreasing} & \text{if } \Delta < 0; \\ \text{constant} & \text{if } \Delta = 0; \\ \text{increasing} & \text{if } \Delta > 0, \end{cases}$$

as function of γ within matching intervals.

Monotonicity of entropy

The numerics for $h_{top}(Q_\gamma)$ and $h_\mu(Q_\gamma)$ suggest self-similarities in the graphs.



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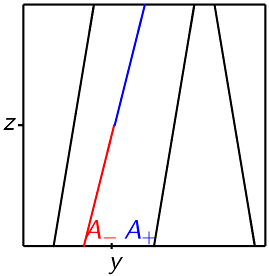

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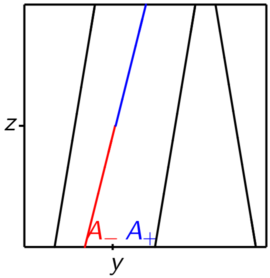

Corollary: R preserves Lebesgue measure m.

Monotonicity of entropy

Proof-sketch of the monotonicity theorem:

- $\int_J \tau \; dm$ increases by an amount proportional to $\eta := \Delta \times$ increased proportion of $|A_+|/|A_-|$.
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- but their period increases by Δ as A_+ absorbes them (when they previously belonged to A_-).
- Hence the topological entropy decreases accordingly.

Pseudo-centers

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- Define the interval $I_\xi := (\xi_L, \xi_R)$ containing ξ where,
- $\xi_L := .\overline{\check{v}v}, \quad \xi_R := .\overline{w}.$
- Also define the "degenerate" interval $I_1 := (2/3, +\infty)$.

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If $\xi = 1/2$ then $w = 10$, $v = 1$ and $\xi_L = .\overline{01}$, $\xi_R = .\overline{10}$.

(01) $w = u01 \Rightarrow \xi_L = .\overline{u001\check{u}110}$;

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ξ	ξ_R	ξ_L
$\frac{1}{2} = .10$	$\frac{2}{3} = .\overline{10}$	$\frac{1}{3} = .\overline{01}$
$\frac{1}{4} = .01$	$\frac{1}{3} = .\overline{01}$	$\frac{2}{9} = .\overline{001110}$
$\frac{7}{32} = .001110$	$\frac{2}{9} = .\overline{001110}$	$\frac{7}{33} = .\overline{0011011001}$
$\frac{3}{16} = .0011$	$\frac{1}{5} = .\overline{0011}$	$\frac{2}{11} = .\overline{0010111010}$
$\frac{9}{64} = .001001$	$\frac{1}{7} = .\overline{001}$	$\frac{4334}{16383} = .\overline{00100011101110}$
$\frac{1}{8} = .0010$	$\frac{2}{15} = .\overline{0010}$	$\frac{1}{9} = .\overline{000111}$

Pseudo-centers

Theorem:

- All matching intervals have the form I_ξ, where $\xi \in \mathbb{Q}_{\mathrm{dyd}}$ are precisely the pseudo-centers of the components of $[0, \frac{2}{3}] \setminus \mathcal{E}$.
- The matching index is

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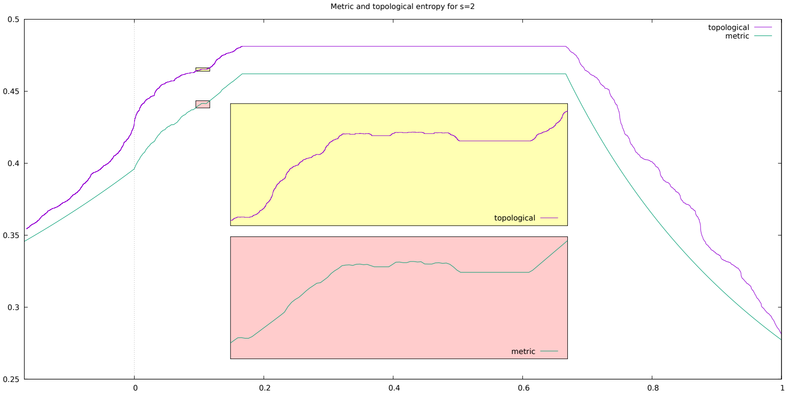
Proposition: If $g^k(\gamma) \geq \frac{1}{6}$, then $|w_0| = |w|_1$. In particular, all matching intervals in $(\frac{1}{6}, \frac{2}{3})$ have matching index $\Delta = 0$.

Pseudo-centers



Remark: This proposition explains constant entropy on all matching intervals in $(\frac{1}{6}, \frac{2}{3})$. A no devil's staircase argument would give:

$$h_\mu(Q_\gamma) = \log(\frac{1+\sqrt{5}}{2}) \quad \text{and } h_{top}(Q_\gamma) = \frac{2}{3}\log 2, quad(\alpha = 0)$$

for all $\gamma \in [\frac{1}{6}, \frac{2}{3}]$.

Pseudo-centers (period doubling)

Pseudo-center $\xi = .w$ (even expansion) and $1 - \xi = .v$ (odd exp).
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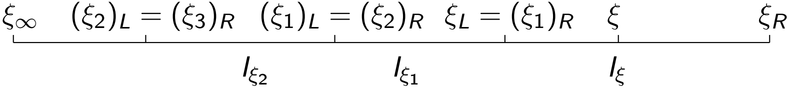

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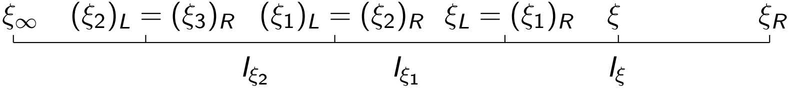

Lemma: The pseudo-center of the next period doubling can be obtained from the previous using the substitution:

$$\chi : \begin{array}{ll} w \mapsto \check{v}v & \check{w} \mapsto v\check{v} \\ v \mapsto vw & \check{v} \mapsto \check{v}\check{w} \end{array} .$$

Thus the limit ξ_∞ has s-adic expansion

$$\xi_\infty = .\check{v}\check{w}v\check{v}vw\check{v}\check{w}vw\check{v}v\check{v}\check{w}\dots$$

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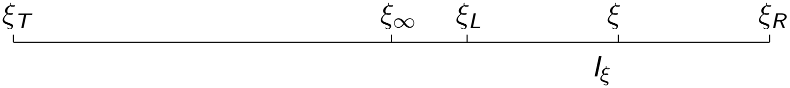

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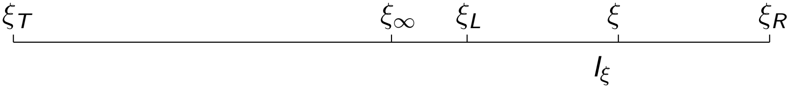

Theorem: Let $K(\xi_T) = \{x : g^k(x) \geq \xi_T \ \forall k\}$.
Then $x \in K(\xi_T) \cap T_\xi$ if and only if

$$x = .\sigma_1\sigma_2\sigma_3\sigma_4\ldots$$

for $\sigma_1 \in \{w, \check{v}\}$, $\sigma_j \in \{w, v, \check{w}, \check{v}\}$
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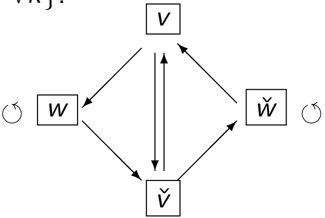

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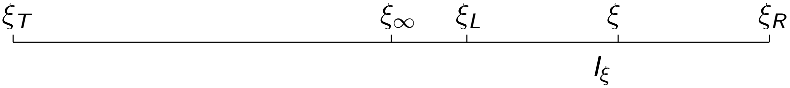

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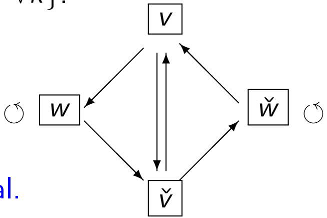
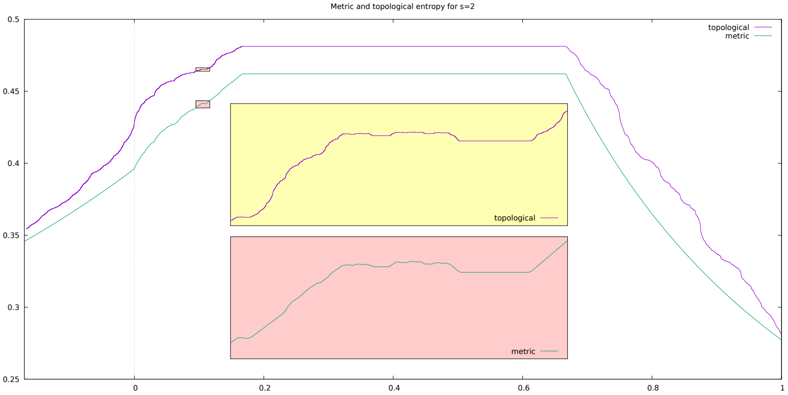
If $\Delta(\xi) = 0$, all matching in T_ξ is neutral.



Shape of the entropy function



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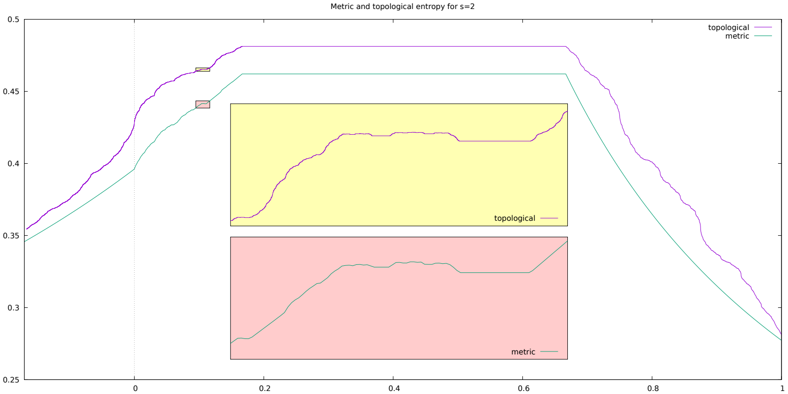

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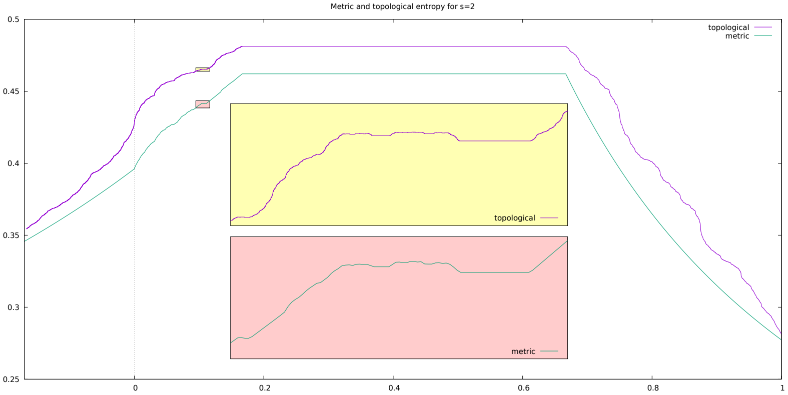

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Conjecture: The neutral tuning windows are exactly the plateaus of (topological and metric) entropy.

References

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