Self-similarity in the entropy graph for a family of piecewise linear maps.

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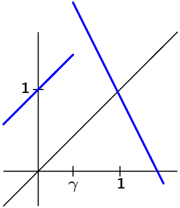
The family Q_γ

For fixed slope $s > 1$ and $\gamma \in [0, 1]$, take:

$$Q_\gamma(x) = \begin{cases} x + 1, & x \leq \gamma \\ 1 + s(1 - x), & x > \gamma \end{cases}$$



There is **matching** if there are $\kappa^\pm \geq 1$ such that

$$\begin{aligned} Q_\gamma^{\kappa^+}(\gamma^+) &= Q_\gamma^{\kappa^-}(\gamma^-), \\ (Q_\gamma^{\kappa^+})'(\gamma^+) &= (Q_\gamma^{\kappa^-})'(\gamma^-). \end{aligned}$$

The number $\Delta = \kappa^+ - \kappa^-$ is the **matching index**.

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  (In this case $Q_\gamma^n(\gamma^+) = Q_\gamma^n(\gamma^-) = 1$ is fixed for all n large enough.)
- The non-matching set $\mathcal{E}$ has Hausdorff dimension

  $$\dim_H(\mathcal{E}) = 1,$$

  but $\dim_H(\mathcal{E} \setminus [0, \delta)) < 1$ for every $\delta > 0$.

The family Q_γ

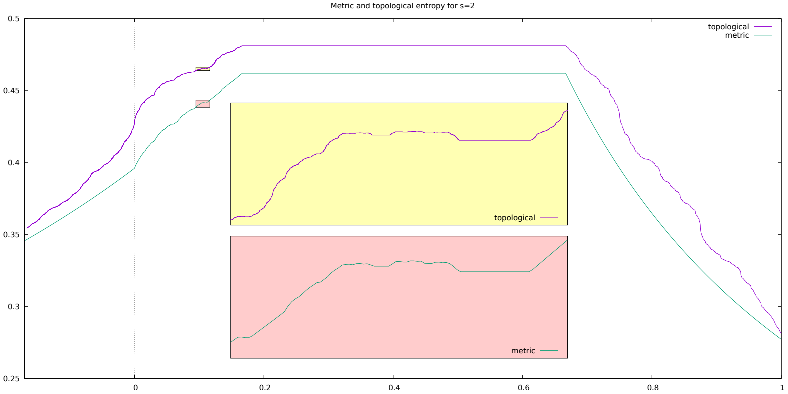


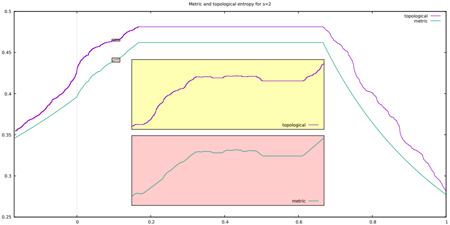
Figure: Topological & metric entropies of Q_γ for $s = 2$ as functions of γ.

The family Q_γ

Theorem: Topological and metric entropy

$$h_\mu(Q_\gamma) \text{ and } h_{top}(Q_\gamma) \text{ are } \begin{cases} \text{decreasing} & \text{if } \Delta < 0; \\ \text{constant} & \text{if } \Delta = 0; \\ \text{increasing} & \text{if } \Delta > 0, \end{cases}$$

as function of γ within matching intervals. (Recall $\Delta = \kappa^+ - \kappa^-$.)



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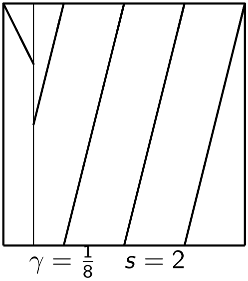
  - pseudo-centers
  - period doubling cascades
  - computing the matching index
- Tuning windows and explanation of the self-similarity of the entropy graphs.

Why there is matching?

Let $g(x) := s(1 - x) \bmod 1$ and $R : (0, 1) \to (0, 1)$ be the first return of Q_γ to $[0, 1)$.

Lemma:

$$R(x) = \begin{cases} g(x) & \text{if } x \in (0, \gamma) \\ g^2(x) & \text{if } x \in (\gamma, 1) \end{cases}$$



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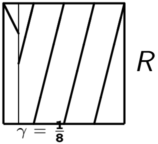

Lemma: For fixed $\gamma \in [0, 1]$, the following conditions are equivalent:

(i) $g^k(\gamma) < \gamma$ for some $k \in \mathbb{N}$;

(ii) matching holds for γ.

In other words, the bifurcation set is

$$\mathcal{E} = \{\gamma \in [0, 1] \ : \ g^k(\gamma) \geq \gamma \ \forall k \in \mathbb{N}\}.$$

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Definition The pseudo-center of an interval $J \subset (0, 1)$ is the (unique) dyadic rational $\xi \in \mathbb{Q}_{\text{dyd}}$ with minimal denominator.

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Theorem For $\xi \in \mathbb{Q}_{\text{dyd}} \setminus \{1\}$, let w be the shortest even binary expansion of ξ and v be the shortest odd binary expansion of $1 - \xi$.

The matching interval containing ξ is

$$I_\xi := (\xi_L, \xi_R)$$

where $\xi_L := .\overline{\check{v}v}$, $\quad \xi_R := .\overline{w}$ (where $\check{v}$ is bitwise negation of v).

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If $\xi = 1/2$ then $w = 10$, $v = 1$ and $\xi_L = .\overline{01}$, $\xi_R = .\overline{10}$.

(01) $w = u01 \Rightarrow \xi_L = .\overline{u001\check{u}110}$;

(11) $w = u11 \Rightarrow \xi_L = .\overline{u101\check{u}010}$;

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ξ	ξ_R	ξ_L
$\frac{1}{2} = .10$	$\frac{2}{3} = .\overline{10}$	$\frac{1}{3} = .\overline{01}$
$\frac{1}{4} = .01$	$\frac{1}{3} = .\overline{01}$	$\frac{2}{9} = .\overline{001110}$
$\frac{7}{32} = .001110$	$\frac{2}{9} = .\overline{001110}$	$\frac{7}{33} = .\overline{0011011001}$
$\frac{3}{16} = .0011$	$\frac{1}{5} = .\overline{0011}$	$\frac{2}{11} = .\overline{0010111010}$
$\frac{9}{64} = .001001$	$\frac{1}{7} = .\overline{001}$	$\frac{4334}{16383} = .\overline{00100011101110}$
$\frac{1}{8} = .0010$	$\frac{2}{15} = .\overline{0010}$	$\frac{1}{9} = .\overline{000111}$

Pseudo-centers (matching index)

Theorem: The matching index is

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Proposition: If $\gamma \geq \frac{1}{6}$, then $|w_0| = |w|_1$. In particular, all matching intervals in $(\frac{1}{6}, \frac{2}{3})$ have matching index $\Delta = 0$.

Pseudo-centers (period doubling)

Pseudo-center $\xi = .w$ (even exp.) and $1 - \xi = .v$ (odd exp.).
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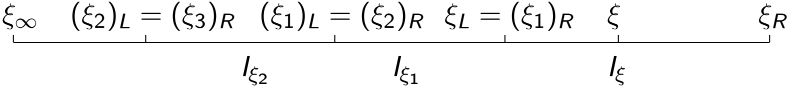
But ξ_L is also the right end-point of I_{ξ_1} for $\xi_1 = .\check{v}v$. We call this "period doubling". It repeats countably often, converging to ξ_∞.



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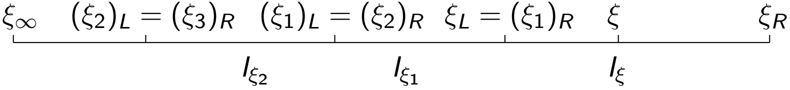

Lemma: The pseudo-center of the next period doubling can be obtained from the previous using the substitution:

$$\chi : \begin{array}{ll} w \mapsto \check{v}v & \check{w} \mapsto v\check{v} \\ v \mapsto vw & \check{v} \mapsto \check{v}\check{w} \end{array}.$$

Thus the limit ξ_∞ has s-adic expansion

$$\xi_\infty = .\check{v}\check{w}v\check{v}vw\check{v}\check{w}vw\check{v}v\check{v}\check{w}\dots$$

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The matching interval is $I_\xi = [\xi_L, \xi_R] = [.\overline{\check{v}v}\ ,\ .\overline{w}]$.
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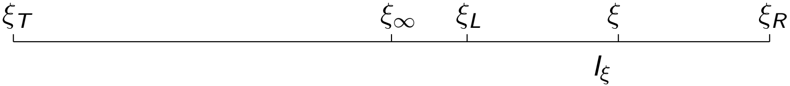

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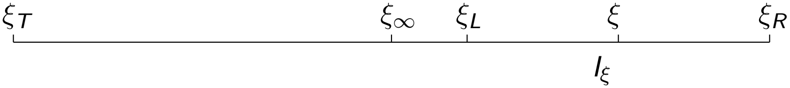
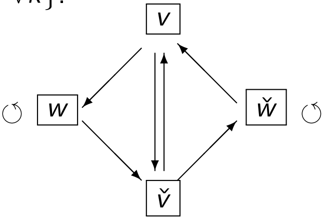


Theorem: Let $K(\xi_T) = \{x : g^k(x) \geq \xi_T \ \forall k\}$.
Then $x \in K(\xi_T) \cap T_\xi$ if and only if

$$x = .\sigma_1\sigma_2\sigma_3\sigma_4 \ldots$$

for $\sigma_1 \in \{w, \check{v}\}$, $\sigma_j \in \{w, v, \check{w}, \check{v}\}$
describing a path in the diagram.



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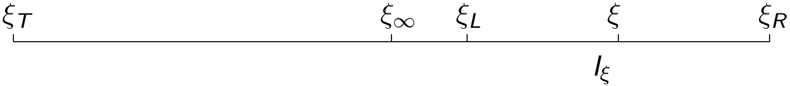

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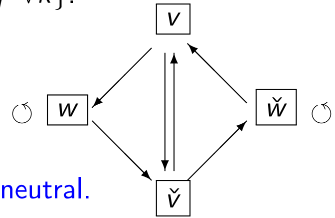
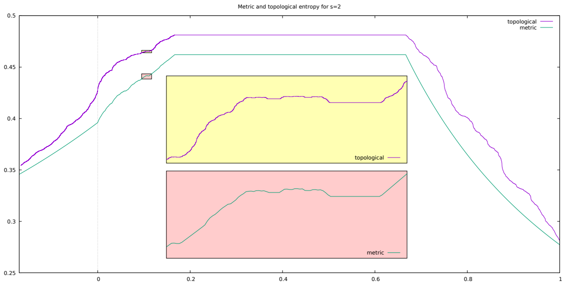
describing a path in the diagram.
If $\Delta(\xi) = 0$, then all matching in T_ξ is neutral.



Pseudo-centers (tuning window)

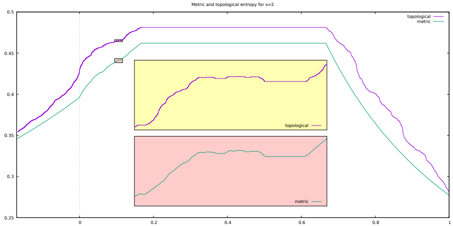


Remark: This theorem explains constant entropy on all matching intervals in $M = (\frac{1}{6}, \frac{2}{3})$. A no devil's staircase argument gives:

$$h_\mu(Q_\gamma) = \log(\frac{1+\sqrt{5}}{2}) \quad \text{and} \quad h_{top}(Q_\gamma) = \frac{2}{3}\log 2,$$

for all $\gamma \in [\frac{1}{6}, \frac{2}{3}]$.

Shape of the entropy function



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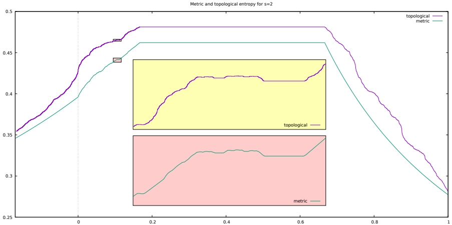

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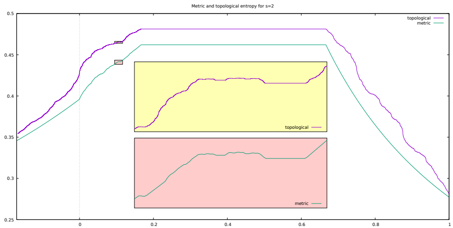

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Corollary: The shape of the entire entropy function (i.e., pattern of increase/decrease) is repeated in every tuning window T_ξ with $\Delta(\xi) > 0$, and reversed in every tuning window T_ξ with $\Delta(\xi) < 0$.