Monotonicity of topologocial entropy in polynomial families of maps.

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(Joint work with Sebastian van Strien and Weixiao Shen)

Topological entropy was introduced by Adler, Konheim and McAndrew (1965). For interval maps, there are easier definitions (Misurewicz, Szlenk).

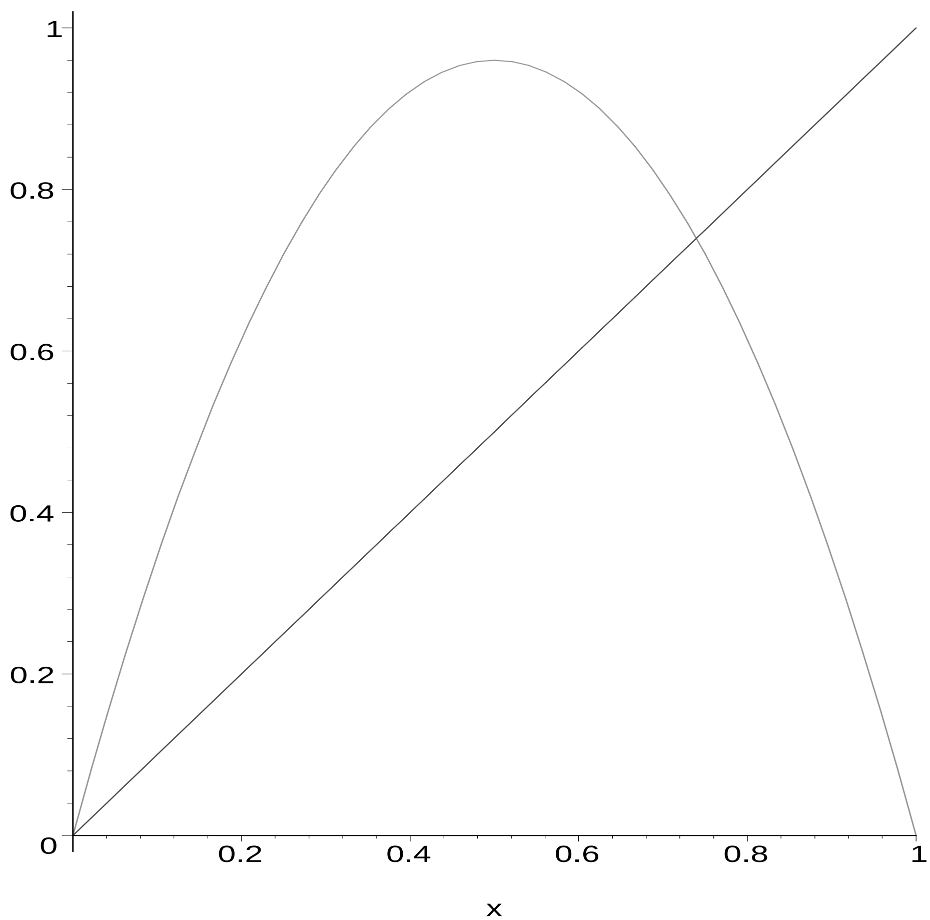
$$l(f^n) = \#\{ \text{ laps of } f^n\}.$$

$$\text{Per}(f^n) = \#\{ \text{ periodic points of } f^n\}.$$

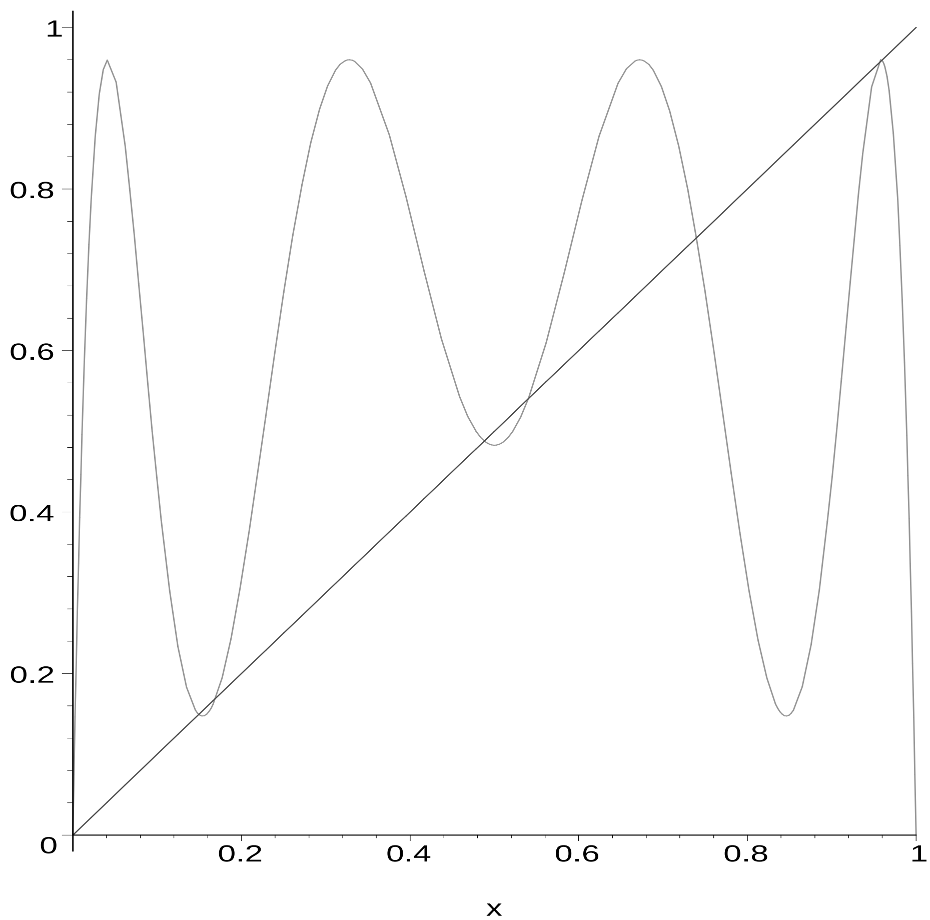
$$\text{Var}(f^n) = \#\{ \text{ variation of } f^n\}.$$

The entropy is the exponential growth rate of all of htese:

$$\begin{aligned} h_{top}(f) &= \lim_{n\to\infty} \frac{1}{n} \log l(f^n) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \text{Per}(f^n) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \text{Var}(f^n) \end{aligned}$$



A quadratic map; $f(x) = ax(1 - x)$, $a = 3.84$.



Renormalization of a quadratic map: $f(x) = ax(1-x)$, $a = 3.84$.

Theorem (Milnor & Thurston 1988)

For the quadratic family $f_c(z) = z^2 + c$,

$$c \mapsto h_{top}(f_c)$$

is monotone decreasing on $[-2, 0]$.

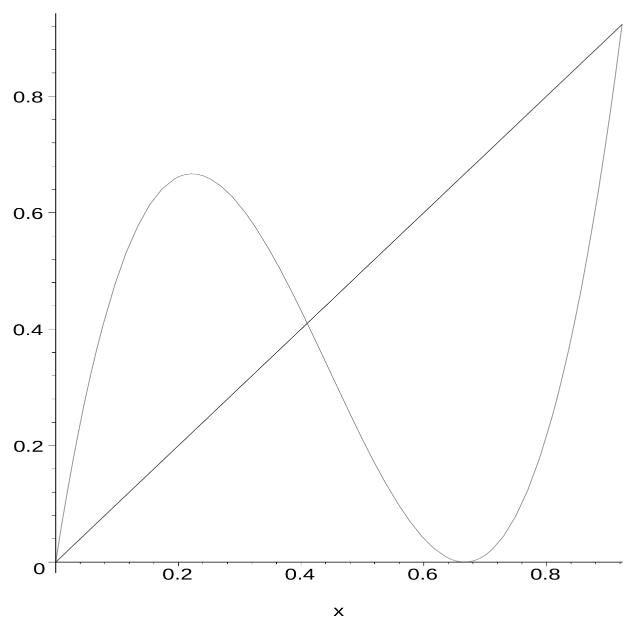
Later proofs due to Douady & Hubbard, and Tsujii.

Theorem (Milnor & Tresser 2000)

For the cubic family $f_{a,b}(x) = ax^3 + bx^2 + (1 - a - b)x$,

$$(a, b) \mapsto h_{top}(f_{a,b})$$

is monotone in the sense that level sets of constant entropy are connected.



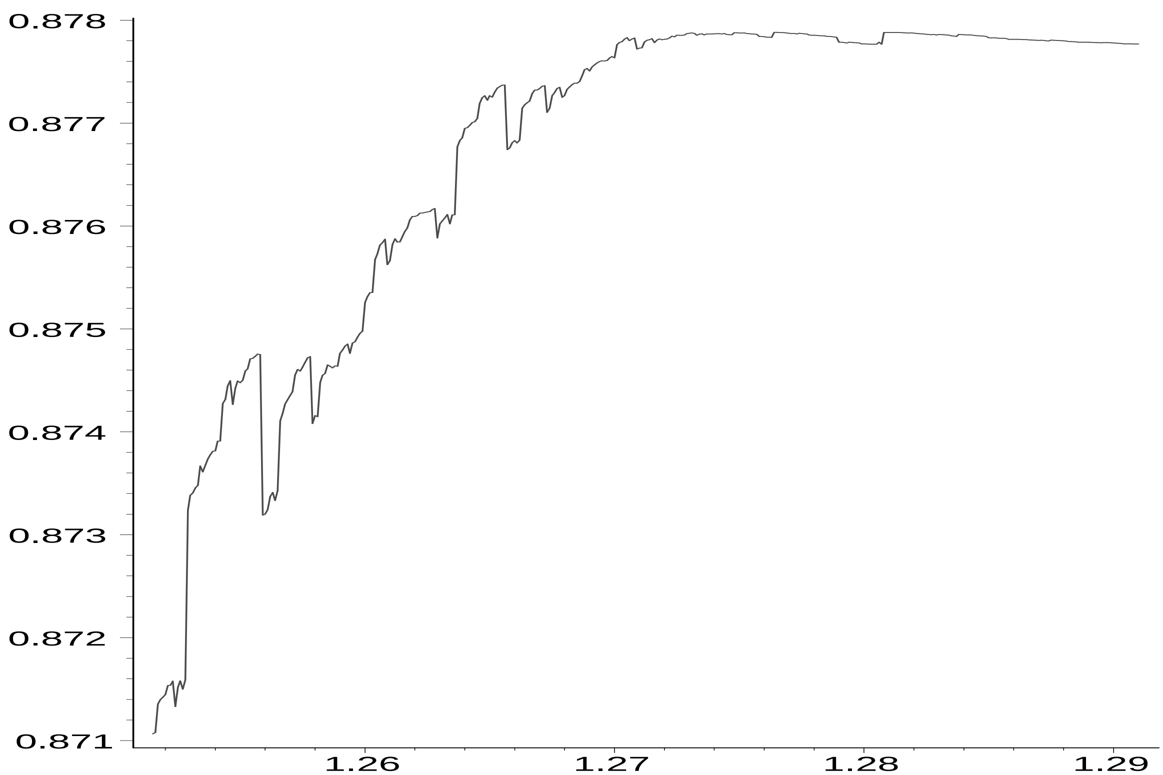
A cubic map $f_{a,b}(x) = ax^3 + bx^2 + (1 - a - b)x$.

Let P^d be the set of d-modal polynomials fixing $\{0, 1\}$ and such that all critical points are real and distinct.

Theorem (Bruin, Shen & van Strien 2005)

For any d, topological entropy is monotone in P^d in the sense that level sets of constant entropy are connected.

The polynomial family P^d can be parametrized by their d critical values $f(c_i)$. However topological entropy need not be monotone with respect any of these parameters separately.



Non-monotonicity of entropy for the map $f_b(x) = 2ax^2 - 3ax^3 + b$ with $a = b + 0.515$.

Ingredients of the proof:

- Multimodal Rigidity Theorem.

- Partially hyperbolic deformation spaces are topological cells.

- Parametrize P^d by stunted seesaw maps.

- Monotonicity of entropy for stunted see-saw maps.

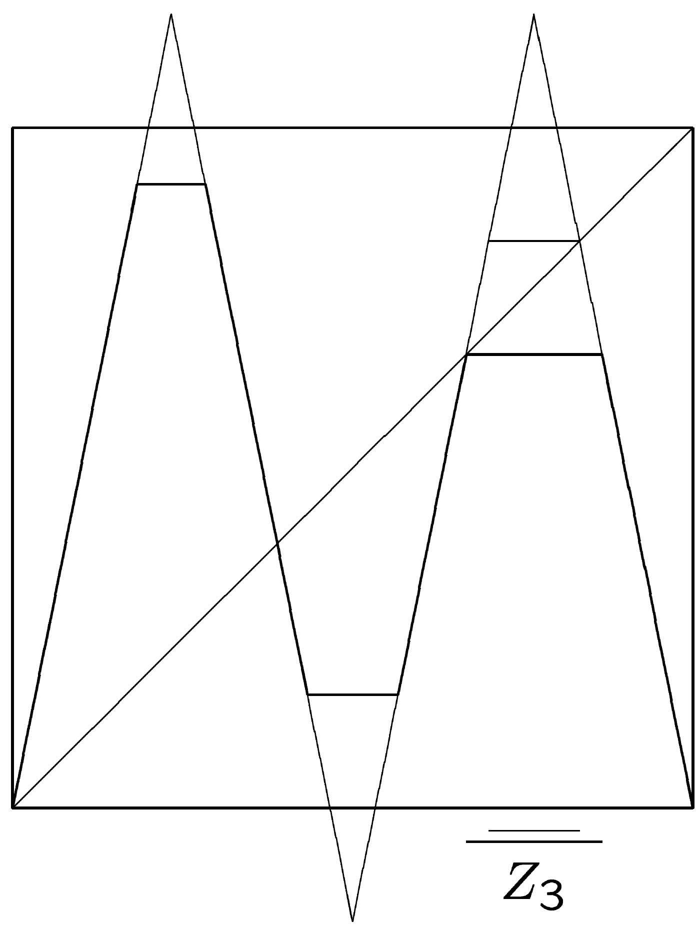
- Investigate the semiconjugacies between maps in P^d and stunted seesaw maps, and their behaviour under bifurcation.

Definition Two maps are **partially conjugate** if they are conjugate away from the basins of periodic attractors.

Rigidity Theorem (Kozlovski, Shen and van Strien 2004) Let f and g be d-modal polynomials with d real and distinct critical points. Assume that f and g are partially conjugate and that each critical point which belongs to the basin of a periodic attractor, is periodic or eventually periodic. Then $f = g$ up to an affine rescaling.

Rigidity for the quadratic family was proved by Lyubich and by Graczyk & Świątek in the mid 1990s.

Stunted Seesaw Maps to parametrize d-modal polynomial families.



The seesaw map S and two stunted seesaw maps (with different third plateaus Z_3)

Theorem 1 *Within the family of d-modal stunted seesaw maps, topological entropy dependents monotonotically on each parameter.*

Proof. Immediate. If any (one) plateau is raised, all the "old" orbits remain unchanged and potentially new orbits are created. □

Define the **kneading invariants** of f as

$$\nu_i = \lim_{x \downarrow c_i} \underline{\mathrm{i}}_f(x)$$

where $\underline{\mathrm{i}}_f(x)$ is the symbolic itinerary with respect to the components $[-1, 1] \setminus \mathrm{Crit}$.

For each ν_i there is a unique point y_i such that

$$\underline{\mathrm{i}}_S(y_i) = \nu_i.$$

Let $\Psi(f)$ be that stunted seesaw map such that y_i is the right boundary point of Z_i.

The map

$$f \mapsto \Psi(f)$$

is discontinuous and neither injective nor surjective.

The crux of the proof is to show that *partially hyperbolic cells* $\mathcal{PH}(f)$ of P^d map appropriately to cells $[\Psi(f)]$ in the class of stunted seesaw maps.

Definition: $f \in \mathcal{A}^*$ if

1. if $f^s : B_i \to B_j$ for some components of the basin of f, then $f^s(B_i \cap \text{Crit})$ lies entirily to the left/right of the left/right-most critical point in B_j.

2. all periodic points are hyperbolical repelling or parabolic.

Lemma: Ψ is 'almost surjective': for each $T \in \mathcal{S}^d_*$ there exists a polynomial $f \in P^d \cap \mathcal{A}^*$ such that $T \in [\Psi(f)]$.

Lemma: The map $\Psi \colon P^d \to \mathcal{S}^d_*$ is 'almost injective': If

$$f_1, f_2 \in \mathcal{A}^* \text{ and } [\Psi(f_1)] \cap [\Psi(f_2)] \neq \emptyset,$$

then

$$\overline{\mathcal{P}H(f_1)} \cap \overline{\mathcal{P}H(f_2)} \neq \emptyset.$$