Monotonicity of entropy for polynomial families

Henk Bruin

University of Surrey

Joint work with
Sebastian van Strien

For interval maps **topological entropy** equals

$$\begin{aligned} h_{top}(f) &= \lim_{n\to\infty} \frac{1}{n} \log \#\{\text{ laps of } f^n\} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \#\{\text{ fixed points of } f^n\}. \end{aligned}$$

For the logistic family $f_a(x) = ax(1-x)$,

$$a \mapsto h_{top}(f_a) \text{ is monotone}$$

Proofs by Sullivan & Thurston (1986),
Milnor & Thurston (1988),
Douady (1995), Tsujii (1993-94).

Monotonicity for other families, e.g. (Tsujii '94)

$$f_a(x) = 1 - a|x|^{2d}, \quad a \in [0,2],\ d \in \mathbb{N},$$

and (Rempe & van Strien, 2010)

$$f_a(x) = a\sin(\pi x), \quad a \in [0,1].$$

All proofs use complex analysis

No completely real proof known!

Multimodal polynomial families:

$$P^d = \left\{ f : [-1,1] \to [-1,1] : \begin{array}{l} \deg(f) = d+1, \\ f(-1) = -1, \\ f(1) \in \{-1,1\} \\ d \text{ distinct critical} \\ \ \text{points in } (-1,1) \end{array} \right\}$$

has d-dimensional parameter space A^d,
e.g. P^d can be parametrized by its d critical values.

Monotonicity of topological entropy means:

Every **isentrope**

$$\{a \in A^d : h_{top}(f_a) = s\}$$

is **connected**

Theorem 1 (Milnor & Tresser, 2000). *In the family of bimodal polynomials, topological entropy is monotone.*

Theorem 2 (Radulescu, 2008). *In the family of trimodal polynomials emerging from the composition of two logistic maps (i.e., $g = f_a \circ f_b$), topological entropy is monotone.*

Theorem 3 (Main Theorem). *Topological entropy is monotone on P^d for every $d \in \mathbb{N}$.*

Conjecture: The isentropes

$$\{f \in P^d_\epsilon \quad ; \quad h_{top}(f) = s\}$$

are contractible.

Question: For $d \geq 3$, are the complements of isentropes are simply connected. (No Alexander horned spheres!?)

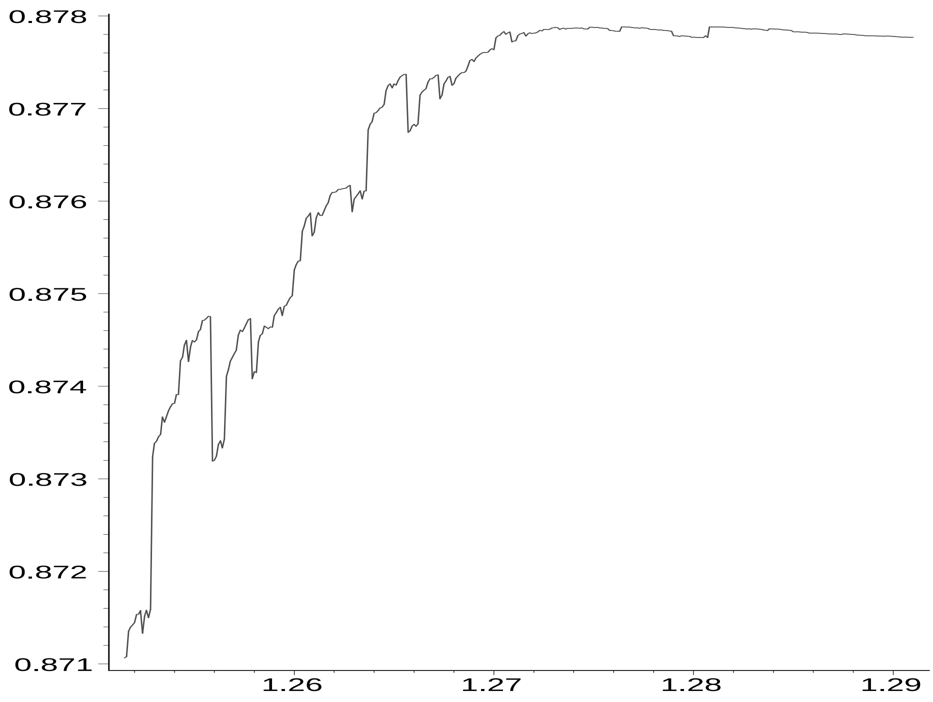


Figure 1: The family $f_b(x) = 2ax^2 - 3ax^3 + b$ with critical values $f(0) = b$ and $f(1) = b - a \equiv -0.515$. The map $b \mapsto h_{top}(f_b)$ is **not** monotone.

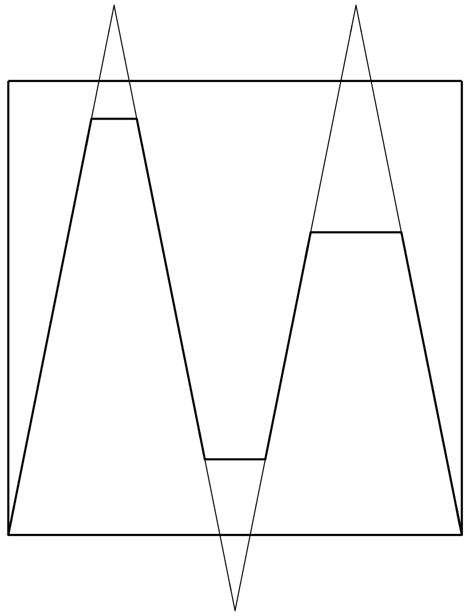
Conjecture: If $d \geq 2$ and all but one of the critical points of $f \in \overline{P}_d$ are attracted to a periodic attractor, then $v \mapsto h_{top}(f_v)$ is non-monotone for each critical value v.

Ingredients of the Monotonicity Proof:

- Multimodal Rigidity Theorem.
- The loci of *partial conjugacy* in P^d of semi-conjugacy are topological cells.
  (Uses complex dynamics!)
- Parametrize P^d by stunted seesaw maps S^d.
- Monotonicity of entropy for stunted seesaw maps (the easy bit).
- Investigate the semiconjugacies between maps in P^d and stunted seesaw maps, and their behaviour under bifurcation. (messy bit).
- Sets of constant entropy in the set

  $$S^d_* = \{T \in S^d : T \text{ has no wandering pair.}\}$$

  of 'good' seesaw maps are contractible in S^d_*.
  (extremely messy bit).



Stunted seesaw maps:

Fix degree $d \in \mathbb{N}$ and $\lambda = d + 2$. A map $T \in S^d$ if

- $T : [-e, e] \to [-e, e]$, $e = \frac{d\lambda}{d+1}$, is continuous;
- $T(-e) = -e$, $T(e) = (-1)^{d+1}e$;
- It has plateaus Z_i, $i = 1, \ldots, d$ around equally spaced critical points c_i;
- It has slope $\pm\lambda$ in between the plateaus.

The map $T(Z_i) \mapsto h_{top}(T)$ is clearly monotone for each i. It follows that entropy is monotone on S^d.

Fact 1: Every multimodal map is entropy-preservingly semiconjugate to some stunted seesaw map.

Proof: Define the **kneading invariants** of $f \in P^d$ as

$$\nu_i = \lim_{x \downarrow c_i} \underline{\mathrm{i}}_f(x)$$

where $\underline{\mathrm{i}}_f(x)$ is the symbolic itinerary with respect to the components $[-1, 1] \setminus \mathrm{Crit}$.

The unstunted map exhibits **all** itineraries, so for each ν_i there is a unique point y_i such that

$$\lim_{x \downarrow y_i} \underline{\mathrm{i}}_S(x) = \nu_i.$$

Let

$$T = \Psi(f)$$

be that stunted seesaw map such that y_i is the right boundary point of Z_i.

Then f and T have the same kneading invariant, and hence the same entropy.

Fact 2: Isentropes in the family of stunted seesaw maps are connected and even contractible.

Proof: Parametrize stunted seesaw maps by $\zeta = (\zeta_1, \ldots, \zeta_d)$ where

$$\zeta_i = \begin{cases} T(Z_{i,T}) & \text{if } S(c_i) \text{ assumes a maximum;} \\ -T(Z_{i,T}) & \text{if } S(c_i) \text{ assumes a minimum.} \end{cases}$$

Then

$$\zeta \mapsto h_{top}(T_\zeta)$$

is increasing in each ζ_i separately.

Fact 2 follows directly.

The map

$$\Psi : P^d \to S^d$$

is neither continuous, nor injective, nor surjective.

Example: Consider the logistic map

$$f_\lambda : x \mapsto \lambda x(1-x).$$

Then

- At $\lambda = \lambda_1 = 2$, c is fixed for f_λ.

  For $0 \leq \lambda \leq 2$, $\nu = 1\overline{0}$, and $\Psi(f_\lambda) = T_0$.

- At $\lambda = \lambda_2$, c has period 2 for f_λ.

  For $\lambda_1 < \lambda \leq \lambda_2$, $\nu = 11\overline{01}$, and $\Psi(f_\lambda) = T_{p_1}$.

- At $\lambda = \lambda_2$, c has period 4 for f_λ.

  For $\lambda_2 < \lambda \leq \lambda_4$, $\nu = 1101\overline{0101}$, and $\Psi(f_\lambda) = T_{p_4}$.

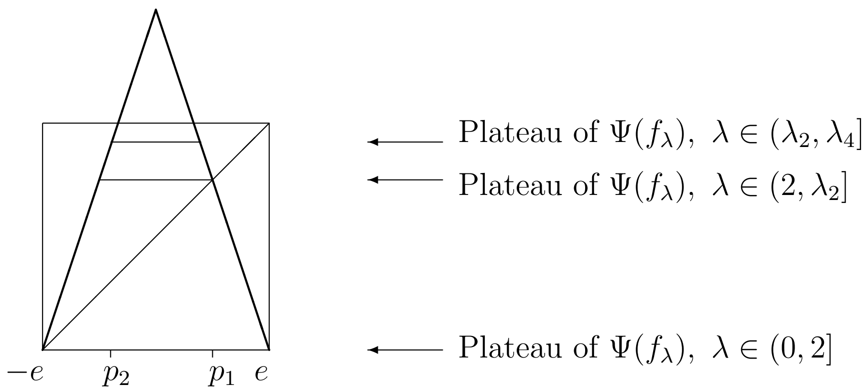


Figure 2: Unimodal sawtooth and stunted sawtooth maps

To solve monotonicity we need to understand the map

$$\Psi : P^d \to S^d \qquad f \mapsto \Psi(f).$$

The crux is that Ψ is not easy to define on cells of *partial conjugacy*; at these, we take a set valued approach: $f \mapsto [\Psi(f)]$.

Proposition 1. *The map $\Psi : P^d \to S^d$ is*

- *almost surjective: $f \in P^d$ has no wandering intervals, so Ψ needs to map into*

  $$S^d_* = \{T \in S^d : T \text{ has no wandering pair.}\}$$

- *almost continuous: yet another issue with $S \setminus S^d_*$;*

- *almost injective: f_1, f_2 not "partially conjugate", then $[\Psi(f_1)] \cap [\Psi(f_2)] = \emptyset$.*

Wandering pairs:

- Two plateaus (Z_i, Z_j) are a *wandering pair* if there exists $n \geq 0$ such that $T^n(\mathcal{J})$ is a point (for $\mathcal{J} := [Z_i, Z_j]$ the convex hull of Z_i and Z_j.)
- If this point is not eventually periodic, then

  $$T \neq \Psi(f)$$

  for every polynomial (or in fact C^2 map).
  The reason is that C^2 multimodal maps have no wandering intervals.
- Let $\mathcal{S}^d_*$ denote the set $T \in \mathcal{S}^d$ for which every wandering pair (Z_i, Z_j), the convex hull $\mathcal{J}$ is eventually mapped into a periodic plateau.

Theorem 4. *The isentropes*

$$\{T \in \mathcal{S}_*^d : h_{top}(T) = s\}$$

are connected and contractible.

Under Ψ^{-1}, we can pull-back the connectedness of isentropes, and this gives the main theorem. It is not clear if contractibility is preserved under Ψ^{-1}, see our Conjecture.

To give an idea of the messiness of Theorem 4, here is the retract:

$$R_t = \begin{cases} \beta_{6t} & \text{for } t \in [0, \frac{1}{6}], \\ \Gamma_{6t-1} \circ \beta_1 & \text{for } t \in [\frac{1}{6}, \frac{2}{6}], \\ \Gamma_1 \circ \hat{\gamma}_{6t-2} \circ \beta_1 & \text{for } t \in [\frac{2}{6}, \frac{3}{6}], \\ \Gamma_1 \circ \hat{\gamma}_1 \circ \beta_1 \circ \delta_{6t-3} & \text{for } t \in [\frac{3}{6}, \frac{4}{6}], \\ \Gamma_1 \circ \hat{\gamma}_1 \circ \beta_1 \circ \Delta_{6t-4} \circ \delta_1 & \text{for } t \in [\frac{4}{6}, \frac{5}{6}], \\ \Gamma_1 \circ \hat{\gamma}_1 \circ \beta_1 \circ r_{6t-5} \circ \Delta_1 \circ \delta_1 & \text{for } t \in [\frac{5}{6}, 1]. \end{cases}$$