Multiple equilibrium states for interval maps

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Setting:

Let $\mathcal{M}$ be the set of invariant measures of a (smooth) unimodal map $f : I \to I$. Let c be the critical point and assume $D^{\ell-1}f(c) \neq 0$ for some $\ell > 1$.

Definition: Given a potential function

$$\varphi : X \to \mathbf{R}$$

an **equilibrium state** is a measure $\mu_\varphi \in \mathcal{M}$ that achieves the **pressure**:

$$P(\varphi) := \sup_{\mu \in \mathcal{M}} h_\mu + \int_X \varphi d\mu = h_{\mu_\varphi} + \int_X \varphi d\mu_\varphi.$$

We are interested in $\varphi := \varphi_t = -t \log |Df|$.

Ledrappier's result:

It is known that for $t = 1$, i.e., $\varphi = -\log |Df|$, the pressure $P(\varphi) = 0$.

Ledrappier (1981) proved:

If the Lyapunov exponent

$$\lambda(\mu) := \int |Df| d\mu > 0,$$

then

μ **is equilibrium state iff** $\mu \ll$ **Leb.**

Furthermore, there is only one such measure.

Current work with Mike Todd for $t \approx 1$:

If we restrict to

$$\mathcal{M}_+ = \{\mu \in \mathcal{M} : \lambda(\mu) > 0\}$$

and $Df(c_1)$ increases at least polynomially:

$$|Df^n(c_1)| \geq Cn^\alpha$$

for $c_1 = f(c)$ and $\alpha > \ell(1 + \frac{1}{t_0}) - 1$ for some $t_0 \in (0, 1)$, then

- there exists exactly one equilibrium measure for φ_t and $t \in (t_0, 1]$ sufficiently close to 1

- If $t < 1$, then this measure has exponential decay of correlations and satisfies CLT.

Equilibrium states with $\lambda(\mu) = 0$?

Because $h_\mu \leq \lambda(\mu)$ we have

- For $t = 1$, any measure with $\lambda(\mu) = 0$ is equilibrium state.

- If there is an $\mathcal{M}_+ \ni \mu \ll$ Leb, then

  $$P(\varphi_t) \geq (1-t)\lambda(\mu) > 0 \text{ for } t \in (0,1),$$

  and no measure with $\lambda(\mu) = 0$ is equilibrium state.

Remark: If f satisfies the Collet-Eckmann condition:

$$|Df^n(c_1)| \geq Ce^{\alpha n} \text{ for some } C, \alpha > 0,$$

then $\mathcal{M} = \mathcal{M}_+$.

Cutting times and Kneading map

The n-th iterate of f has **central branch**

$$f^n : J \to f^n(J),$$

where J is a maximal interval adjacent to c on which f^n is monotone.

The number n is a **cutting time** if $c \in f^n(J)$.

Cutting times are denoted as

$$1 = S_0 < S_1 < S_2 < \dots$$

There is a map (called **kneading map**)

$$Q : \mathbf{N} \to \mathbf{N} \cup \{0\}$$

such that

$$S_k - S_{k-1} = S_{Q(k)}.$$

Not every map Q serves as kneading map, but if Q is non-decreasing, then Q is kneading map of some unimodal map.

Examples:

The **Feigenbaum-Coullet-Tresser** map:

$$Q(k) = k - 1, \; S_k = 2^k.$$

The **Fibonacci** map:

$$Q(k) = \max\{0, k - 2\}.$$

The S_k are the Fibonacci numbers.

Basic asumption: $Q(k) \to \infty$

Under this assumption, we have the properties:

- $\omega(c)$ is a minimal Cantor set.

- f is not Collet-Eckmann.

- $\lambda(\mu) = 0$ iff $\text{supp}(\mu) = \omega(c)$.

- $h_{top}(f|\omega(c)) = 0$.

Main Theorem (Bruin J. Difference Eq. Appl. **9** (2003) 305–318.

Let Q is be non-decreasing and $Q(k) \to \infty$.

(a) If $k - Q(k) \leq C \cdot \sqrt{k}$, then $(\omega(c), f)$ is uniquely ergodic.

(b) For any $\gamma > 2$, there is a kneading map Q with

$$k - Q(k) \asymp k^{\frac{\gamma}{\gamma+1}}$$

such that $(\omega(c), f)$ has two ergodic invariant measures.

Remarks:

- Part (a) compares to a result of Barat, Downarowicz and Liardet (2002) which claims unique ergodicity if
$$\sup_k S_k \sum_{m \geq k} \frac{1}{S_m} < \infty.$$
For non-decreasing kneading maps, our result is stronger, but BDL applies to more general Cantor systems.

- The example of (b) can be extended to any number of invariant measures.