The one-sided Bernoulli property for one-dimensional dynamical systems

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Bernoulli Shifts.

For alphabet $\mathcal{A} = \{1, \ldots, n\}$, $n \geq 2$, let $(\Omega, \mathcal{D}, \rho; \sigma)$ is the two-sided (respectively one-sided) Bernoulli shift. Here

- $\Omega = \mathcal{A}^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{N}}$ is the two-sided and one-sided sequence space, with left-shift σ;
- $p = \{p_1, \ldots, p_n\}$, $p_k > 0$ is a probability vector;
- $\mathcal{D}$ is the σ-algebra generated by cylinder sets;
- ρ is the product measure determined by p.

Measure-Theoretic Isomorphism.

An *isomorphism* ψ between $(X_1, \mathcal{B}_1, \mu_1; T_1)$ and $(X_2, \mathcal{B}_2, \mu_2; T_2)$ is a measurable a.e.-bijection such that

$$\begin{array}{ccc} (X_1, \mathcal{B}_1, \mu_1) & \xrightarrow{T_1} & (X_1, \mathcal{B}_1, \mu_2) \\ \downarrow \psi & & \downarrow \psi \\ (X_2, \mathcal{B}_2, \nu) & \xrightarrow{T_2} & (X_2, \mathcal{B}_2, \mu_2) \end{array}$$

commutes.

More precisely

- There are $Y_1 \subset X_1$, $Y_2 \subset X_2$ of full measure such that $\psi : Y_1 \to Y_2$ is a bijection.

- $T_2 \circ \psi = \psi \circ T_1$ for all $x \in Y_1$.

- $\psi^{-1} B \in \mathcal{B}_1$ and $\mu_1(\psi^{-1} B) = \mu_2(B)$ for all $B \in \mathcal{B}_2$.

The Two-sided Bernoulli Property.

Definition 1. An invertible measure preserving transformation $(X, \mathcal{B}, \mu; T)$ is (two-sided) Bernoulli if it is isomorphic to a two-sided Bernoulli shift.

For two-sided Bernoulli shifts, and hence, invertible measure preserving transformations, entropy is a complete invariant. Proofs by Ornstein (1973) and later simplified by Keane & Smorodinsky.

Definition 2. An (non-invertible) measure preserving transformation $(X, \mathcal{B}, \mu; T)$ is **one-sided** Bernoulli if it is isomorphic to a one-sided Bernoulli shift.

For one-sided Bernoulli shifts, entropy is an invariant, but not a complete invariant.

Noninvertible Bernoulli Properties.

Let $(X, \mathcal{B}, \mu; T)$ be a non-invertible measure preserving transformation. There are several ways of relating it to Bernoulli shifts.

(a) The natural extension is Bernoulli.

(b) $(X, \mathcal{B}, \mu; T)$ is weakly Bernoulli (def. later).

(c) $(X, \mathcal{B}, \mu; T)$ is one-sided Bernoulli.

The implications are as follows:

$$(c) \Rightarrow (b) \Rightarrow (a)$$

but the reverse implications are both false.

Weakly Bernoulli.

Definition 3. Let $(X, \mathcal{B}, \mu; T)$ be a measure preserving endomorphism. Let $\zeta = \{P_1, P_2, \cdots\}$ and $\eta = \{Q_1, Q_2, \cdots\}$ be partitions. The partition ζ *is independent of* η if

$$\sum_{i,j} |\mu(P_i \cap Q_j) - \mu(P_i)\mu(Q_j)| = 0$$

and $\varepsilon-$*independent of* ζ if

$$\sum_{i,j} |\mu(P_i \cap Q_j) - \mu(P_i)\mu(Q_j)| \leq \varepsilon.$$

A partition ζ is *weak Bernoulli* if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \geq 1$,

$$\bigvee_0^m T^{-i}\zeta \quad \text{is} \quad \varepsilon - \text{independent of} \bigvee_N^{N+m} T^{-i}\zeta.$$

The system $(X, \mathcal{B}, \mu; T)$ is *weakly Bernoulli* if it has a **generating** weak Bernoulli partition.

Theorem 4 (Friedman & Ornstein)**.** If $(X, \mathcal{B}, \mu; T)$ be an invertible measure preserving system, and η is a weak-Bernoulli partition such that

$$\zeta_{-\infty}^{\infty} \equiv \bigvee_{i=-\infty}^{\infty} T^{-i}(\zeta)$$

generates $\mathcal{B}$, then T is isomorphic to a two-sided Bernoulli shift.

Therefore, if a measure preserving endomorphism is weakly Bernoulli, its natural extension is two-sided Bernoulli ($b \Rightarrow a$).

Among endomorphisms shown to be weakly Bernoulli are:

- β-transformations (Parry, 70s)
- Toral endomorphisms (Adler & Smorodinsky, '72)
- Various interval maps with acips (Ledrappier, '81)
- Equilibrium states for rational maps of $\bar{\mathbb{C}}$ with supremum gap (Haydn, '00)

Decomposing n-to-one Endomorphisms.

Due to Rohlin (1952) proved that a ***n*-to-one** (for $n = 2, 3, \ldots, \aleph_0$) measure preserving endomorphism $(X, \mathcal{B}, \mu; T)$ has a proper factor $(Y, T^{-1}\mathcal{B}, \nu; T)$ with factor map φ such that

$$\begin{array}{ccc} (X, \mathcal{B}, \mu) & \xrightarrow{T} & (X, \mathcal{B}, \mu) \\ \downarrow \varphi & & \downarrow \varphi \\ (Y, T^{-1}\mathcal{B}, \nu) & \xrightarrow{T} & (Y, T^{-1}\mathcal{B}, \nu) \end{array}$$

commutes, where $\nu = \mu|_{T^{-1}\mathcal{B}}$.

Thus we can decompose

$$\mu(B) = \int_Y \mu_y(B) d\nu(y) \tag{0.1}$$

where for ν-a.e. $y \in Y$, $\mu_y = \mu_{[T^{-1}x]}$ is a measure that is nonsingular for T, purely atomic (since T is at most countable-to-one), and its support is contained in the set of points $\{T^{-1}x\}$ such that $[y] = [T^{-1}x]$.

The Index of a Point.

Definition 5. For a nonsingular endomorphism T, the *index function* (or index) $ind_T(x)$ is defined to be, (μ mod 0), the cardinality of the support of $\mu_{[T^{-1}x]} = \mu_{[\varphi(x)]}$ for $x \in X$.

If $(X, \mathcal{B}, \mu; T)$ is one-sided Bernoulli, then the index is constant n.

Moreover the Jacobians

$$J(x) = \frac{d\mu \circ T}{d\mu}(x) \in$$

satisfy

$$\{J_{\mu T}(y)\}_{y \in supp(\mu_{[T^{-1}x]})} = \{1/p_1, 1/p_2, \ldots, 1/p_n\}.$$

for μ-a.e. x.

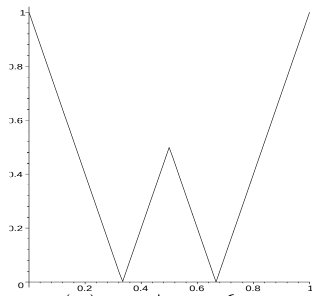

FIGURE 1. The map $T(x) = |\min\{3x - 1, 2 - 3x\}|$ preserves an acip μ with $\frac{d\mu}{dm} = \frac{4}{3}$ on $[0, \frac{1}{2})$ and $\frac{d\mu}{dm} = \frac{2}{3}$ on $(\frac{1}{2}, 1]$.
T is bounded-to-one w.r.t. Lebesgue, but 2-to-1 w.r.t. Hausdorff measure supported on the middle thirds Cantor set.

Rohlin Partitions

An bounded-to-one measure preserving endomorphisms $(X, \mathcal{B}, \mu; T)$ has an ordered partition $\zeta = \{A_1, A_2, A_3, \dots\}$ satisfying:

(1) $\mu(A_i) > 0$ for each i;

(2) the restriction of T to each A_i, which we will write as T_i, is one-to-one (μ mod 0);

(3) each A_i is of maximal measure in $X \setminus \cup_{j<i} A_j$ with respect to property 2;

(4) T_1 is one-to-one and onto X (μ mod 0) by numbering the atoms so that
$$\mu(TA_i) \geq \mu(TA_{i+1})$$
for $i \in \mathbb{N}$.

Non-uniqueness of Rohlin Partitions.

- For the angle doubling map (preserving Lebesgue measure), any partition
  $$\zeta_t = \{A_0 = [0, t) \cup (t + \frac{1}{2}, 1], \quad A_1 = [t, t + \frac{1}{2})\}$$
  is a Rohlin partition.

- ζ_t generates $\mathcal{B}$ for all $t \in (0, \frac{1}{2})$ except $t = \frac{1}{4}$.

- The coding map π_t is surjective but not injective for all $t \in (0, \frac{1}{2})$.

  For $t = 0$, π_t is injective, but no point has code $111\dots$;

For the map $T_{p,t}$ below, Lebesgue measure is one-sided $\{p, 1-p\}$-Bernoulli, except for $t = \frac{1}{4}$.

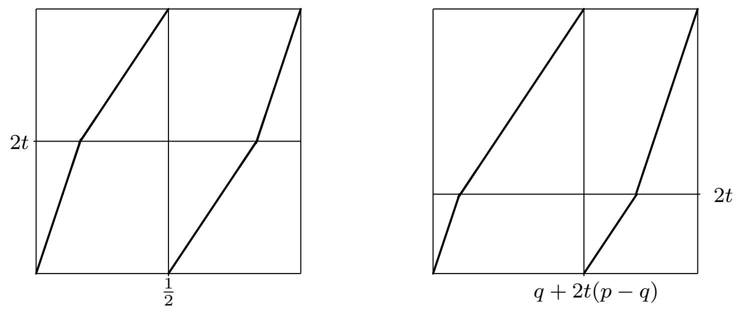

FIGURE 2. The map $T_{p,t}$ is not one-sided Bernoulli for $t = \frac{1}{4}$ (left) but it is for e.g. $t = \frac{3}{20}$ (right).

Commuting Automorphisms

Theorem 6. Suppose $p \neq \frac{1}{2}$:

(1) Let σ on (Ω, ρ) be the one-sided $\{p, 1-p\}$ Bernoulli shift. Then there exists no nontrivial nonsingular automorphism $\varphi : (\Omega, \rho) \to (\Omega, \rho)$ with $\varphi \circ \sigma = \sigma \circ \varphi$ (μ mod 0).

(2) If T on $(X, \mathcal{B}, \mu)$ is a one-sided $\{p, 1-p\}$ Bernoulli endomorphism, then there is no nontrivial nonsingular T-commuting automorphism $\varphi : (X, \mu) \to (X, \mu)$.

Corollary 7 (Parry)**.** Suppose $(X, \mathcal{B}, \mu; T)$ is a measure preserving 2-to-one endomorphism. If there exists a non-trivial nonsingular automorphism φ commuting with T, then T is not isomorphic to a one-sided $\{p, 1-p\}$ Bernoulli shift.

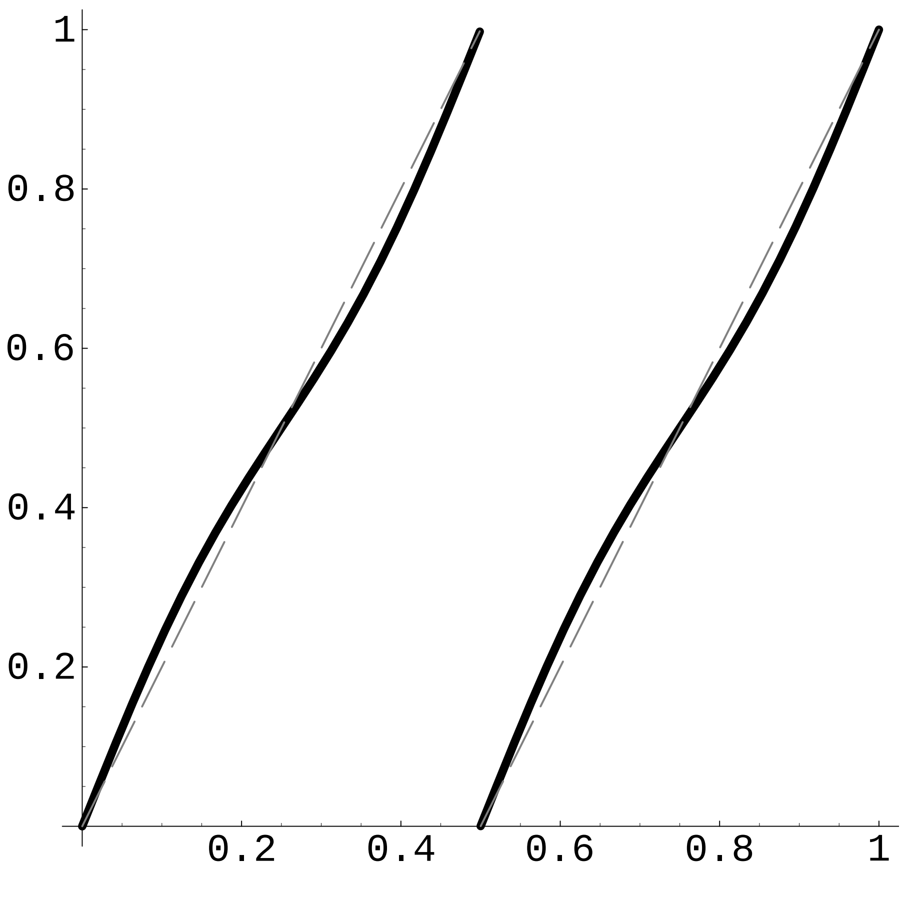

FIGURE 3. $T(x) = 2x + \varepsilon \sin 4\pi x$ preserves an acip μ but is not one-sided Bernoulli, because of it symmetry $x \mapsto 1 - x$.

A Rigidity Result

Theorem 8. Let $T : I = [0,1] \longrightarrow I$ be a piecewise C^2 n-to-1 map and assume T preserves a probability measure $\mu \sim m$.

Assume that the Radon-Nikodym derivative

$$g(x) = \frac{d\mu}{dm}$$

is continuous and bounded away from 0.

Then T is one-sided Bernoulli on $(I, \mathcal{B}, m)$ if and only if T is C^1-conjugate to a map $S : I \longrightarrow I$ whose graph consists of n linear pieces, with slopes $\pm\frac{1}{p_i}$ such that $h_\mu(T) = -\sum_{i=1}^{n} p_i \log p_i$.

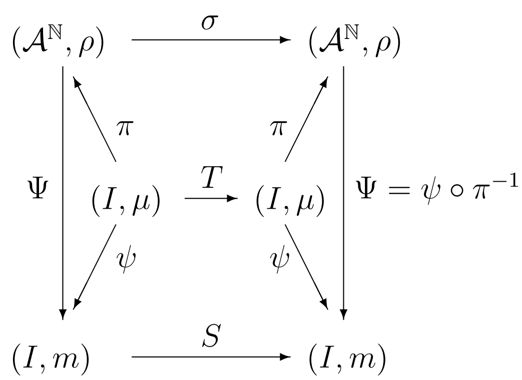

FIGURE 4. Commutative diagram to construct $\Psi = \psi \circ \pi^{-1}$.

Rational Maps on the Riemann Sphere.

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$R(z) = \frac{p(z)}{q(z)}$$

be a rational map of degree $d = \max\{\deg(p), \deg(q)\}$. The Julia set $\mathcal{J} = \mathcal{J}(R)$ supports

- a measure of maximal entropy $\mu_{\max}$
  - Lyubich ('83) and Mañé (85) proved that $(\mathcal{J}, \mu_{\max}, R^k)$ is one-sided Bernoulli for some $k \geq 1$.
  - Heicklen & Hoffman ('02) proved that $(\mathcal{J}, \mu_{\max}, R)$ itself is one-sided Bernoulli.
- an invariant measure $\mu_\alpha \ll m_\alpha$, the α-conformal measure (where ideally $\alpha = \dim_H(\mathcal{J})$).
  Weak-Bernoulli results exist in some cases for $(\mathcal{J}, \mu_{\max}, R)$

Hyperbolic Rational Maps.

For hyperbolic rational maps, $\alpha = \dim_H(\mathcal{J}(R))$ is the correct conformal exponent to work with: the invariant measure μ_α is equivalent to α-dimensional Hausdorff measure.

Theorem 9. If R is a hyperbolic rational map of degree $d \geq 2$ with connected Julia set $\mathcal{J}(R)$, then $(\mathcal{J}, \mu_\alpha, R)$ is not one-sided Bernoulli, unless R is conformally equivalent to $z \mapsto z^{\pm d}$.

For $f_c : z \mapsto z^2 + c$, $c \in \mathbb{C} \setminus \mathcal{M}$, the Julia set $\mathcal{J}$ is a hyperbolic Cantor set (so not connected). Write μ_c for the invariant measure equivalent to α-conformal measure (= α-dimensional Hausdorff measure).

Theorem 10. If $c \in \mathbb{C} \setminus \mathcal{M}$ satisfies
(i) $c \notin (\frac{1}{4}, \infty)$,
(ii) $Re(c) \neq \frac{1}{2}$,
(iii) $2|1 + c| \neq 1 + 2c \pm \sqrt{1 - 4c}$,
then $(\mathcal{J}, \mu_\alpha, R)$ is not one-sided Bernoulli.

Applications to Postcritically Finite Maps:

Any degree n Chebyshev system is one-sided Bernoulli.

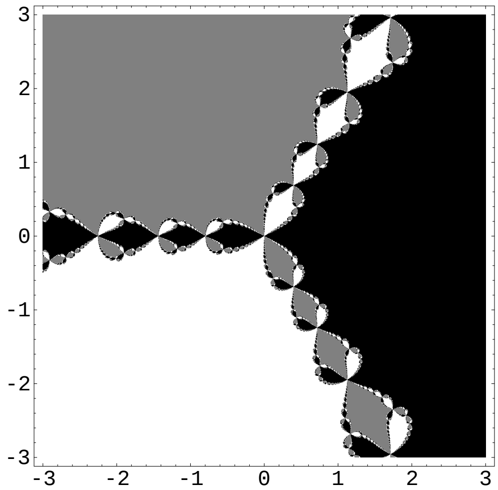

FIGURE 5. The Julia set separating basins of super-attracting fixed points for the rational function of Newton's root-finding algorithm for $z^3 - 1$.

Let $T : \mathbb{C} \to \mathbb{C}$ be the rational map associated the Newton algorithm for finding the roots of the equation $z^d - 1 = 0$:

$$T(z) = z - \frac{z^d - 1}{dz^{d-1}} = \frac{(d-1)z^d + 1}{dz^{d-1}}.$$

Then T preserves a measure $\mu \ll m_t$, where $t = \dim_H(\mathcal{J})$ and m_t is t-conformal measure.

The dihedral group $\mathcal{G}$ generated by $z \mapsto e^{2\pi i/d} z$ and $z \mapsto \overline{z}$ is the group of symmetries of $\mathcal{J}$, which also transitively permutes the atoms of the Rohlin partition $\{A_1, \ldots, A_d\}$.

The system $(\mathcal{J}, \mathcal{B}, \mu; T)$ is not one-sided Bernoulli.

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