Dynamic and ergodic properties of rotated odometers.

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The staircase model

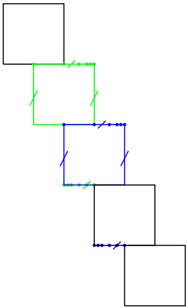

Figure: Five steps of the staircase with identifications.

The staircase flow has Poincaré map $\hat{f} : I \times \mathbb{Z} \to I \times \mathbb{Z}$ with cocycle:

$$\phi : I \to \mathbb{Z}, \qquad x \mapsto \begin{cases} +1 & x < \frac{1}{2}; \\ -1 & x > \frac{1}{2}. \end{cases}$$

Let $\xi : I \times \mathbb{Z} \to \mathbb{Z}$ be the canonical projection.

Weights

The letters in the substitution words get weights ± 1 according to whether they correspond to the flow going up or down a square.

The middle symbol is split in two: a^+ for the part $< \frac{1}{2}$ and a^- for the part $> \frac{1}{2}$.

Extend weights to words by summing the weights of their letters.

Proposition If the weights

$$w(\chi(a)) = 0 \qquad \text{for all } a,$$

then the cocycle ϕ is cohomologous to 0, and all flow-lines in the $\mathbb{Z}$-extension are bounded.

Diffusion coefficient

The **diffusion coefficient** is typical value of

$$\sup_n \frac{\log |\xi \circ \hat{f}^n(z) - \xi(z)|}{\log n}$$

for the projection $\xi : I \times \mathbb{Z} \to \mathbb{Z}$.

The normal diffusion coefficient (of e.g. standard symmetric random walks Brownian motion) is $\frac{1}{2}$.

Proposition The diffusion coefficient of the $\mathbb{Z}$-extension of (I, R_π) with a stationary **covering** permutation π. Then the diffusion coefficient

$$\gamma \leq \max \left\{ \frac{\log |\lambda_2|}{\log \lambda_1}, 0 \right\},$$

where λ_i are the eigenvalues of the associate matrix of the substitution χ.

Essential values

To decide on recurrence and ergodicity of $\mathbb{Z}$-extension $(I \times \mathbb{Z}, R_\pi, \phi)$, one can study the essential values.

Definition: We say that $e \in \mathbb{Z}$ is an essential value of the cocycle ϕ if for every positive measure $A \in \mathcal{B}$ there exists an $n \in \mathbb{N}$ such that

$$\mu\left(A \cap R_\pi^{-n}(A) \cap \{x \in [0,1) : \sum_{j=0}^{n-1} \phi \circ R_\pi^j(x) = e\}\right) > 0.$$

Additionally, ∞ is an essential value if for every $N \in \mathbb{N}$, and every positive measure $A \in \mathcal{B}$ there exists an $n \in \mathbb{N}$ such that

$$\mu\left(A \cap R_\pi^{-n}(A) \cap \{x \in [0,1) : |\sum_{j=0}^{n-1} \phi \circ R_\pi^j(x)| \geq N\}\right) > 0.$$

The set E_ϕ of all essential values forms a subgroup of $\mathbb{Z} \cup \{\infty\}$.

Recurrence/Ergodicity

Known Facts:

The flow is recurrent if and only if $0 \in E_\phi$.
NB: Lebesgue $\times$ counting measure is infinite, so recurrence does not immediately follow from the invariance of μ.

The flow is ergodic if and only if $\mathbb{Z} \subset E_\phi$, see [3].

If $E_\phi = \{0, \infty\}$, then the flow has uncountably many ergodic components.

Example (012):

$\pi = (012)$		$\pi' = (012)$	covering
Substitution	weight	Associated Matrix	char. polynomial
$0 \to 0221^-$	-2	$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 0 \end{pmatrix}$	$x^4 - 2x^3 - 8x^2$ with eigenvalues $4, -2, 0, 0$
$1^+ \to 0221^-$	-2		
$1^- \to 0221^-$	-2		
$2 \to 001^+1^+$	$+4$		

R_π is uniquely ergodic.

The extension is recurrent, non-ergodic, $E_\phi = \{0, \infty\}$, diffusion coefficient $\gamma = \frac{1}{2}$.

Example (021):

$\pi = (021)$		$\pi' = (021)$	covering
Substitution	weight	Associated Matrix	char. polynomial
$0 \to 01^+1^-221^+1^-220$	-2	$\begin{pmatrix} 2 & 2 & 2 & 4 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$x^4 - 2x^3 - 8x^2$ with eigenvalues $4, -2, 0, 0$
$1^+ \to 0$	$+1$		
$1^- \to 0$	$+1$		
$2 \to 0$	$+1$		

R_π is uniquely ergodic.

The extension is recurrent, ergodic?, $E_\phi \supset \{0, \infty\}$, diffusion coefficient $\gamma = \frac{1}{2}$.

Example (01234):

$\pi = (01234)$		$\pi' = (01234)$	covering
Substitution	weight	Associated Matrix	char. polynomial
$0 \to 03$	0	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 8 & 4 & 4 & 4 & 8 \end{pmatrix}$	$x^6 - 10x^5 + 16x^4$ with eigenvalues $8, 2, 0, 0, 0, 0$
$1 \to 03$	0		
$2^+ \to 03$	0		
$2^- \to 03$	0		
$3 \to 03$	0		
$4 \to 042^-2^+2^+11$ $1431431430-$ $42^-042^-042^-$ 2^+2^+11143	0		

R_π is not uniquely ergodic. $(I_{\min}, R_\pi)$ is dyadic odometer.

The extension is recurrent, non-ergodic, $E_\phi = \{0\}$, diffusion coefficient $\gamma = 0$: Cocycle ϕ is cohomologous to 0.

Example (0516234):

$\pi = (0516234)$		$\pi' = (0516234)$	not covering
Substitution	weight	Associated Matrix	char. polynomial
$0 \to 03^+21$	$+4$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$x^7 - 2x^6 - 6x^5$ with eigenvalues $1 \pm \sqrt{7}$ and 0 (multiplicity 5)
$1 \to 03^+21$	$+4$		
$2 \to 001$	$+3$		
$3^+ \to 011$	$+3$		
$3^- \to 011$	$+3$		
$4 \to 01$	$+2$		
$5 \to 01$	$+2$		
$6 \to 01$	$+2$		

Lebesgue is not ergodic. $0 \in \overline{I_{per}}$, $(I_{\min}, R_\pi)$ is substitution shift with dyadic odometer as maximal equicontinuous factor.

Lifted measure on I_{np} is transient to $+\infty$. Periodic orbits lift to?

Example (02431):

$\pi = (02431)$		$\pi' = (02431)$	covering
Substitution	weight	Associated Matrix	char. polynomial
$0 \to 042^+12^-$	$+1$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 3 & 5 & 2 & 1 & 8 & 5 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$	$x^6 - 10x^5 + 8x^4$ $+58x^3 + 47x^2 + 8x$ with eigenvalues $8, 2 \pm \sqrt{5}$, $-1, -1, 0$
$1 \to 042^+$	$+1$		
$2^+ \to 04012^-$	$+1$		
$2^- \to 0402^-$	$+1$		
$3 \to 04013341$ 3342^+0133 413342^+12^-	-4		
$4 \to 012^-$	$+1$		

R_π is not uniquely ergodic. $(I_{\min}, R_\pi)$ is a substitution shift.

Lifted Lebesgue transient to $-\infty$ with diffusion coefficient $\gamma = \frac{\log 2+\sqrt{5}}{\log 8}$; lifted minimal system has $\gamma \approx 1/4$.

References

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