Ergodic considerations on Selmer's continued fraction algorithm in higher dimensions

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The Euclidean Algorithm

An example from very old Greeks:

Let $x < y$ be positive real numbers.

The Euclidean algorithm to approximate $\frac{x}{y}$ by rationals goes by iterating:

$$(x, y) \to \begin{cases} (x, y - x) & \text{if } x < y - x, \\ (y - x, x) & \text{if } x > y - x. \end{cases}$$

If we scale the largest coordinate to 1, we get the Farey map:

$$f(x) = \begin{cases} \frac{x}{1-x} & \text{if } x < \frac{1}{2}, \\ \\ \frac{1-x}{x} & \text{if } x > \frac{1}{2}. \end{cases}$$

The Gauß map

To speed up this algorithm, define

$$\tau(x) = 1 + \min\{n \geq 0 : f^n(x) \in (\frac{1}{2}, 1]\}.$$

The induced map $G = f^\tau$ is the **Gauß map**

$$G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$$

with invariant measure $d\nu = \frac{1}{\log 2} \frac{1}{1+x} \, dx$.

The Gauß map

The measure $d\nu = \frac{1}{\log 2}\,\frac{1}{1+x}\,dx$ does **not** pull back to an f-invariant **probability** measure.
Instead, the Farey map preserves the infinite density

$$d\mu = \frac{1}{x}\,dx$$

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Basic problem: Do such continued fraction algorithms have invariant measures, and what are their properties?

Algorithms in higher dimension.

Let $\vec{x} = (x_1, \ldots, x_d)$ be a d-tuple of positive reals. and π a permutation on $\{1, \ldots, d\}$.
Any subtractive algorithm can be composed of basic maps

$$T_\pi(\vec{x}) = \pi \circ (x_1, \ldots, x_{d-1}, x_d - x_1)$$

and then iterated

$$T^n(\vec{x}) = T_{\pi_n} \circ T_{\pi_{n-1}} \circ \cdots \circ T_{\pi_1},$$

where the permutations may depend on the argument $\vec{x}$, (for example, to `sort` in increasing order).

Algorithms in higher dimension.

$$T^n(\vec{x}) = T_{\pi_n} \circ T_{\pi_{n-1}} \circ \cdots \circ T_{\pi_1},$$

You can scale to unit size (say $\max x_j = 1$) at any moment:

$$f^n(\vec{x}) = \frac{1}{\max \hat{x}_j}\, \hat{x} \quad \text{for } \hat{x} = T^n(\vec{x}).$$

Thus f acts on

$$\Delta_d = \{\vec{x} = (x_1, \ldots, x_{d-1}) : 0 \leq x_i \leq 1\}.$$

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- The boundary $x_1 \equiv 0$ of Δ_d consists of neutral fixed points.
- The shear of f can be much worse than the expansion.
- Further complications can arise from the lack of Markov partition.

Selmer's Algorithm

Let $a \in \mathbb{N}$ and define:

$$T(\vec{x}) = \texttt{sort}(x_1, \ldots, x_a, x_{a+1} - x_1)$$

Here `sort` means: rearrange in increasing order.

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That means: typically. If there are rational relations between the coordinates, e.g. $x_{a+1} = x_1$, then x_1 can become zero in finitely many steps.

Selmer's Generalised Algorithm

Let $a, b \in \mathbb{N}$, $d = a + b$.

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Trapping Theorem The r-th coordinate of $\vec{x}^\infty := \lim_{n\to\infty} T^n(\vec{x})$ is zero

$$\begin{cases} \text{almost surely} & \text{if } r \leq a+1, \\ \text{with probability strictly between 0 and 1} & \text{if } a+1 < r \leq \min\{a+b, 2a\}. \end{cases}$$

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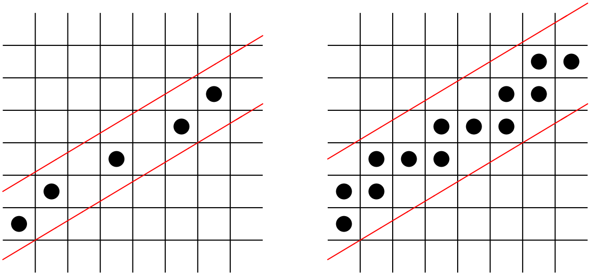
For $r > 2a$ there is no Markov partition. Numerical experiments suggest that the r-th coordinate is positive for Lebesgue-a.e. $\vec{x}$.

Selmer's Generalised Algorithm

Selmer's algorithm finds applications in percolation problems.

Disconnected dots　　　　Connected dots



The minimal width between red lines so that the pattern of dots is connected depends on the answer to Question 1.

Trapping regions

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For the case $a = 1, b = 2$, *i.e.*,

$$T(x_1, \ldots x_3) = \textbf{sort}(x_1, x_2 - x_1, x_3 - x_1),$$

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Therefore, if at some iterate $\eta > 0$, then $x_3^{\infty} = \eta > 0$.

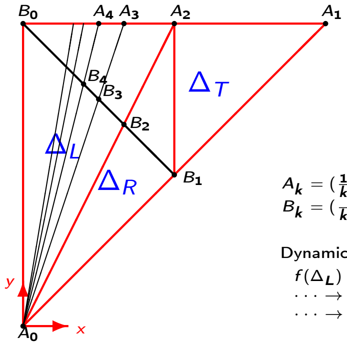
In particular, Lebesgue measure is not ergodic.

We call $\{\vec{x} \in \mathbb{R}_+^3 : x_1 + x_2 < x_3\}$ the trapping region.

Trapping regions

Scaling $x_3 = 1$, this map reduces to $f : \Delta_3 := \Delta \to \Delta$

$$f(x,y) = \begin{cases} \left(\frac{y-x}{x}, \frac{1-x}{x}\right) & \text{if } (x,y) \in \Delta_T, \\ \left(\frac{y-x}{1-x}, \frac{x}{1-x}\right) & \text{if } (x,y) \in \Delta_R, \\ \left(\frac{x}{1-x}, \frac{y-x}{1-x}\right) & \text{if } (x,y) \in \Delta_L, \end{cases}$$



$$A_0 = (0,0)$$
$$B_k = (0,1)$$

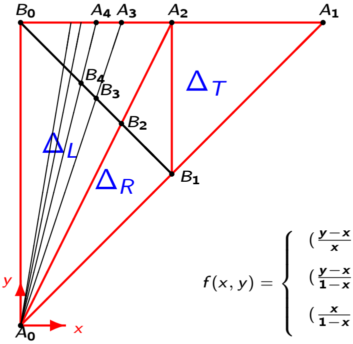
$$\left.\begin{aligned} A_k &= \left(\tfrac{1}{k}, 1\right) \\ B_k &= \left(\tfrac{1}{k+1}, 1 - \tfrac{1}{k+1}\right) \end{aligned}\right\} \text{ for } k \geq 1$$

Dynamics of f:

$$f(\Delta_L) = f(\Delta_R) = f(\Delta_T) = \Delta$$
$$\cdots \to A_3 \to A_2 \to A_1 \to A_0 \circlearrowleft$$
$$\cdots \to B_3 \to B_2 \to B_1 \to B_0 \circlearrowleft$$

Figure: The Markov partition for $f : \Delta \to \Delta$

Trapping regions



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Theorem (Nogueira '95, Bruin '13): Lebesgue-a.e. f-orbit converges to $(0,0)$. This convergence is chaotic:
Lebesgue measure is totally dissipative, ergodic and exact.

Trapping regions

Recall Selmer's generalised algorithm

$$\begin{cases} T(\vec{x}) = \mathtt{sort}(x_1, \ldots, x_a, x_{a+1} - x_1, \ldots, x_d - x_1), \\ f(\vec{x}) = \frac{1}{\hat{x}_d}\ \vec{x} \qquad \hat{x} = T(\vec{x}). \end{cases}$$

for $a, b \in \mathbb{N}$, $d = a + b$.

The r-**th Trapping Region** is

$$\mathcal{T}_r = \{\vec{x} \in \Delta_d : \frac{1}{r-a} \sum_{j \leq r} x_j < x_r\}.$$

If $\vec{x} \in \mathcal{T}_r$, then $x_1, \ldots, x_{r-1}$ combined are too small to pull x_r to zero.

Trapping regions

Recall that $\mathcal{T}_r = \{\vec{x} \in \Delta_d : \frac{1}{r-a} \sum_{j \le r} x_j < x_r\}$ is the *r*-**th Trapping Region**.

If $r \le 2a$ then

$$\begin{cases} \mathcal{V}_r := \mathcal{T}_{r+1} \setminus \mathcal{T}_r & \text{is } T_r\text{-invariant,} \\ \mathcal{V}_r \cap \{x_r = 1\} & \text{is } f_r\text{-invariant.} \end{cases}$$

Let $\mathfrak{c}$ be the last coordinate such that

$$x_{\mathfrak{c}}^{\infty} := (\lim_{n \to \infty} T^n(\vec{x}))_{\mathfrak{c}} = 0.$$

Thus (from now on) we can restrict to

$$f : \mathcal{V}_{\mathfrak{c}} \to \mathcal{V}_{\mathfrak{c}}$$

Markov Partitions

The map T (and hence f) has discontinuous derivatives at the folding planes $\{x_i = x_{i+1}\}$, due to the permutations π.

Preimages of folding planes provide a Markov partition.

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Preimages of folding planes provide a Markov partition.

Definition: $\mathcal{P} = \{P_i\}_{i \geq 0}$ of Δ is a Markov partition if

- $\cup P_i = \Delta \pmod 0$;
- $f|P_i$ is injective;
- and $\mathcal{P}$ is preserved under f:

$$f(P_i) \cap P_j \neq \emptyset \Rightarrow f(P_i) \supset P_j$$

Markov Partitions

Most folding planes map to $\partial\Delta$, but not $\{x_a = x_{a+1}\}$.

Lemma: For $a \geq b$, $\{x_a = x_{a+1}\}$ has a finite (set-valued) orbit.

Thus the complemetary domains of

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Challenge: What to do if $\nexists$ Markov partition?

Invariant Measures

For $d = 2a$, decompose

$$\Delta = \cup_{\mathfrak{c}=a+2}^{d} \mathcal{V}_{\mathfrak{c}}$$

where for $\vec{x} \in \mathcal{V}_{\mathfrak{c}}$, $\mathfrak{c}$ is the last coordinate such that

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Measure Theorem: If $a \geq \max\{b, 2\}$, then the restriction

$$f_{\mathfrak{c}} : \mathcal{V}_{\mathfrak{c}} \to \mathcal{V}_{\mathfrak{c}}$$

preserves a probability measure $\mu_{\mathfrak{c}}$, which is equivalent to $Leb|_{\mathcal{V}_{\mathfrak{c}}}$ with density bounded away from 0.

Remarks on the Proof: Inducing

For generalised Selmer, we select the set $Y \subset \Delta$, bounded away from $\{x_1 = 0\}$ (i.e., away from neutral fixed points), at which some coordinate x_j overtakes x_1.

Induce: $\begin{cases} \tau(x) = 1 + \min\{n \geq 0 : f_{\mathfrak{c}}(x) \in Y\}. \\ G = f_{\mathfrak{c}}^{\tau} \text{ is "first passage through } Y\text{" map:} \end{cases}$

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- Distor. J_G + (Markov ór uniform expansion) $\Rightarrow$ Distor. J_{G^n}.

Remarks on the Proof: G expanding?

Shears can keep G from being expanding: $DG(\vec{x}) = \frac{1}{x_j - m_j x_1} \cdot$

$$
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\frac{m_j x_{j+1} - m_{j+1} x_j}{x_j - m_j x_1} & \cdots & & \cdots & 0 & \frac{m_{j+1} x_1 - x_{j+1}}{x_j - m_j x_1} & 1 & \cdots & 0 \\
\vdots & & & & \vdots & \vdots & 0 & \ddots & \\
\frac{m_j x_{\mathfrak{c}-1} - m_{\mathfrak{c}-1} x_j}{x_j - m_j x_1} & & & & \vdots & \frac{m_{\mathfrak{c}-1} x_1 - x_{\mathfrak{c}-1}}{x_j - m_j x_1} & & 0 & 1 \\
\frac{m_j - m_{\mathfrak{c}} x_j}{x_j - m_j x_1} & 0 & \cdots & & 0 & \frac{m_{\mathfrak{c}} x_1 - 1}{x_j - m_j x_1} & & & 0
\end{array}\right)
$$

- DG is uniformly expanding for $\mathfrak{c} \leq 6$;
- DG^2 is probably uniformly expanding for $\mathfrak{c} \leq 20$;
- Challenge: Spectral properties of transfer operator $\mathcal{L}_G$?

Remarks on the Proof: Pulling back ν

For

$$\mu(A) = \sum_{n\geq 1} \sum_{k=0}^{n-1} \nu(\{\tau(x) = n\}) \cap \ f^{-k}(A))$$

to be finite, we need

$$\Lambda = \sum_{n\geq 1} n\nu(\{\tau(x) = n\}) = \sum_{n\geq 1} \nu(\underbrace{\{\tau(x) \geq n\}}_{\text{tail}}) < \infty.$$

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For this reason *e.g.*

- the Farey map
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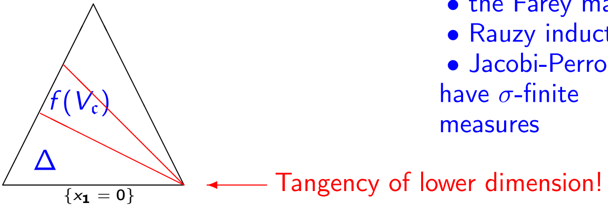

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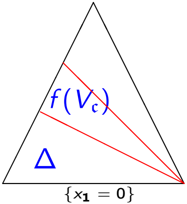

← Tangency of lower dimension!

For this reason *e.g.*

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However, generalised Selmer has a finite measure μ for $a \geq 2$.

Rational Approximations

The map T is piecewise linear; each iterate T^k is given by an integer matrix A_k that depends on $\vec{x}$. Its inverse

$$A_k^{-1} = \begin{pmatrix} p_{1,k} & p_{1,k-1} & \cdots & \cdots & p_{1,k-d+1} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ p_{d-1,k} & p_{d-1,k-1} & \cdots & \cdots & p_{d-1,k-d+1} \\ q_k & q_{k-1} & \cdots & \cdots & q_{k-d+1} \end{pmatrix}.$$

is also integer and has non-negative entries.

In projective space, the columns of A_k^{-1} approximate $\vec{x}$, **provided** $T^k(\vec{x}) \to \vec{0}$.

Rational Approximations

Hence,

$$\left(\frac{p_{1,k-j}}{q_{k-j}}, \ldots, \frac{p_{d-1,k-j}}{q_{k-j}}\right) \qquad \text{for each } 0 \leq j < d,$$

are rational approximations of

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To see this: Each column of A_k^{-1} is orthogonal to all rows of A_k, except one. Each column therefore spans the orthogonal complement of $d-1$ rows. If $\lim_{k\to\infty} A_k\vec{x} = \vec{0}$, then $\vec{x}$ is nearly orthogonal to all rows of A_k.

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Therefore, in projective space, $\vec{x}$ is close to the column vectors of A_k^{-1}. The quality of the approximation depends on the rate of convergence of $T^k(\vec{x}) \to \vec{0}$; if $\lim_k T^k(\vec{x}) \neq \vec{0}$, then T gives no approximations at all.

Qualifty of Approximations

Dirichlet's Theorem states that every vector $\vec{x}$ has infinitely many rational approximations $\vec{w}$ of denominator $q = q(\vec{w})$ such that

$$\|\vec{w} - \vec{x}\| \leq q^{-(1+1/(d-1))},$$

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In higher dimension, there is no algorithm known that finds all best approximants, or even achieves infinitely many approximants with $\|\vec{w} - \vec{x}\| \leq q^{-(1+1/(d-1))}$.

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Following Lagarias '93, let

$$\eta(\vec{w}, \vec{x}) = \frac{-\log \|\vec{w} - \vec{x}\|}{\log q}$$

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$$\eta(\vec{w}, \vec{x}) = \frac{-\log \|\vec{w} - \vec{x}\|}{\log q}$$

The **best approximation exponent** is

$$\eta(\vec{x}) = \limsup_{k \to \infty} \sup_{0 \le i < d} \eta(\vec{w}_{k,i}, \vec{x})$$

The **uniform approximation exponent** is

$$\eta^*(\vec{x}) = \inf_k \frac{\min_{0 \le i < d} - \log \|\vec{w}_{k,i} - \vec{x}\|}{\max_{0 \le i < d} \log q_{k-i}}.$$

Qualifty of Approximations

These are chosen such that we can conclude

$$\|\vec{w}_{k,i} - \vec{x}\| \leq \begin{cases} q_{k-i}^{-\eta(\vec{x})} & \text{infinitely often} \\ q_{k-i}^{-\eta^*(\vec{x})} & \text{for all } k, i. \end{cases}$$

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Thus Dirichlet's Theorem states that

$$\eta(\vec{x}) \geq 1 + 1/(d-1)$$

for every $\vec{x}$, provided the algorithm finds infinitely many of the best approximations.

Qualifty of Approximations

Main Theorem: For Selmer's Generalised Algorithm with $a \geq \max\{2, b\}$, Lebesgue-a.e. vector $\vec{x} \in \mathcal{V}_d$ satisfies

$$\eta(\vec{x}) = \eta^*(\vec{x}) = 1 - \frac{\lambda_2}{\lambda_1} > 1,$$

where $\lambda_1 > 0 > \lambda_2$ are the largest two typical Lyapunov exponents of the cocycle A_k^{-1}.

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Remark: If all negative Lyapunov exponents are equal, then

$$1 - \frac{\lambda_2}{\lambda_1} = 1 + 1/(d-1).$$

Finding an algorithm with this equality of Lyapunov exponents is extremely unlikely.

Qualifty of Approximations

Remarks on the Proof:

The Main Theorem follows from the work of Lagarias '93, based on Oseledec' Theorem on Lyapunov exponents of matrix-valued cocycles (here A_k^{-1}).

We need:

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- Challenge: Estimate λ_1 and λ_2;
- Challenge: What about non-typical $\vec{x}$?

Enough Remarks

Thank you for your attention!

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