Lorentz gas with small scatterers; some non-standard Limit Theorems

Henk Bruin

Joint work with
Péter Bálint (TU Budapest) and Dalia Terhesiu (Leiden Univ)

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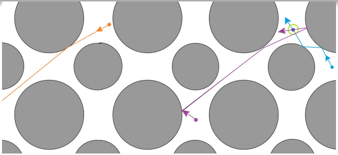
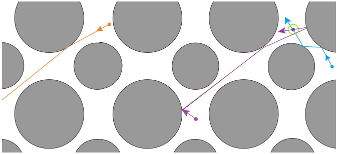


Figure: finite horizon

Dispersive Mathematical Billiard



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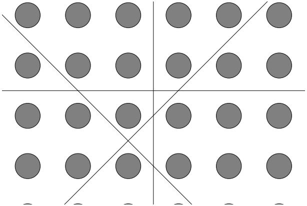

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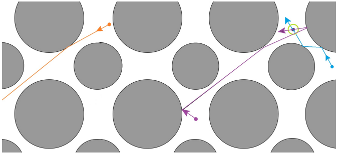
Figure: infinite horizon

- Unit speed movement of particles with elastic collisions.
- Planar region filled convex obstacles (scatterers Ω_i)
- Models electron movement in metal.

Dispersive Mathematical Billiard



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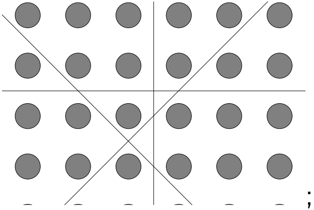
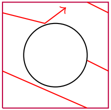


Figure: infinite horizon

- Unit speed movement of particles with elastic collisions.
- Planar region filled convex obstacles (scatterers Ω_i)
- Models electron movement in metal.

- $\partial\Omega_i$ is C^4 smooth and curvature $> 0 \Rightarrow$ hyperbolicity
- singularity at grazing collision
- billiard map vs. billiard flow.
- finite vs. infinite horizon

Lorentz gas



Sinaĭ billiard: single round scatterer on torus

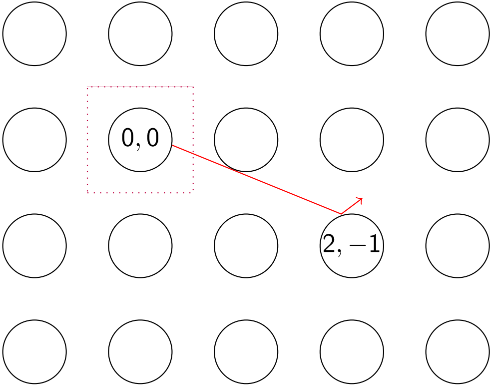
Phase space billiard map: $M = \partial\Omega \times (-\frac{\pi}{2}, \frac{\pi}{2})$

Coordinates $r \in \partial\Omega$ for collision point

and θ for ougoing angle with normal vector

Invariant density: $d\mu = \sin\theta dr d\theta$

Lorentz gas



Lorentz gas: round scatterers at $\mathbb{Z}^2 \subset \mathbb{R}^2$

$\mathbb{Z}^2$-extension over Sinaĭ billiard

Displacement function

$\kappa : M \to \mathbb{Z}^2$
(here $\kappa = (2, -1)$)

Lorentz gas (recap)

- Lorentz gas: $\mathbb{Z}^2$-extension of Sinaĭ billiards $T : M \to M$; of the **dynamics on the cell** M by the free flight (displacement) function κ. Map on $\mathbb{Z}^2$-extension:

$$\hat{T}(z, n) = (T(z), n + \kappa(z))$$

preserves $\hat{\mu} = \mu \times$ counting measure.

- Although deterministic, displacement function κ satisfies stochastic laws, as a random process on probability space (M, μ). The **dependent** sequence of r.v. is $\{\kappa \circ T^j\}_{j \geq 0}$.
- Analogous laws as a random walk on $\mathbb{Z}^2$ with transition probabilities of $z \mapsto z + n$ equal to $\mu(\kappa = n)$.

Stochastic laws for **Finite Horizon**

- κ is **bounded**
- **CLT**: $\frac{\kappa_n}{\sqrt{n}} \to^d \mathcal{N}(0, \Sigma)$ where $\det \Sigma \neq 0$ (Green Kubo). Bunimovich-Sinaĭ 1981.
- **Exponential mixing** for Hölder observables: there are $C = C(\psi_1, \psi_2) > 0$ and $\theta \in (0,1)$ s.t. for all $n \geq 1$:

$$Cor_n(\psi_1, \psi_2) := \left| \int \psi_1 \cdot \psi_2 \circ T^n \, d\mu - \mathbb{E}_\mu(\psi_1)\mathbb{E}_\mu(\psi_2) \right| \leq C\theta^n.$$

  Young 1998, Chernov 1999
- **Local Limit Theorem (LLT)** for κ: there is $C > 0$ s.t.

$$\frac{n}{C}\, \mu(\kappa_n = N) \to_{n\to\infty} \Psi_\Sigma(N) \qquad \text{for all } N \in \mathbb{Z}^2,$$

  for density Ψ_Σ of the Gaussian $\mathcal{N}(0, \Sigma)$. Szász-Varjú 2004

Stochastic laws for **Infinite Horizon**

- κ **unbounded**, $\kappa \notin L^2$.
- **CLT** with nonstandard normalizing: $\frac{\kappa_n}{\sqrt{n \log n}} \to^d \mathcal{N}(0, \Sigma)$, where $\det \Sigma \neq 0$ and Σ comes from summation over corridors
- **Exponential mixing** for κ: there exist $C > 0$ and $\theta \in (0,1)$ s.t. for all $n \geq 1$:

$$\left| \int \kappa \cdot \kappa \circ T^n \, d\mu \right| \leq C\theta^n.$$

  Note that $\mathbb{E}_\mu(\kappa) = 0$ by symmetry. Chernov-Dolgopyatt 2009 (using coupling argument of standard pairs).
- **LLT** for κ: $\frac{n \log n}{C} \mu(\kappa_n = N) \to_{n\to\infty} \Psi_\Sigma(N)$, for all $N \in \mathbb{Z}$. Szász-Varjú, 2007.
  (Exploit Young tower and results from Bálint-Gouëzel 2006)

The size of the scatterer $\rho \to 0$

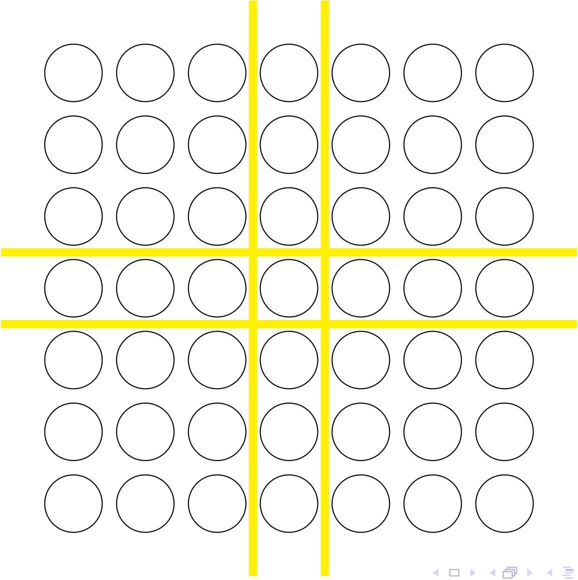
Let ρ be the radius of the scatters, so $\kappa \to \kappa_\rho$.

Question: What happens when $\rho \to 0$?

- Chance to hit a scatterer decreases: need for scaled time $\tilde{t} \sim \rho t$ (i.e., time measured in units of mean collision time).
- Curvature of scatterers and therefore hyperbolicity increases.
- ρ-dependence of constants in mixing & CLT: Σ_ρ, C_ρ and θ_ρ.

  NB: So far, ρ-dependence of $\theta_\rho \in (0, 1)$ remains non-tractable.
- Number of infinite horizon directions increases (corridors open up).

Corridors open up as $\rho \to 0$



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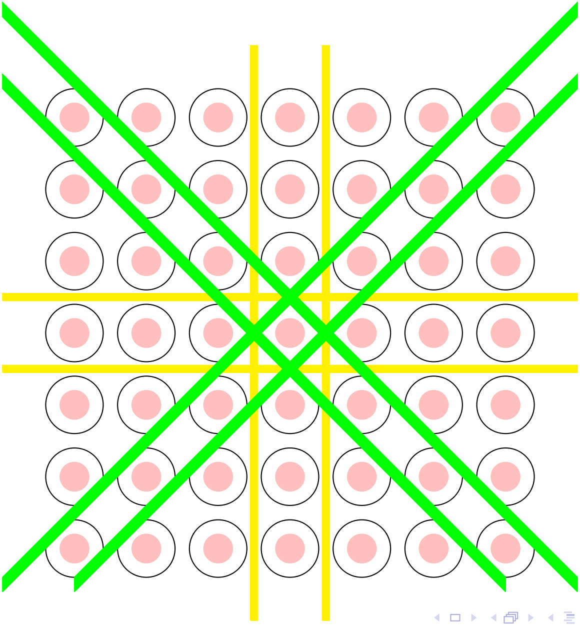

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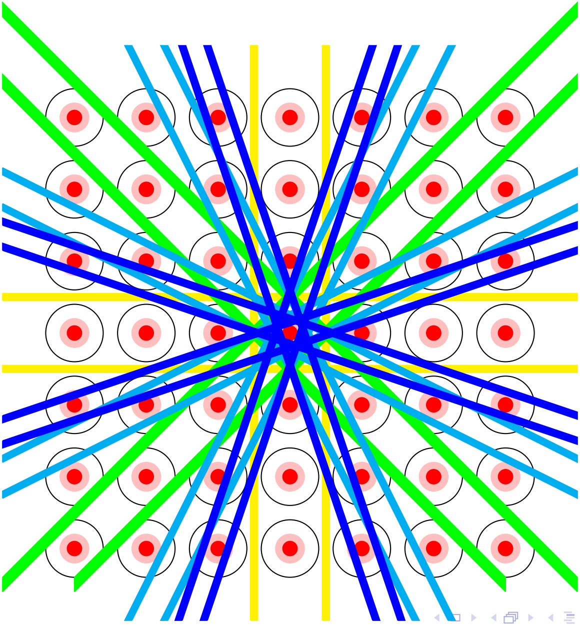

Boltzmann-Grad limit

Szász-Varjú 2007 and Chernov-Dolgopyatt 2007 proved a conjecture by Bleher 2005:

$$\frac{\kappa_{n,\rho}}{\sqrt{n \log n}} \Rightarrow^d \mathcal{N}(0, \Sigma_\rho) \qquad \text{as } n \to \infty.$$

for $\kappa_{n,\rho} = \sum_{j=0}^{n-1} \kappa_\rho \circ T_\rho^j$ and a ρ-dependent variance matrix.

Marklof-Tóth 2016 obtained CLT when $\rho \to 0$:

$$\frac{\kappa_{n,\rho}}{(2\rho\sqrt{\pi})^{-1}\sqrt{n \log n}} \Rightarrow^d \mathcal{N}(0, \Sigma)$$

as $\rho \to 0$ followed by $n \to \infty$. Here

$$\Sigma = \frac{1}{\pi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{so} \quad \Sigma_\rho \sim \frac{1}{2\rho\sqrt{\pi}}\Sigma.$$

New results

Question: *How is CLT in the joint limit* $\rho \to 0, n \to \infty$*?*

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Theorem (CLT)

Let $\kappa_{n,\rho}$ and Σ be as before. Take scaling sequence:

$$b_{n,\rho} = \frac{\sqrt{n \log(n/\rho^2)}}{2\sqrt{\pi}\ \rho}.$$

There exists a function $M(\rho) \to \infty$ as $\rho \to 0$ such that

$$\frac{\kappa_{n,\rho}}{b_{n,\rho}} \implies \mathcal{N}(0, \Sigma), \text{ as } n \to \infty \text{ and } \rho \to 0 \text{ such that}$$

$$M(\rho) = o(\log n).$$

Form of M:

$$M(\rho) = C + C_\rho (1 - \theta_\rho)^{-2},$$

where $\theta_\rho \in (0,1)$ is unknown, C_ρ is a polynomial function in ρ and C is a uniform constant. Note that:

$$M(\rho) = o(\log n) \quad \Longrightarrow \quad b_{n,\rho} \sim \frac{\sqrt{n \log n}}{2\sqrt{\pi}\rho}$$

Remark: Similar results by Lutsko-Tóth 2020 for a purely stochastic model (random direction after collision, exponentially distributed flight time between collisions).

New results

Theorem (LLT)

Let ψ_1, ψ_2 be sufficiently smooth observables. Then as $n \to \infty$ and $\rho \to 0$ such that $M(\rho) = o(\log n)$,

$$\left| \int_{\{\kappa_{n,\rho}=N\}} \psi_1 \cdot \psi_2 \circ T_\rho^n \, d\mu - \frac{\mathbb{E}_\mu(\psi_1)\, \mathbb{E}_\mu(\psi_2)}{(b_{n,\rho})^2} \Psi_\Sigma \left(\frac{N}{b_{n,\rho}} \right) \right| \to 0,$$

uniformly in $N \in \mathbb{Z}^2$ as $\rho \to 0$.

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uniformly in $N \in \mathbb{Z}^2$ as $\rho \to 0$.

Corollary for the infinite measure system $(\hat{M} = M \times \mathbb{Z}^2, \hat{\mu})$:

$$\lim_{\rho \to 0} \left| (b_{n,\rho})^2 \int_{\hat{M}} \psi_1 \cdot \psi_2 \circ \hat{T}_\rho^n \, d\hat{\mu} - \mathbb{E}_{\hat{\mu}}(\psi_1) \mathbb{E}_{\hat{\mu}}(\psi_2) \right| = 0.$$

Ingredients of proofs

- Young towers and standard pairs for mixing estimates. Still no explicit $\theta_\rho \in (0,1)$.
- Transfer operator $\mathcal{L}_\rho$ of the billiard map T_ρ:

$$\int \mathcal{L}_\rho \psi_1 \cdot \psi_2 \, d\mu = \int \psi_1 \cdot \psi_2 \circ T_\rho \, d\mu.$$

  (continuous function analogue of stochastic matrix)
- ... acting on **anisotropic** Banach spaces.

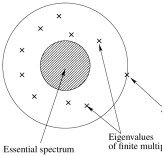
  Recall that T_ρ is (non-uniformly) hyperbolic, so has expanding and contracting directions. The norm of an **anisotropic** Banach space treats functions differently in the stable and the unstable direction.

Ingredients of proofs

- Lasota-Yorke (= Doeblin-Fortet) inequality

$$\|\mathcal{L}_\rho^n \psi\|_s \leq \theta_\rho^n \|\psi\|_s + C\|\psi\|_w \qquad (1)$$

requires a strong Banach space compactly embedded in a weak Banach space. We use **anisotropic Banach spaces** as in Demers-Zhang (2012/2014), as they allows us to track ρ-dependence.



- The **spectral gap** of $\mathcal{L}_\rho$ follows from (1), and leads to the decomposition:

$$\mathcal{L}_\rho^n = \lambda_\rho^n \Pi_\rho + Q_\rho^n,$$

where $\lambda_\rho = 1$ (isolated in spectrum of $\mathcal{L}_\rho$), Π_ρ is projection on leading eigenspace, and $\|Q_\rho^n\| \ll \gamma_\rho^n$ for some $\gamma_\rho \in (0, 1)$.

Main ingredients of proofs

- **Spectral/Nagaev method** using the **twisted** transfer operator:
$$\mathcal{L}_\rho(u)\psi := \mathcal{L}_\rho(e^{i\langle u,\kappa\rangle}\psi),$$
($\int \mathcal{L}_\rho^n(u)1\,d\mu$ is analogue of characteristic function) so as to use Lévy Continuity Theorem in the end.
- **Control of continuity in** u for $\mathcal{L}_\rho(u)$ in norm and thus eigenvalues $\lambda_\rho(u)$ and eigenprojections $\Pi_\rho(u)$
- Crucial for spectral method to work: asymptotics of the form
$$1 - \lambda_\rho(u) \sim D(\rho)u^2 \log(1/u) + E(\rho, u) \quad \text{as } u \to 0,$$
for $D(\rho)$ bounded, $|E(\rho, u)| \ll |u|^2$. New even for fixed ρ
- Equally problematic: track the ρ-dependence in $\|\mathcal{L}_\rho(u) - \mathcal{L}_\rho(0)\|$, which allows an explicit control of $E(\rho, u)$.

More details on the Banach space

Strong and weak Banach spaces $\mathcal{B}_s$ and $\mathcal{B}_w$:

- distributions acting on test functions supported on admissible stable leaves W (depends on ρ).
  We take $|W| \leq c\,\rho^{\nu}$, with ν specified later and $c < 1$.
- weak and strong norm defined via integrals on stable leaves.
- $C^{p_0} \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^{q_0})'$
- the unit ball of $\mathcal{B}_s$ is relative compact in $\mathcal{B}_w$

To carry out Nagaev method, at least control $\|\mathcal{L}_\rho(u) - \mathcal{L}_\rho(0)\|_w$

Growth Lemma or One-step Expansion Estimate

Crucial ingredient for Lasota-Yorke (spectral gap in $\mathcal{B}$) and continuity estimates: **Growth Lemma**.

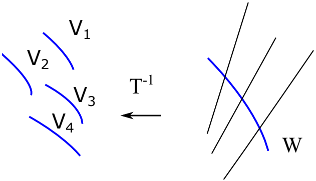


Figure: Pre-images of a stable leaf

- $\sum \frac{|TV_i|}{|V_i|} < \vartheta^* < 1$; can be iterated.
- $\sum |\kappa(V_i)|^\nu \frac{|TV_i|^\varsigma}{|V_i|} \ll \rho + |W| \cdot \rho^{-\nu}$ for $\nu \in (1/3, 1/2)$
- Growth of corridors (as $\rho \to 0$) complicates estimates

Continuity estimate

For every $\rho > 0$ and for every $\nu \in (1/3, 1/2)$, there exists a constant C independent of ρ so that

$$\|\mathcal{L}_\rho(u) - \mathcal{L}_\rho(0)\|_s \leq C\rho^{-\nu} u^\nu.$$

Problem: ν too small to be used directly for the control of $1 - \lambda_\rho(u)$.

Crucial ingredient: i) decomposition of several quantities entering the expression of $1 - \lambda_\rho(t)$ and ii) use of the standard pair type arguments to control several quantities

Main technical result

For $u \in \mathbb{R}^2$, let $\bar{A}(t,\rho) = \sum_{|\xi| \leq 1/(2\rho)} \frac{d_\rho(\xi)^2 \langle t, \xi \rangle^2}{|\xi|}$. Then

$$\lim_{\rho \to 0} \frac{\rho}{2} \bar{A}(u,\rho) = \frac{|u|^2}{\pi} = \langle \Sigma u, u \rangle \qquad \text{for } \Sigma = \frac{1}{\pi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem

Let $\nu \in (\frac{1}{3}, \frac{1}{2})$ and δ small enough (a precise characterization in terms of ρ). Then for $u \in B_{\delta_0}(0)$,

$$1 - \lambda_\rho(u) = \bar{A}(u,\rho) \frac{\log(1/|u|)}{8\pi\rho} + E(u,\rho),$$

where $|E(u,\rho)| \leq C_\rho \, \theta_\rho^{-2} \, |u|^2 + C|u|^2 \rho^{-2}$ for C_ρ and θ_ρ as in the characterization of $M(\rho)$ in the main result and some uniform C.