Matching for generalized β-transformations

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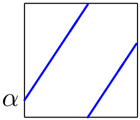
Generalized β-transformations

The generalized (= translated) β-transformation is defined as

$$T_{\beta,\alpha} : x \mapsto \beta x + \alpha \pmod 1$$



For this talk, we will fix β and vary α. Hence we write $T_\alpha(x)$ (or just T).

For $|\beta| > 1$, T has an acip μ.

Matching

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The set $\bigcup_{1 \leq j < n} T_\alpha^j(\{0,1\})$ is the prematching set.

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Definition: We speak of prevalent matching if the set of α where matching occurs has full Lebesgue measure, and its complement, the non-matching or bifurcation set A_β, is nowhere dense.

Consequences of matching

- Theorem If a generalized β-transformation with $|\beta| > 1$ has matching, then it has an invariant density h which is constant on the components of $[0,1]\backslash$prematching set.

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- For T_α, the entropy is $\log \beta$, due to constant slope β. For other piecewise linearfamilies, with non-constnt slope, entropy is monotone on matching intervals (and constant if the matching is **neutral**)

Matching for quadratic Pisot integers

The quadratic Pisot integers are those $\beta > 1$ satisfying

$$\beta^2 - k\beta \pm d = 0 \quad \text{with } \begin{cases} k > d+1 & \text{if } +d, \\ k > d-1 & \text{if } -d. \end{cases}$$

Theorem 1: For β as above, $\dim_H(A_\beta) = \frac{\log d}{\log \beta}$.

For all other quadratic numbers, no matching occurs.

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Note: $d = 1$ (quadratic Pisot units) gives $\dim_H(A_\beta) = 0$. We conjecture that this is the only situation where $\dim_H(A_\beta) = 0$.

Matching for non-quadratic algebraic integers

The examples we have of prevalent matching all relate to β being Pisot. However, matching can occur at non-Pisot numbers, e.g. the quartic Salem number satisfying

$$\beta^4 - \beta^3 - \beta^2 - \beta + 1 = 0$$

has matching at some non-trivial intervals.

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Numerical simulations give the following table

β	minimal polynomial	$\dim_B(A_\beta)$
tribonacci	$\beta^3 - \beta^2 - \beta - 1 = 0$	0.66...
tetrabonacci	$\beta^4 - \beta^3 - \beta^2 - \beta - 1 = 0$	0.76...
plastic	$\beta^3 - \beta - 1 = 0$	0.93...

Towards a proof of matching

Note

$$\begin{aligned} T_\alpha^n(0) &= (\beta^{n-1} + \cdots + 1)\alpha - a_{n-2}\beta^{n-2} - \cdots - a_1\beta - a_0, \\ T_\alpha^n(1) &= (\beta^{n-1} + \cdots + 1)\alpha + \beta^n - b_{n-1}\beta^{n-1} - \cdots - b_1\beta - b_0. \end{aligned}$$

Therefore matching at (minimal) iterate n requires

$$0 = T_\alpha^n(1) - T_\alpha^n(0) = \beta^n + \sum_{j=0}^{n-1} \beta^j (b_j - a_j).$$

Hence β has to be an algebraic integer.

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The integers b_j, a_j depend on α, but change only at a finite set.
Hence, if matching occurs, it occurs on an entire parameter interval.

Towards a proof of matching

Since β is an algebraic integer of order n, we can write

$$T_\alpha^j(0) - T_\alpha^j(1) = \sum_{k=1}^{n} \frac{e_k(j)}{\beta^k} \qquad e_k(j) \in \mathbb{Z}.$$

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Lemma (Sample Lemma)

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Lemma (Sample Lemma)

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Proof.

If $|T^j(0) - T^j(1)| = \ell/\beta$, then $T^j(0)$ and $T^j(1)$ belong to branch-domains of T that are $|\ell|$ domains apart, and their images are the same. □

Matching for quadratic Pisot integers

Back to Theorem 1. We sketch the proof for $\beta^2 - k\beta + d = 0$, $k \in \mathbb{N}$, so $k - 1 < \beta < k$ and T_α has k or $k + 1$ branches.

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Hence, take $\alpha \in [0, k-\beta)$ and call the domains of the branches $\Delta_0, \dots, \Delta_{k-1}$. Compute

$$T_\alpha(1) = \beta + \underbrace{\alpha}_{=T(0)} - (k-1) = T_\alpha(0) + \underbrace{\beta - (k-1)}_{\gamma}.$$

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Lemma: If $T^\ell(0) \in \Delta_i$ and $T^\ell(1) \in \Delta_{i+(k-1)-d}$ for $1 \le \ell < n, i = i(\ell)$, then

$$T^n(1) - T^n(0) = \gamma.$$

Matching for Pisot integers: $\beta^2 - k\beta + d = 0$

Recall: $T^n(1) - T^n(0) = \gamma$.

Lemma: If $T^n(0) \in \Delta_i$ and $T^n(1) \in \Delta_{i+k-d}$ then the distance $|T^{n+1}(1) - T^{n+1}(0)| = \frac{d}{\beta}$ and there is matching in 2 steps.

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Hence, to avoid as matching, $T^\ell(0)$ has to avoid the sets

$$\begin{aligned} V_i &:= \{x \in \Delta(i) : x + \gamma \in \Delta(i+k-d)\} \\ &= \Big[\frac{i+k-d-\alpha}{\beta_k} - \gamma \ , \ \frac{i+1-\alpha}{\beta_k}\Big). \end{aligned}$$

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Lemma: If

$$T^n(0) \in V = \cup_{i=0}^{d-1} V_i$$

then there is matching in two steps.

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Lemma: The map $g_\alpha : [0, k-\beta] \to [0, k-\beta]$,

$$g_\alpha(x) := \begin{cases} k-\beta & \text{if } x \in V, \\ T_\alpha(x) & \text{otherwise.} \end{cases}$$

is a non-decreasing degree d circle endomorpism, and $g_\alpha^n(0) \in V$ for some $n > 1$ precisely if $k-\beta$ is periodic.

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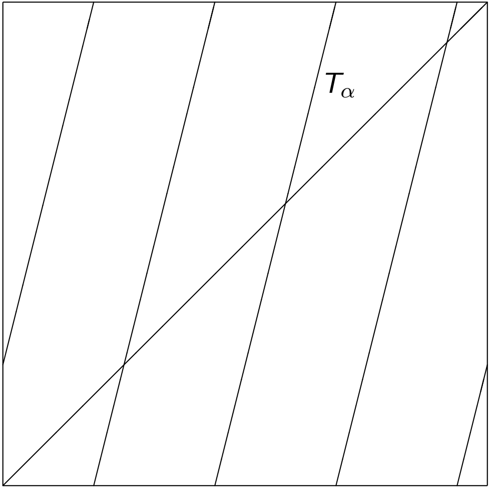

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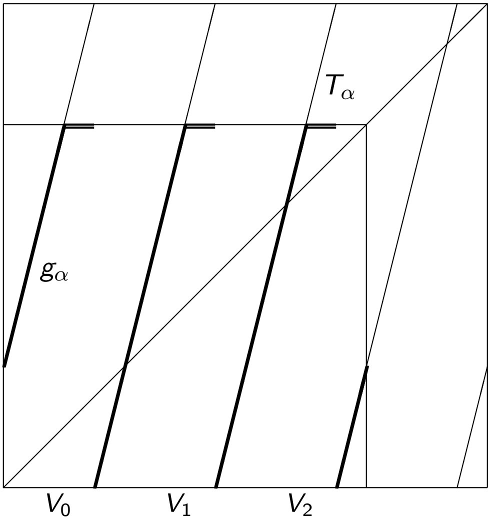

and g_α

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Lemma: Define

$$X_\alpha = \{x \in \mathbb{S}^1 \,:\, g_\alpha^n(x) \notin V \ \text{ for all } n \geq 0\}.$$

If there is no matching, then $\dim_H(X_\alpha) = \frac{\log d}{\log \beta}$.

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Idea of Proof.
For each n, we cover X_α by $O(d^n)$ intervals of length β^{-n}. $\square$

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Proof of Theorem 1 for $\beta^2 - k\beta + d = 0$.

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- There is a one-to-one correspondence between intervals J in the cover of X_α and interval U of cover of A_β, and $|J|$ and $|U|$ are comparable (independently of n).
- Therefore, for each n, the set A_α can be covered by $O(d^n)$ intervals of length $O(\beta^{-n})$.

□

Matching for non-Quadratic Pisot Units

There is another frequently used class of Pisot units, namely leading solutions β_k of

$$\beta^k - \beta^{k-1} - \beta^{k-2} - \cdots - 1 = 0.$$

for $k \geq 3$.

Theorem (Non-Quadratic Pisot Units)

For β_3 (tribonacci), there is prevalent matching. The non-matching set satisfies $0 < \dim_H(A_\beta) < 1$.

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We expect the same result for β_k, $k \geq 4$, but at the moment, we have no proof.

Matching for non-Quadratic Pisot Units

Lemma: For every $k \geq 2$ and $j \geq 0$ we have

$$|T_\alpha^j(0) - T_\alpha^j(1)| \in \left\{ \frac{e_1}{\beta} + \frac{e_2}{\beta^2} + \cdots + \frac{e_k}{\beta^k} \ : \ e_1, \dots, e_k \in \{0, 1\} \right\}.$$

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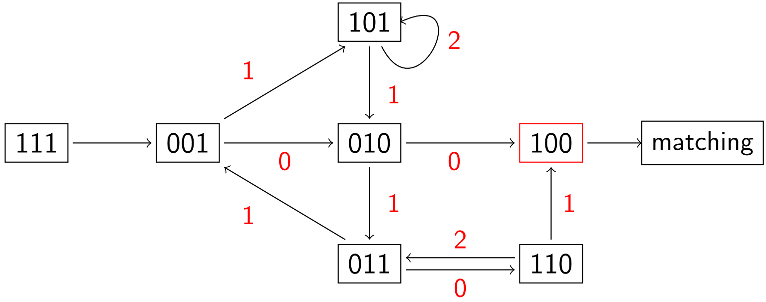
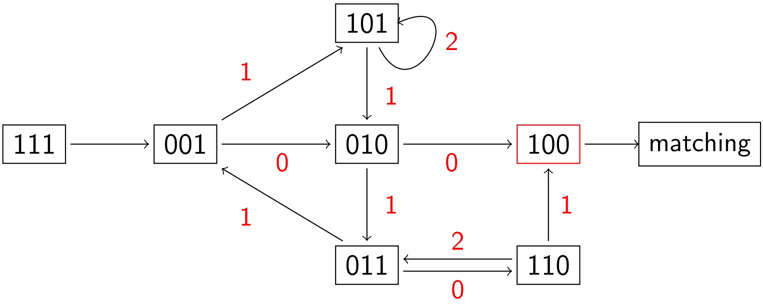


Figure: The transition graph for the tribonacci number β_3. The red numbers indicate the difference in branch between $T_\alpha^j(0)$ and $T_\alpha^j(1)$.

Matching for non-Quadratic Pisot Units



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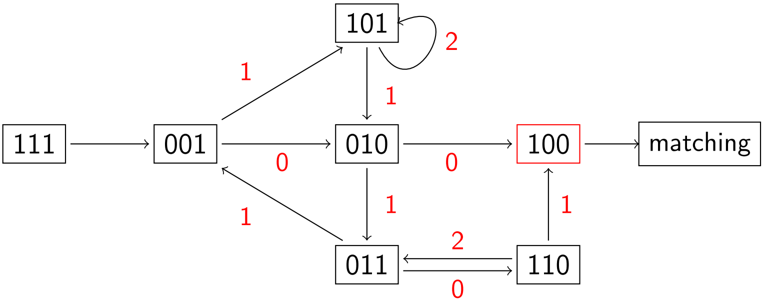

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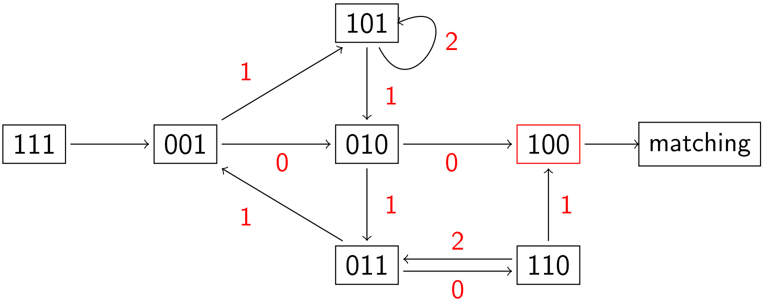

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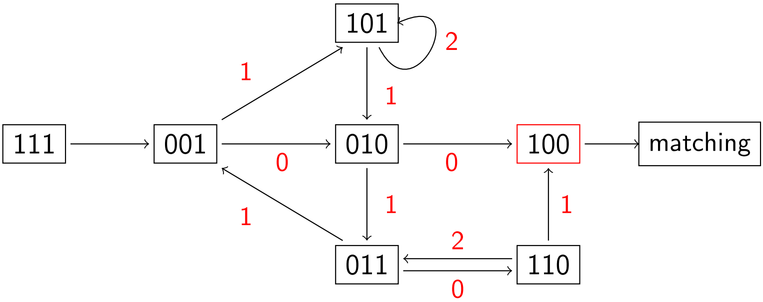

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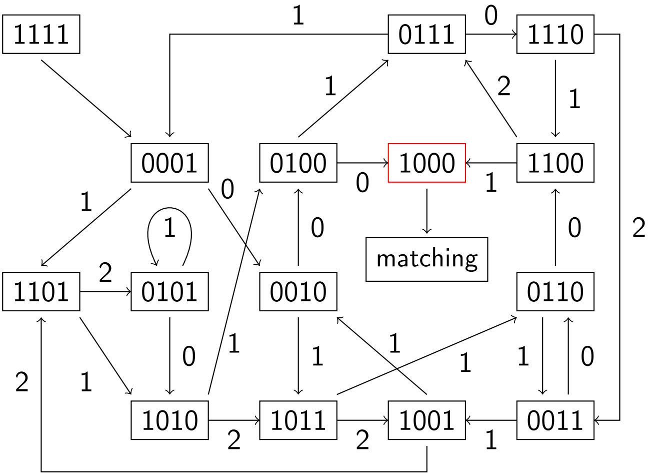

Figure: The transition graph for the Pisot number β_4 is similar but too complicated to handle.