Exercise sheet for Introduction to Topology, WS2022

Exercise 1 *Show that with the discrete metric of on a space X:*

- *Every subset of X is open;*
- *Every subset of X is closed;*
- *Every subset of X has an empty boundary;*
- *Every map $f : X \to X$ is continuous.*

Exercise 2 *Let X be the space of all closed subsets of the Euclidean space $\mathbb{R}^n$. Let $B_\varepsilon(A) = \bigcup_{a\in A} B_\varepsilon(a)$ be an ε-neighborhood of the set A.*

- *Show that the so-called Hausdorff metric*

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon(B) \text{ und } B \subset B_\varepsilon(A)\}$$

  is indeed a metric on X.

- *Is d_H also a metrik on the space of* **all** *subsets of $\mathbb{R}^n$? Justify your answer.*

Exercise 3 *The metric d_p for $p \in (0, \infty)$ is defined on $\mathbb{R}^n$ as*

$$d_p(x, y) = \left(\sum_i |x_i - y_i|^p\right)^{1/p}.$$

- *Sketch the unit ball in $\mathbb{R}^2$ for $p = \frac{1}{2}, 1, 2$ uand 100.*
- *Verify that d_p is really a metric for $p \geq 1$.*
  (Hint: Minkovski inequality.)
- *The metric d_p is also applicable on sequence spaces $\ell_p = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{C}, \sum_n |x_n|^p < \infty\}$. Show that $\ell_p \subset \ell_q$ for $1 \leq p < q$. Do d_p and d_q generate the same topology on ℓ_p?*

Exercise 4 *Let $\ell_\infty = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{C},\ \sup_n |x_n| < \infty\}$ be the space of bounded sequences equiped with the metric $d_\infty(x, y) = \sup_n |x_n - y_n|$. For $p \in [1, \infty)$, let (ℓ_p, d_p) the metric space from Exercise 3. In (ℓ_∞, d_∞) and in (ℓ_2, d_2), find the interior, closure and boundary of the following sets:*

- $Y = \{(y_n)_{n\in\mathbb{N}} : |y_n| < \frac{1}{n}\}$;
- $Z = \{(z_n)_{n\in\mathbb{N}} : \exists\ n \in \mathbb{N}\ x_n = 0\} \cap \ell_2$;

Exercise 5 *Let X be the space of 0-1-sequences $x \in \{0, 1\}^{\mathbb{N}}$, and for every integer $a \geq 2$, let*

$$\rho_a(x, y) = \begin{cases} a^{-m(x,y)} & m = \min\{n \in \mathbb{N} : x_n \neq y_n\} \text{ falls } x \neq y; \\ 0 & \text{if } x = y. \end{cases}$$

- *Show that ρ_a is a metric.*

- *Are ρ_2 and ρ_3 equivalent in the sense that there is a constant $C \geq 1$ such that*

$$\forall\, x, y \in X \quad \frac{1}{C}\,\rho_2(x,y) \leq \rho_3(x,y) \leq C\rho_2(x,y)?$$

- *Are ρ_2 and ρ_3 equivalent in the sense that the identity $I : (X, \rho_2) \to (X, \rho_3)$ is a homeomoorphism? (A homeomorphism is a continuous map with a well-defined continuous inverse.)*

- *Do ρ_2 and ρ_3 generate the same Topology?*

Exercise 6 *Let X be a topological space with subsets A, B. Prove the following statements, or find counter-examples:*

1. $\partial\partial A = \partial A$ *and* $A^\circ = (A^\circ)^\circ$.
2. $\overline{A^\circ} = (\overline{A})^\circ$ *and* $\overline{\partial A} = \partial(\overline{A})$.
3. $\partial(A^\circ) = \partial A$ *(maybe $\partial(A^\circ) \subset \partial A$ or $\partial(A^\circ) \supset \partial A$ holds?)*
4. $\partial A \cup \partial B = \partial(A \cup B)$.
5. $\partial(A \cap B) = \partial A \cap \partial B$.
6. $(\partial A)^\circ = \emptyset$.

Exercise 7 *In the co-finite topology on a space X, the open sets are U entweder $U = \emptyset$ or the cardinality $\#(X \setminus U) < \infty$.*

1. *Show that the co-finite topology indeed satisfies the axioms of a topology.*
2. *Show that in finite spaces, the co-finite topology coincides with the discrete topology.*
3. *Has the co-finite topology the Hausdorff property? (That is, are there for all $x \neq y \in X$ disjoint neighbourhoods $U \ni x$ and $V \ni y$. Is the co-finite topology metrisable?*

Exercise 8 *Order the following topologies on $X = [0, 1]$ according to finer/coarser.*

1. *the trivial topology;*
2. *the discrete topology;*
3. *Euclidean topology;*
4. *the co-finite topology;*
5. *co-countable topology: U is open if $U = \emptyset$ or $X \setminus U$ is countable;*
6. *the topology defined by $U = X$ or $0 \notin U$;*

Which of these topologies are metrisable (or just Hausdorff)?

Exercise 9 *Find countably infinite metric spaces (X, d) so that*

1. (X, d) *has* $N \in \mathbb{N}$ *isolated points;*

2. (X, d) *is complete, and another one that is not complete.*

Exercise 10 *Find a countable basis and subbasis for the following topological spaces, or show that such don't exist.*

1. $\mathbb{R}^2$ *with Euclidean topology;*

2. *The Sorgenfrey line;*

3. $\mathbb{Z}$ *with co-finite topology.*

Exercise 11 *Set* $X = \mathbb{R}^n$. *In the Zariski-topology on* X, *the closed sets are the sets of zeros of polynomials in* $x = (x_1, \dots, x_n)$. *That is,* $G \subset X$ *is closed if there is a polynomial* $p : X \to \mathbb{R}$ *such that* $p(x) = 0$ *if and only if* $x \in G$.

1. *Show that the Zariski-topology satisfies the axioms of a topology (for the closed sets approach).*

2. *Show that singletons are closed sets in the Zariski-topology.*

3. *Does the Zariski-topology have the Hausdorff property?*

4. *Are there closed sets with a non-empty interior?*

5. *Show that in dimension* $n = 1$, *the Zariski-topology coincides with the co-finite topology.*

Exercise 12 *An arithmetic sequence on* $\mathbb{Z}$ *is a set of the form* $S(a, b) := \{an + b : n \in \mathbb{Z}\}$ *where* $a, b \in \mathbb{N}$, $a > 0$. *Define on* $X = \mathbb{Z}$ *the topology* τ *in which the non-empty open sets are the unions of arithmetic sequences. Show that*

1. τ *is indeed a topology;*

2. *every non-empty is infinite;*

3. *the arithmetic sequences* $S(a, b)$ *are both closed and open;*

4. $\bigcup_{p \text{ is prime}} S(p, 0) = \mathbb{Z} \setminus \{-1, 1\}$.

5. *Conclude there are infinitely many primes.*

Exercise 13 *The Niemytzki space (or Moore plane) is the half-plane* $\mathbb{R} \times [0, \infty)$ *so that the points* p *have the following neighbourhoods ases:*

$$\begin{cases} \{B_\varepsilon(p) : \varepsilon > 0\} & \text{if } p = (x, y), y > 0, \\ \{B_\varepsilon(p) : \varepsilon > 0\} \cup \{B_\varepsilon(x, \varepsilon) \cup \{p\} : \varepsilon > 0\} & \text{if } p = (x, 0). \end{cases}$$

1. *Show that this system indeed gives neighbourhoodbases.*

2. *Determine the closures of the subsets* $A := \mathbb{Q} \times \{0\}$ *and* $B := \mathbb{Q} \times \{1\}$.

3. *Describe the relative topology of Niemytzki topologie for the above subsets A und B as well as* $C := \{(x, x^2) : x \in \mathbb{R}\}$.

Exercise 14 *Let* (X, τ) *be a* **separable** *topological space, that is: X has a countable dense subset A.*

1. *Show that* $\mathcal{B} := \{B_{1/n}(a) : 1 \leq n \in \mathbb{N}, a \in A\}$ *is a basis.*

2. *Is a subset* $E \subset X$ *with relative topology separable again?*
   Hint: Exercise 13.

Exercise 15 *According to the Theorem of Heine-Borel on* $\mathbb{R}^n$ *with Euclidean Topology, one characterize compact by closed and bounded. How can you characterize compact on* R^n *in*

1. *discrete topology;*

2. *co-finite topology;*

3. *co-countable topology?*

Exercise 16 *Let* $X = \mathbb{R}^2$ *be equiped with Zariski topology.*

- *Find a basis* $\mathcal{B}$ *of the topology consisting of sets that are not co-finite.*
- *Show that* $\overline{G} = X$ *for all non-empty open sets G.*
- *What is the closure of* $\{(\frac{1}{n}, 0) : 1 \leq n \in \mathbb{N}\}$*? And of* $\{(\frac{1}{n}, \sin(\frac{1}{n})) : 1 \leq n \in \mathbb{N}\}$*?*
- *Characterise the compact sets in Zariski topology.*

Exercise 17 *Let* $X = \mathbb{R}^2$ *be the plane mit Sorgenfrey product-topologie.*

- *Are the following sets open?*

$$(i) \quad [0,1) \times (0,1] \qquad (ii) \quad \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}.$$

- *Describe the relative topology for the following subspaces:*

$$(i) \quad \{(x,y) \in \mathbb{R}^2 : x + y = 0\} \qquad (ii) \quad \{(x,y) \in \mathbb{R}^2 : x - y = 0\}$$

$$(iii) \quad \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Exercise 18 *Let* $H = [0,1]^{\mathbb{N}}$ *be the Hilbert cube, with product topology. Thus the sets of the form*

$$B = \{(x_n) \in H : \exists N \in \mathbb{N} \text{ and open intervals } I_j \subset [0,1], j \leq N \text{ such that } x_j \in I_j\}$$

constitute a basis of this topology.

1. *Show that* $d(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} |x_n - y_n|$ *is a the metric on H.*

2. *Show that d induces the product topology.*
   Hint: Show that balls in metric d are basis element and vice versa, every element in $\mathcal{B}$ is the union of balls in d.

3. *Do d_{prod} und d_∞ (defined by $d_\infty(x,y) = \sup_{n\in\mathbb{N}} |x_n - y_n|$) generate the same topology?*

Exercise 19 1. *Show that for a subset A of a topological space X holds*

$$\overline{A} = A \cup \{x \in X : x \text{ is accumulation point of } A\}.$$

2. *Let A' denote the set of accumulation points of A. Find a subset $A \subset \mathbb{R}$ (Euclidean) such that $A' \neq A''$. Does $A'' = A'''$ hold?*

Exercise 20 *Let τ_1 and τ_2 be topologies on a set X and for every $x \in X$ let $\mathcal{B}^1(x)$ and $\mathcal{B}^2(x)$ be neighborhood bases of x w.r.t. τ_1 and τ_2, respectively.*

1. *Show that*

$$\tau_1 \text{ is coarser as } \tau_2 \iff \forall x \in X\ \forall B^1 \in \mathcal{B}^1(x)\ \exists B^2 \in \mathcal{B}^2(x) : B^2 \subseteq B^1.$$

2. *Give an example to show that the inclusion above can be strict.*

Exercise 21 *Take $X = \mathbb{R}$. Find topologies on X with bases $\mathcal{B}$ such that*

1. *B is countable and every open set can be written as countable union of basis-elements.*

2. *B is uncountable and every open set can be written as countable union of basis-elements (but there is no basis for which 1. holds).*

3. *B is uncountable and there are open sets that can only be written as* **un***countable union of basis-elements.*

Exercise 22 *Consider the square $Q := [0,1]^2$ with lexicographical order: $(x,y) \le (x',y')$ if $x < x'$ or if $x = x'$ and $y \le y'$. Let τ be the order topology w.r.t. this order.*

1. *Describe neighbourhood bases for $(x,y) \in Q$, especially for the points $(0,0)$, $(x,0)$, $(x,1)$ und $(1,1)$. Does every point have a countable neighbourhood basis?*

2. *Has (Q,τ) a countable basis and/or a countable dense subset?*

Exercise 23 *Let (X,τ) be a topological Hausdorff space. The co-compact topology is defined as*

$$\tau_{cc} = \{U \subset X : X \setminus U \text{ is compact}\} \cup \emptyset.$$

1. *Show that τ_{cc} is really a topology.*

2. *When is it true that $\tau_{cc} = \tau$? In general, is τ_{cc} coarser or finer as τ?*

3. *If $A \subset X$ is compact in τ, is the boundary ∂A in τ compact too?*

Exercise 24 *Let $(X, \leq_X)$ and $(X, \leq_Y)$ be well-ordered sets. Show that the identity mapping the only order-preserving bijection is between X and itself.*

Exercise 25 *Two sets $A \subset X$ und $B \subset Y$ in topological spaces X and Y are called homeomorphic if there is a homeomorphism $\psi : A \to B$ (in the relative topology on A and B). Which of the following subsets of $\mathbb{R}^2$ Euclidean are homeomorphic and which are not (with justification).*

$$\mathbb{R}^2 \qquad \mathbb{R} \times \{0\} \qquad [0,1] \times \{0\} \qquad \{(x,y) : x^2 + y^2 < 1\} \qquad \{(x,y) : \max\{|x|,|y|\} < 1\}$$

Exercise 26 *Let $f : X \to Y$ be a continuous mapping between topological spaces.*

1. *If $A \subset X$ is compact, show that $f(A)$ is compact in Y.*
2. *If additionally Y is Hausdorff and $f : A \to f(A)$ is injective, show that $f : A \to f(A)$ is a homeomorphism.*
3. *Show by means of an example that the assumption that A is compact is essential.*

Exercise 27

1. *Show that in a Hausdorff space, every sequence has at most one limit.*
2. *Give an example of a sequence with two limit points (that is, in a non-Hausdoff space).*

Exercise 28 *Take $X = Y = \mathbb{R}$ with Euclidean or Sorgenfrey topology, and $f : X \to Y$ as below. Is f continuous for these four possible choice of the topology on X, Y?*

$$(i) \quad f(x) = -x \qquad (ii) \quad f(x) = \lfloor x \rfloor \qquad (iii) \quad f(x) = x - \lceil x \rceil$$

Exercise 29 *Let X be a topological space with subbasis $\mathcal{S}$, and $f : X \to X$ a surjective ap.*

1. *Show that: $f^{-1}(S) \in \mathcal{S}$ for all $S \in \mathcal{S}$ implies that f is continuous.*
2. *Does the converse hold? Proof or counter-example.*

Exercise 30 *Let X be a topological space and $\sim$ an equivalencerelation on X. We denote the quotient space $X/\sim$ with quotient topology as Y. Show that*

1. *if the equivalence classes $[x]$ are not closed, then Y is not Hausdorff.*
2. *if $X = [-1,1]$ with Euclidean topology and $x \sim x'$ if $x = x'$ or $x = 1 - \frac{1}{n}$ und $x' = -1 + \frac{1}{n}$ for some $2 \leq n \in \mathbb{N}$, then allvequivalence classes are closed, but Y is still not Hausdorff.*

Exercise 31 *We saw in Example 5.6,2 of the class notes, that there is no sequence in $\Omega_0 = [0, \omega_1)$ that converges to ω_1. However, $\omega_1 \in \overline{\Omega_0}$, and therefore there must be a net in Ω_0 that converge to ω_1. Construct such a net.*

Exercise 32 *We saw in Example 5.6,2 of the class notes, that in $\mathbb{R}^{\mathbb{R}}$ with topology of pointwise convergence, the function $g(x) \equiv 1$ belongs to the closure of the set $E = \{g \in \mathbb{R}^{\mathbb{R}} \mid g(x) \neq 0$ for finitely many $x\}$, but is not the limit of any sequence $(f_n)_{n \in \mathbb{N}}$ aus E auftreten kann. Construct a net that converges to g.*

Exercise 33 *Indicate for each of the folllowing topological spaces if they are AA1, AA2 and/or separable.*

1. *The Niemytzki space (see Execise 13).*
2. $\mathbb{R}$ *with cofinite topology.*
3. *The order interval* $\Omega = [0, \omega_1]$ *in which* ω_1 *is the first uncoutable ordinal.*

Exercise 34 *Let* $X = \prod_{\lambda \in \Lambda} X_\lambda$ *be a product space with product topology. Does the statement*

X is separable if and only if every factor space X_λ *is separable*

hold if Λ *is (i) finite, (ii) countable, (iii) arbitrary?*

Exercise 35 *Let* $\mathbb{R}_S$ *be the set of real number with Sorgenfrey Topology (i.e.,* $\{[a; b) : a < b \in \mathbb{R}\}$ *is a basis of the topology).*

1. *Are the sequences* $(\frac{1}{n})_{n \in \mathbb{N} \setminus \{0\}}$ *und* $(-\frac{1}{n})_{n \in \mathbb{N} \setminus \{0\}}$ *convergent? If yes, what are their limits?*
2. *Argue which of the following sets are open/closed/compact. (Compact means that every open cover has a finite subcover.):*

$$
\begin{array}{ll}
(a, b) & [a, b] \\
[a, b) & \{0\} \cup \{\frac{1}{n})_{n \in \mathbb{N} \setminus \{0\}}\} \\
(a, b] & \{0\} \cup \{-\frac{1}{n})_{n \in \mathbb{N} \setminus \{0\}}
\end{array}
$$

3. *Show that the Sorgenfrey plane, that is* $\mathbb{R}_S \times \mathbb{R}_S$ *with product topology, is a normal space.*

Exercise 36 *Let* $\mathcal{U}(x)$ *be the neighborhood system of a point* x *in a topological space* (X, τ)*. We equip* $\mathcal{U}(x)$ *with the directed order* $U \leq_{\mathcal{U}(x)} V$ *if* $V \subseteq U$*, and for every* $U \in \mathcal{U}(x)$ *choose a point* $x_U \in U$*. Is the net* $(x_U)_{U \in \mathcal{U}(x)}$ *an ultranet? Justify your answer.*

Exercise 37 *Let* $Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ *be the unit square with Euclidean topology.*

1. *By identifying opposite sides of Q pairwise and in an orientation preserving way, one gets the torus. Which other surfaces can be obtained by identifying sides of Q (orientation preserving or reversing)? Which of these surfaces are orientable.*
2. *Consider* $Q/\sim$ *for* $x \sim x'$ *if* $x = x'$ *or* x *and* x' *lie both on the boundary of* Q*. Construct a homeomorphism between* $Q/\sim$ *and the unit sphere* $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ *(where* $\| \ \|$ *indicates the Eucliden norm).*
   Hint: Show that both $(0, 1) \times (0, 0)$ *and* $\mathbb{S}^2 \setminus (0, 0, 1)$ *are homeomorphic to* $\mathbb{R}^2$ *(stereographic projection).*

Exercise 38 *Show by means of an example that a torus satisfies at least a "seven-colour theorem". That is, find a graph on* $\mathbb{T}^2$ *that can only be coloured with seven colours.*

Exercise 39 *Take* $K = \{\frac{1}{n} : 1 \leq n \in \mathbb{N}\}$ *and let* $\mathbb{R}_K$ *be the set of real numbers with topology given by the basis* $\{(a,b) : a < b \in \mathbb{R}\} \cup \{(a,b) \setminus K : a < b \in \mathbb{R}\}$.

1. *Show that the interval* $[0,1]$ *is not compact.*

2. *Show that* $\mathbb{R}_K$ *has the Hausdorff property, but there is a closed set* A *and* $x \notin A$*, so that* A *and* $\{x\}$ *don't have disjoint open supersets.*

3. *Show that* $\mathbb{R}_K$ *connected but not arc-connected is.*

Exercise 40 *We consider the following subsets* $\mathbb{R}^2$*:*

$$\begin{array}{rl} \text{the } \sin\frac{1}{x}\text{-continuum:} & A := \{(x, \sin\frac{1}{x}) : 0 < x \leq 1/\pi]\} \cup (\{0\} \times [-1,1]); \\ & B := A \cup (\{0\} \times [-2,2]); \\ \text{the Warsaw circle:} & W := A \cup \{(x, 1 + \sin\pi^2 x) : 0 < x \leq 1/\pi]\} \cup (\{1/\pi\} \times [0,1]). \end{array}$$

1. *Which of these sets are connected and/or arc-connected? Are they continua (i.e. compact, connected metric spaces)?*

2. *The arc-component of a point* p *is the union of all connected subsets that contain* p*. What are the arc-components of* A, B *and* W*?*

3. *The composant of a point* p *is the union all proper subcontinua that contain* p*. What are the composants of* A, B *and* W*?*

Exercise 41 *Let* $f : X \to Y$ *be a continuous funktion between topological spaces, and* $A \subset X$*. Show that or give a counter-example:*

1. A *is arc-connected implies* $f(A)$ *is arc-connected.*

2. A *is locally connected implies* $f(A)$ *is locally connected.*

A set A *is called* **locally connected** *if for every* $x \in A$ *holds: for every neighborhood* $V \ni x$ *there is an open connected neighborhood* $x \in U \subset V$*.*

Exercise 42 *The Knaster continuum* K *is (any set homeomorphic to the following) subset* $\mathbb{R}^2$*:*

- *We start with* $C \times \{0\}$ *where* $C \subset [0,1]$ *is the middle-third Cantor set.*

- *For each* $x \in C$*, connect* $(x,0)$ *to* $(1-x,0)$ *by a semi-circle arched upward.*

- *Let* $J_i = [3^{-i} - 3^{-i+1)}, 3^{-i}]$ *for* $i \geq 0$*. For each* $x \in J_i$*, connect* $(x,0)$ *to* $(\frac{5}{3}3^{-i} - x, 0)$ *by a semi-circle arched downwards.*

1. *Show that* K *is connected, but not arc-wise connected.*

2. *Show that* K *has uncountably many arc-components, that all lie dense in* K*. Are all these arc-components homeomoprhc to each other?*

3. *Show that these arc-components coincide with the composants of* K*.*

Exercise 43 *Let (X, τ) be a topological space. A G_δ-set is a countable intersection of open sets. Prove or give a counter-example:*

1. *The boundary of an open set is nowhere dense.*
2. *The boundary of a G_δ-set is nowhere dense.*
3. *The boundary of a G_δ-set is meager.*

Exercise 44 *Show that the set $\mathbb{Q}$ of rational numbers with Euclidean topology is not a Baire space.*

Exercise 45 *Our metric space will be $X = [0, 1]$ with Euclidean metric, and all fractions are taken in lowest terms (so 4/6 is avoided in favor of 2/3). A number $x \in X$ is called* **Diophantine of order** ν *if*

$$\liminf_{p/q \in \mathbb{Q} \cap [0,1]} |x - \frac{p}{q}| q^{2+\nu} < \infty$$

and x is **Diophantine** *if it is Diophantine of order ν for some real order $\nu > 0$. Denote the sets of Diophantine numbers and Diophantine numbers of order ν as $\mathcal{D}$ and $\mathcal{D}_\nu$. So $\mathcal{D} = \cup_{\nu>0} \mathcal{D}_\nu$.*

The purpose of this exercise is to show that $\mathcal{D}$ is large in topological sense, but small in the sense of Lebesgue measure Λ. To simplify the argument, we will argue with definition

$$\liminf_{p/q \in \mathbb{Q} \cap [0,1]} |x - \frac{p}{q}| q^{2+\nu} < 1 \text{ instead of } < \infty.$$

Those who want can try to do the exercise with the offical definition; the basic argument is the same, just lengthier.

1. *Show that for every $r \in \mathbb{N}$ and $\nu > 0$ the set $U_r := \cup_{0 \leq p \leq q, q > r} B_{q^{-(2+\nu)}}(\frac{p}{q})$ is open and dense.*
2. *Show that $\mathcal{D}_\nu = \cap_{r \in \mathbb{N}} U_r$.*
3. *Conclude that the complements of both $\mathcal{D}_\nu$ and $\mathcal{D}$ are meager.*
4. *For the remainder of this exercise, you don't really need details of Lebesgue measure λ. It suffice to apply the*

   Borel-Cantelli Lemma: *If $(Y_r)_{r \in \mathbb{N}}$ is a sequence of subsets of X such that $\sum_{r \in \mathbb{N}} \lambda(Y_r) < \infty$, then the set of points $x \in X$ such that $x \in Y_r$ for infinitely many $r \in \mathbb{N}$ has Lebesgue measure zero.*

   Using this lemma, show that the set of $x \in X$ such that $x \in U_r$ for infinitely many $r \in \mathbb{N}$ has Lebesgue measure zero.

5. *Show that $\lambda(\mathcal{D}_\nu) = 0$ and $\lambda(\mathcal{D}) = 0$. (You may use the fact that coutable unions of sets of Lebesgue measure zero have Lebesgue measure zero.)*

Exercise 46 *Verify the Euler characteristic of the projective plane, and the tori with two (pretzel) and three holes, by means of triangulation.*

Exercise 47 *Compute the Euler characteristic and genus of the surface that emerges from a hexagon by identifying its opposite sites in an orientation preserving way.*

Exercise 48 *Show or find a counter-example:*

1. X_i *is connected for all* $i \in I \Rightarrow$ *the product space* $\prod_{i \in I} X_i$ *is connected.*
2. X *is connected and* $A \subset X \Rightarrow A$ *with relative topology is connected.*
3. X *is connected and* $\sim$ *an equivalence relation* $\Rightarrow$ *the quotient space* $X/\sim$ *is connected.*

Exercise 49 *Are the following topological spaces connected and/or arc-connected? Justify your answer.*

1. $\mathbb{R}$ *with co-finite topology.*
2. $[0,1]^2$ *with orderingstopology w.r.t. the lexicographical Order:* $(x_1, y_1) < (x_2, y_2)$ *if (i)* $x_1 < x_2$ *or (ii)* $x_1 = x_2$ *and* $y_1 < y_2$.

Exercise 50 *Are the following topological spaces Baire space?*

1. $\mathbb{R}$ *mit ko-endlicher Topologie.*
2. $(\mathbb{Q} \times \{0\}) \cup (\mathbb{R} \times (0,1]$ *with euclidean topology.*
3. $[0,1]^2$ *with topologie* $\tau = \{[0,x) : x \in (0,1]\} \cup \{\emptyset\}$.

Justify your answer.

Exercise 51 *Let* $\{q_n\}_{n \in \mathbb{N}}$ *be a denumeration of the rational umbers, and* $U_k = \bigcup_{n \in \mathbb{N}} (q_n - k^{-n}, q_n + k^{-n})$ *for* $k = 1, 2, 3, \dots$. *Are the following statement tryue or false?*

- U_k *is* G_δ*-dense;*
- $\bigcap_{k \geq 1} U_k \supset \mathbb{Q}$*;*
- $\left(\bigcap_{k \geq 1} U_k\right) \setminus \mathbb{Q} = \emptyset$.

Justify your answer.

Exercise 52 *Let* (X, τ) *be a compact Hausdorff space.*
(a) Show that every point is closed.
(b) Show that for every point $x \in X$ *and non-empty closed set* $A \not\ni x$*, there are disjoint open sets* U_x *and* U_A *so that* $x \in U_x$ *and* $A \subset U_A$.
(c) Show that every non-empty open set $U \subset X$ *has an open nonempty subset* V *with closure* $\bar{V} \subset U$.