On functions with a conjugate

Joint work with Mike Eastwood

Two functions $f, g : (M^n, c) \to \mathbf{R}$ **are conjugate iff at each point of** M^n

$||\text{grad}\, f|| = ||\text{grad}\, g||$ and $\langle \text{grad}\, f, \text{grad}\, g \rangle = 0$.

Semi-conformal maps

$\varphi : (M, \mathcal{G}) \to (N, h)$ **is semi-conformal if and only if**

$\exists$ **continuous function** $\lambda : M \to \mathbf{R}\,(\geq 0)$ **such that**

$$\mathrm{d}\varphi_x \circ (\mathrm{d}\varphi_x)^* = \lambda(x)^2 \mathrm{Id}|_{T_{\varphi(x)}N} \,.$$

if and only if in local coordinates (x^i) **and** (y^α)

$$\mathcal{G}^{ij}\varphi_i^\alpha \varphi_j^\beta = \lambda(x)^2 h^{\alpha\beta}$$

$f, g : (M, \mathcal{G}) \to \mathbf{R}$ **are conjugate if and only if**

$\varphi = (f, g) : (M, \mathcal{G}) \to \mathbf{R}^2$ **semi-conformal**

***Question*: What differential condition on a function** f **ensures that it admits a conjugate** g**?**

Answer when $n = 2$**: iff** f **is harmonic.**

A necessary condition

f **admits conjugate** $\Rightarrow \exists$ **closed** 1**-form** ω **s.t**

$f^j\omega_j = 0$ **and** $\omega^j\omega_j = f^j f_j$

$\Rightarrow f^{ij}\omega_j + f^j{\omega^i}_j = 0$ **and** $\omega^{ij}\omega_j + f^{ij}f_j = 0$

$$\Rightarrow \left\{ \begin{array}{rcl} f^{ij}\omega_i\omega_j + f^{ij}f_i f_j & = & 0 \\ \omega^j\omega_j + f^j f_j & = & 0 \end{array} \right. \textbf{ - two quadratics in } \omega_i\textbf{.}$$

This gives the necessary condition:

$$||\nabla f||^2(\Delta f)^2 \leq (n-2)\left[||\nabla f||^2 \mathrm{Tr}\left((\nabla^2 f)^2\right) - 2||(\nabla^2 f)\nabla f||^2\right]$$

$\dim M = 3$: some conformal invariants

$J := ||\nabla f||^2 = f^i f_i$ **(weight -2)**

$X := 2 f_i{}^j f_j f^{ik} f_k - f^i f_i f^{jk} f_{jk} + f^i f_i (f^j{}_j)^2$

$Z := f^{ij} f_i f_j + f^i f_i f^j{}_j$ **(3-Laplacian)**

$Y := Z^2 - 2JX$

Normalization at a point x

$$f_1 = f_2 = f_{12} = 0 \Rightarrow \omega_3 = 0$$

$J = {f_3}^2$

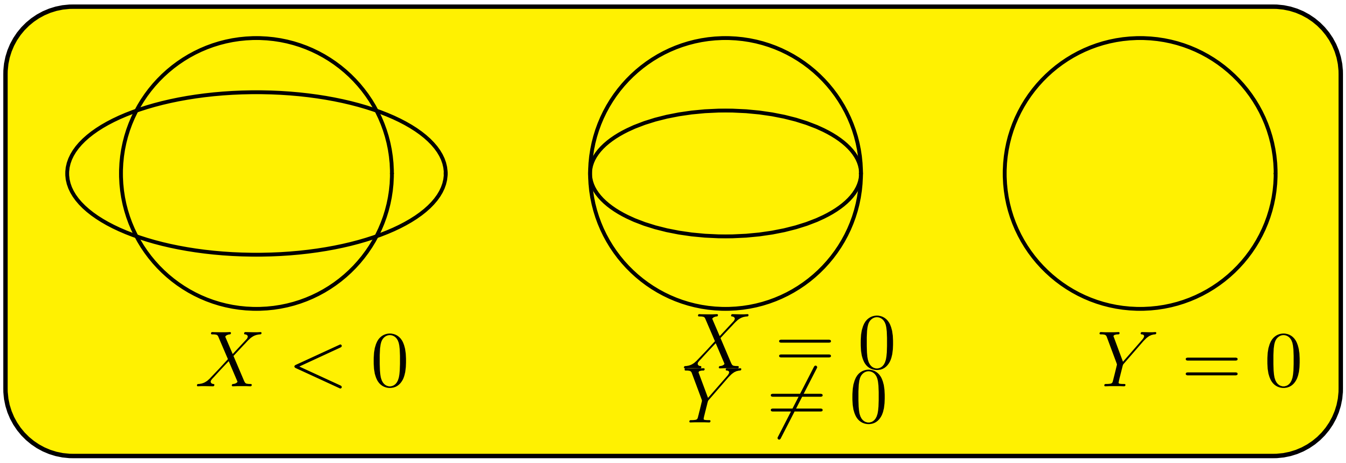
$X = 2{f_3}^2(f_{11} + f_{33})(f_{22} + f_{33})$

Conjugate directions

$$X \neq 0 \iff \exists\ 4 \textbf{ distinct solutions}$$

$$X = 0 \textbf{ and } Y \neq 0 \iff \exists\ 2 \textbf{ distinct solutions}$$

$$X = 0 \textbf{ and } Y = 0 \iff \exists\ \infty\textbf{-many solutions.}$$



Integrability of conjugate direction

The generic case $X < 0$

$$\begin{aligned} & \omega \textbf{ integrable} \\ \Leftrightarrow \quad & \omega_{ij} \textbf{ symmetric in its indices} \\ \Leftrightarrow \quad & u^i v^j (\omega_{ij} - \omega_{ji}) = 0 \textbf{ with } u^i, v^j \textbf{ lin. ind.} \end{aligned}$$

Take $u^i, v^i \in \{f^i, \omega^i, f^{ij}\omega_j\}$ l. i. since $X < 0$.

$f^i \omega^j (\omega_{ij} - \omega_{ji}) = f^{ij} f_i f_j + f^{ij} \omega_i \omega_j$ vanishes by assumption. Differentiate RHS:

$$\begin{aligned} 0 &= f^i \nabla_i (f^{jk} f_j f_k + f^{jk} \omega_j \omega_k) \\ &= (\text{terms involving } \omega^i) + 2 f^{jk} \omega_{ij} f^i \omega_k \end{aligned}$$

Latter term is first component of symmetry cond: $f^i f^{jk} \omega_k (\omega_{ij} - \omega_{ji}) = 0$, which holds iff:

$$0 = (\text{terms involving } \omega^i) + 2 f^{jk} \omega_{ji} f^i \omega_k$$

Now replace $\omega_{ji} f^i$ with $-f_{ji} \omega^i$.

A first criterion for $X < 0$.

$$
\begin{aligned}
0 &= f^{ijk} f_i f_j f_k + f^{ijk} f_i \omega_j \omega_k + 2 f^{ij} f_j{}^k f_i f_k - 2 f^{ij} f_j{}^k \omega_i \omega_k \\
0 &= f^{ijk} f_i f_j \omega_k + f^{ijk} \omega_i \omega_j \omega_k + 4 f^{ij} f_j{}^k f_i \omega_k
\end{aligned}
$$

Let η be the other conjugate direction (defined up to sign)

$$
\sqrt{Y} \eta_i = 2(f^{jk} f_j \omega_k) f_i + (Z - 2 f^{jk} f_j f_k) \omega_i - 2 J f_i{}^j \omega_j
$$

$$
\begin{aligned}
p^+ &:= f^{ijk} f_i f_j f_k + f^{ijk} f_i \omega_j \omega_k + 2 f^{ij} f_j{}^k f_i f_k - 2 f^{ij} f_j{}^k \omega_i \omega_k \\
p^- &:= f^{ijk} f_i f_j f_k + f^{ijk} f_i \eta_j \eta_k + 2 f^{ij} f_j{}^k f_i f_k - 2 f^{ij} f_j{}^k \eta_i \eta_k \\
q^+ &:= f^{ijk} f_i f_j \omega_k + f^{ijk} \omega_i \omega_j \omega_k + 4 f^{ij} f_j{}^k f_i \omega_k \\
q^- &:= f^{ijk} f_i f_j \eta_k + f^{ijk} \eta_i \eta_j \eta_k + 4 f^{ij} f_j{}^k f_i \eta_k
\end{aligned}
$$

f **admits a conjugate** $\Leftrightarrow$

$p^+ = q^+ = 0$ **or** $p^- = q^- = 0 \Leftrightarrow$

$p^+ p^- = 0$**,** $q^+ q^- = 0$**,** $(p^+ q^-)^2 + (p^- q^+)^2 = 0$

More conformal invariants

$$\left.\begin{array}{l}\psi \text{ inv. wt. } v \\ \varphi \text{ inv. wt. } w\end{array}\right\} \Rightarrow v\psi\nabla_i\varphi - w\varphi\nabla_i\psi \text{ inv.}$$

1-form of wt. $v + w$. Then

$\sigma_i = J\nabla_i Z - 2Z\nabla_i J$ and $\tau_i = J\nabla_i X - 3X\nabla_i J$

are inv. 1-forms $\Rightarrow R = f^i\sigma_i,\ S = f^i\tau_i$ inv.

$$E = \varepsilon^{ijk} f_i \omega_j f_k{}^l \omega_l$$

is invariant: $E^2 = -J^2X/2$. Then E changes sign under $\omega \leftrightarrow \eta$.

Q^{ij} quadratic form:

$$Y(Q^{ij}\omega_i\omega_j - Q^{ij}\eta_i\eta_j) := 4Ev$$

For $Q^{ij} = f^{ijk}f_k - 2f^{ik}f_k{}^j$, define $V := 4Jv$ odd inv. depending only on f and its derivatives.

Eliminating ω from polynomial expressions $F(f_i, f_{ij}, f_{ijk}, \ldots) = 0$

Two identities

Let Q^{ij} be any symmetric form. Then

$$
\begin{aligned}
Y(Q^{ij}\omega_i\omega_j + Q^{ij}\eta_i\eta_j) &= 2Q^{ij}f_if_j(JX - Z^2) \\
&\quad +2J^2{Q_j}^j(Z{f_l}^l - X) \\
&\quad -2J^2ZQ^{ij}f_{ij} + 4JZQ^{ij}{f_i}^kf_kf_j \\
\sqrt{Y}Q^{ij}\omega_i\eta_j &= -ZQ^{ij}f_if_j + 2JQ^{ij}f_i{f_j}^kf_k \\
&\quad +J^2({f_k}^k{Q_l}^l - Q^{kl}f_{kl})
\end{aligned}
$$

$$\Rightarrow$$

$$
\begin{aligned}
Y(p^+ + p^-) &= ZS - 2XR + 2XY \\
Y(p^+ - p^-) &= EV/J
\end{aligned}
$$

Answer for the generic case

$$4Y^2p^+p^- = Y^2(p^+ + p^-)^2 - Y^2(p^+ - p^-)^2 \Rightarrow$$

$$p^+p^- = 0 \Leftrightarrow$$

$$P := 2(ZS - 2XR + 2XY)^2 + XV^2 = 0$$

$$q^+q^- = 0 \Leftrightarrow$$

$$\begin{aligned} Q \quad := \quad & \tfrac{1}{6}JZB - \tfrac{1}{4}JU - \tfrac{1}{4}ZS^2 \\ & + X(XZ^3 - JX^2Z + 6W + \tfrac{1}{4}JM \\ & - \tfrac{2}{7}ZXR + \tfrac{5}{7}RS - \tfrac{15}{7}N + \tfrac{2}{9}ZA \\ & - \tfrac{9}{10}F - \tfrac{2}{21}ZK + \tfrac{10}{21}T + \tfrac{6}{25}G - \tfrac{17}{42}JD) \\ & \qquad = 0 \end{aligned}$$

$$(p^+q^-)^2 + (p^-q^+)^2 = 0 \Leftrightarrow .....$$